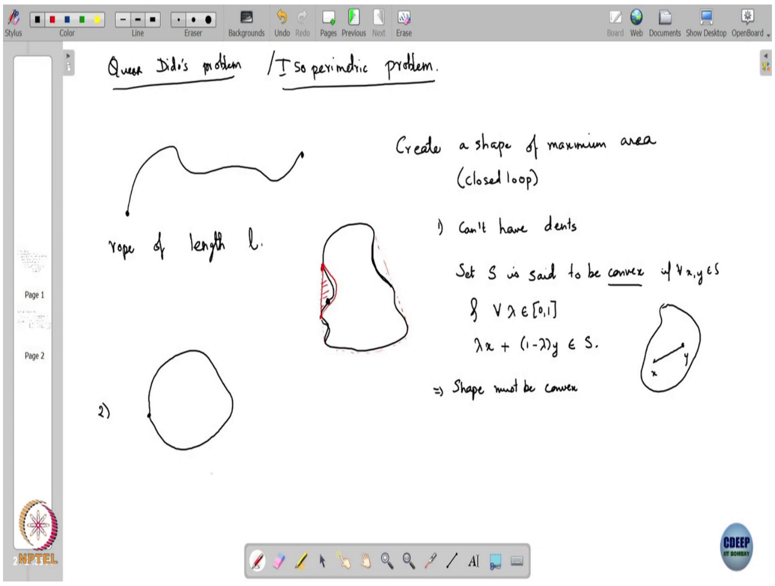
Optimization from Fundamentals
Prof. Ankur Kulkarni
Department of Systems and Control Engineering
Indian Institute of Technology, Bombay

Lecture – 1B
Isoperimetric problem

Let me give you an example of an approach to optimization which is a kind of classical approach that does not use any modern arguments, ok.

(Refer Slide Time: 00:38)



So, this is this problem is it is it has known by several names, one name for it is what is called Queen Dido's Problem. Another name for it is what is called the Isoperimetric Problem.

So, the problem is the following. You have been you have given a rope, ok; this is some rope you have been given of a certain length of a length say l. The given a rope like this, you have to create a shape, ok; you have to create a shape out of this, create a shape of maximum area.

So, you have a plane on which there is lying with this rope. You have to create a loop out of this of maximum area ok that encloses the maximum area. So, by shape by this is I mean a closed loop that enclose the, so the enclosed area has to be maximum, ok.

Now, so do does anyone know the solution of this?

Student: (Refer Time: 02:17).

It is the circle, ok. Why is it a circle?

Student: (Refer Time: 02:20).

So, you could of course, use a there is a there is a mathematical way of doing this, but like what I want to show you is this is sort of a, this is how sort of primitive people did optimization without even knowing it, ok.

So, yes this is an (Refer Time: 02:35), but the reason I want to show you this is not because to teach you out as a technique, but rather to point out something else. So, but anyway this is also a pretty argument. So, let me let us just go through this, ok.

So, first, so we want to create some shape, right. So, suppose this is some shape that we create, ok. So, this is your start and end point, ok. Now, is there some way you can argue that this is certainly not optimal?

Student: Yeah.

So, suppose I in addition to this rope ok, I also give you a rubber band what you do is you take this rubber band stretch it and fit it on top of this shape. Now, how would that rubber band fit on top of all this? So, I am just going to draw it with that I do not know if I, I think I can change color here. Let me draw it with red. How would that thing fit?

So, it would be tote around these points this way, then it will kind of trace the shape here and then again it will become tote here and maybe tote here like this. Now, look at these sort of points here, right, where the rubber band has become tote like this. Here, the rubber band is basically a straight line, right.

Now, from this dot to this dot, the shortest path is the path taken by the rubber band. It is the it is the straight line distance; whereas, the black part which the rope has taken is has to therefore, be longer.

So, instead of keeping the doing this if I simply allow, if I simply took the rope, let the rope follow the straight line path then I would have used less of the rope and increase the area because now on this part would have become part of the area, right. So, without in the same rope budget I would have increased the area for sure, right.

I have a rope budget of length l without violating that I would have increased the area. So, by what does what this means is the first we have here we have our first observation which is that we cannot have dents like this, this kind of a dent, this sort of a dent, this kind of a dent, all of these dents can be flattened out, right, ok. We cannot have dents like this.

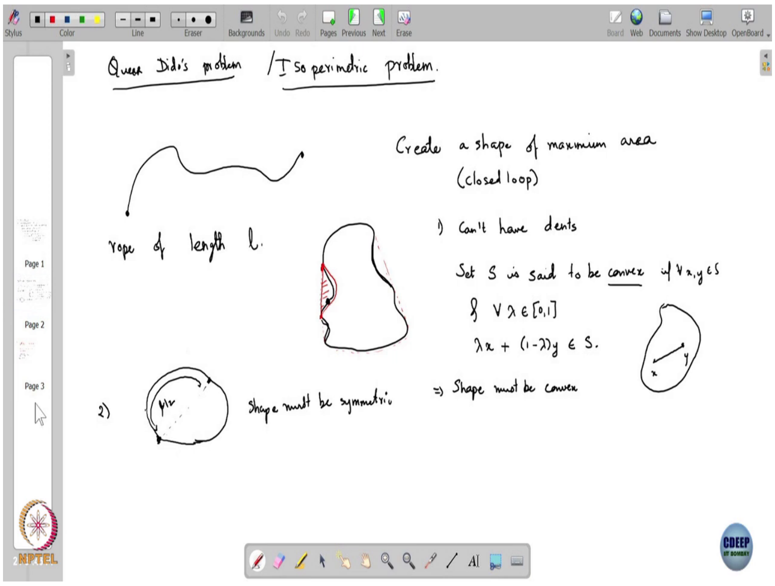
So, all these dents can be flattened out. And this can be made formal. So, a set let us call the set S, the set S is said to be convex if for all x, y in S and for all lambda in the unit interval lambda x plus 1 minus lambda times y belongs to S.

Now, what is this as lambda goes from 0 to 1 what are we doing here? You have you have taken some set S and here are two points x and y unit. As lambda goes from 0 to 1, what are we spanning here? We are spanning the entire segment joining x and y, right. So, what this says is, so the set is said to be convex if for all x, y in, for if for all x, y in S the entire segment joining x and y is belong to S, right.

So, what our first observation which is in which we basically colloquially said that it cannot have dents, what it is effectively saying is that the shape whatever is the shape of maximum area must be convex, ok. So, our first observation is that the shape must be convex, right, ok.

Now, here is the second observation. So, you create some shape. Now, we know it has to be convex, but we do not know what it looks like. So, here is I have drawn some convex shape, ok. This is now suppose I do the following I consider, so this whole perimeter is length l. So, I start from here, I and I start from this end point, start from this end point and go this is not convex, yeah.

(Refer Slide Time: 08:12)



Suppose, this is my end point and I start from this end point and I go in this direction and I stop at the point where the perimeter, where the length has become l by 2, ok. So, half the rope is on one side, half the rope is on the other side, ok. So, this here, this length all of it is l by 2, the other the other half is also l by 2, ok.

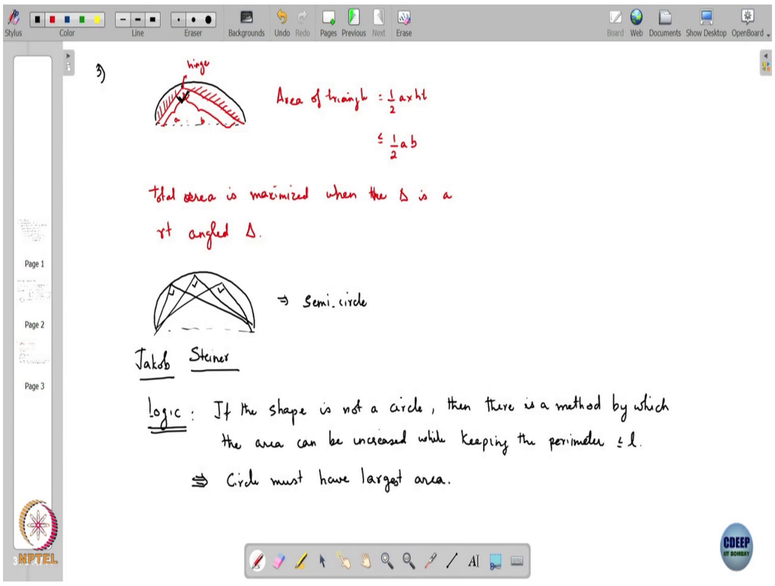
So, now any observation you can make about this? So, if you look at the look at the shape formed by one half you can always replicate it as shape formed by the other half, correct.

So, now, if the left if the left half, whatever you want to call the left half or one half if that has lesser area than the other half then you can simply copy the shape of the other half onto this one and increase the area. So, what this means is around along this line the shape should be symmetric.

Student: (Refer Time: 09:38).

Right. If I take this line which is which joins this the starting point to l to the l by 2 point, around this line the shape must be symmetric, ok. It must be symmetric if it is of largest area. So, this is necessary, ok.

(Refer Slide Time: 10:06)



And now you have the third observation. So, if it is symmetric, then it suffices to just look at half of the shape. So, let us just look at half of the shape and try to say something about it. So, here is your l the diameter or l by 2 line and here you have this other, here you have the, ok.

Now, question now is what should be the shape of this half? Ok. So, any ideas?

Student: (Refer Time: 10:38).

Why circle (Refer Time: 10:42).

Student: (Refer Time: 10:43).

Symmetric does not automatically give that, it is a circle. See it could have been an ellipse for example, and both halves would have looked symmetric.

Student: But it is like a (Refer Time: 10:59).

No, but I have taken the l by 2 line, so I have taken a specific line. I am not arguing that it is a symmetric around every line. So, this would be you should think of sort of calculus free arguments, you know you do not let us not think of infinite smells and so on. This is purely a geometry problem. So, here is one approach.

So, for example, consider this some point here on the on this line ok, and I will and so let us just look at this triangle, starting from here till here. Now, the way it looks like now to my to me, it looks like this angle is obtuse. Now, just imagine I had a I had a hinge here, I had a hinge here and I could move these two parts around this hinge, ok.

These two parts that have been formed I can move them around this way. When I move them, what would happen? These areas this one, this shaded area and this shaded area would remain the same, right. The sides of the triangle what I am hinged by moving thinking of that as a hinge, all I am doing is changing this angle here, this particular angle.

But I am not changing the areas that are highlighted the shaded areas; they remain the same, right. So, I can keep moving it. I am also not changing the lengths of these sides of the triangle, this side length, this side. Let us call this suppose a and this would b, they also remain the same. Correct.

Now, while I when I move this around the shaded areas do not change, but the area in this triangle triangle formed by these a, b, and that dotted base the area of that triangle will change, it may increase, it may decrease. Now, can you tell me what should I make the angle to get the maximum area?

Student: (Refer Time: 13:25) 90 degrees.

Why 90 degrees?

Student: (Refer Time: 13:29).

Yeah. So, the angle, so the area of the triangle is half base into height. So, let us take one of these as the base. Let us take the base as a. So, base is not changing when I change the angle the height is changing. And what is the maximum height can I that I can get? I can make the these other side perpendicular to it and that will give me height equal to b, right.

So, area of triangle this triangle is half a into height and the height is maximum when it is equal to b which happens when that angle is the right angle triangle. And so, this is why atmost half a into b, correct.

So, the shaded areas do not change. Triangle area is maximized when it is a right angle triangle, right. So, total area is maximized when this is a right angle triangle, right. Now, here is the interesting thing. So, now, we, so this what we have concluded is well the shape of maximum area must have a right angle must form a right angle here. But then I did this by choosing an arbitrary point on the circumference and making thing, imagining a hinge at that point. I can do this for every point, right.

So, what it means is all of these, the shape should be such that all of these this this this all of these should be right angles and for that to be that we know happens only for a circle, right. So, then if all of these are right angle, right angle triangle implies that this must be a semi circle.

And the total thing should be a circle, is it clear. So, this is a completely calculus and algebra free argument show of showing what you already know that the solution of the isoperimetric problem is that is the circle, ok. So, this argument is due to Steiner. I do not recall the year, it is due to this is this called Steiner's proof, Jakob Steiner, ok.

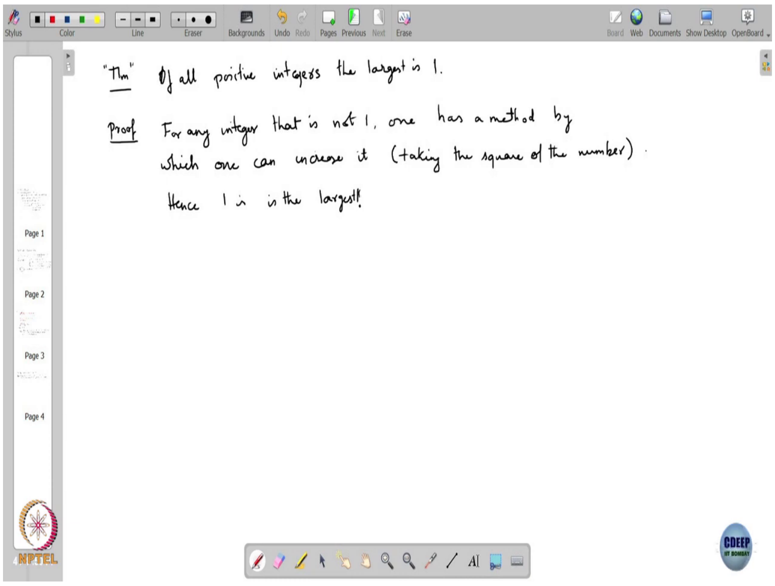
So, now, so there is something interesting that has happened that is the result one is that of course we have got we have got this conclusion, but what there is an there is something slightly more subtle that has also happened. So, what was the logic that we followed if you think about it? What was the logic?

He said a consider a shape, ok, say suppose if this shape is not a circle then it can, then there is a we have what we did is we said we presented a method of manipulating it one was by removing dents ok, second is by symmetrization, and third is by introducing this hinge.

We presented this method of manipulating it in such a way that its area can be increased, right. So, what was the logic? And hence, we concluded that the circle must have the highest largest area. Logic is if there is a and the conclusion is and ok.

Now, there is a problem with this logic. The problem is the following. I can use this logic to prove many kind of things, such as for example.

(Refer Slide Time: 18:07)



I can prove this theorem. The theorem is of all positive integers, the largest is 1. And the proof is what is the proof? If 1 is, if I have an integer that is not 1, then I have a method of increasing it to get an even larger number, larger integer.

Method of increasing it would be say for example, squaring it. And I can keep doing this and eventually conclude there are by this therefore, the 1 is the largest number. If it is not 1, then squaring it will increase the number, so therefore 1 is the large, right.

So, there is you can see that there is something has gone wrong somewhere. I have applied the same logic as I did for this circle problem for any. So, this is obviously, absurd and so the reason it is absurd is that we have implicitly assumed that there is actually a solution to the problem.

So, the problem, earlier problem was that you have a rope; you wanted to find a shape that maximizes the area. You have assume that there is such a shape, likewise here we have assumed that there is such a thing as the largest integer whereas, there is no such thing as the largest integer, right.

In both cases, what the there is a the logic has a gap which is that we have assumed or we have taken for granted the existence of an optimal solution, ok. In the first case, there was no problem because the even though we did not assume it miraculously there actually exists an optimal solution. In the second case, the logic actually catches you, red handed because you did not, you have implicitly assumed it, but there is no solution that actually is no solution, ok.

So, what this means is we have to be careful, ok. So, this is why there is a need for a more rigorous and careful approach to optimization. So, this is this. So, the purpose I am telling you the story is two-fold, one is just show you a very cute and elegant proof of this isoperimetric problem.

The second is also to is to bring to for the fact that one has to read the subject very carefully. So, what it means for a solution? What is the solution? When does it exist? Etcetera, are all question one has to be one has to be one has to take into account, ok.