Optimization from Fundamentals
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Lecture – 7A
Transformation of Optimization Problems - I

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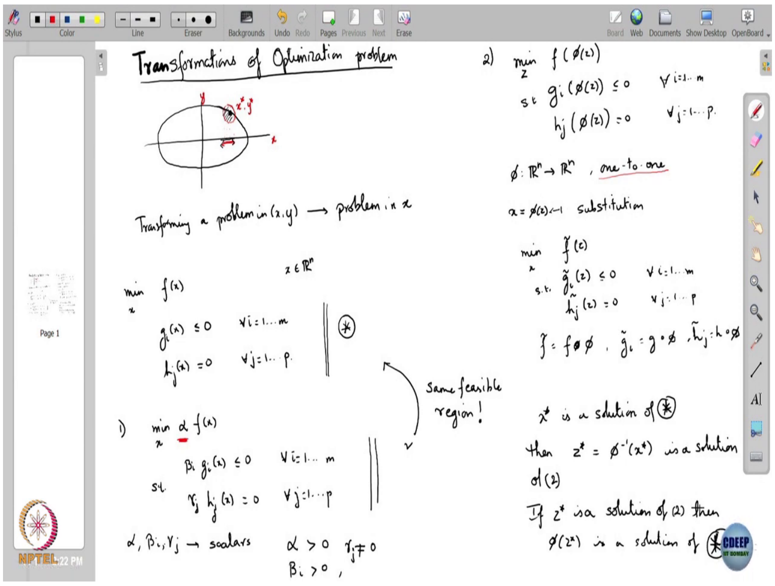

Ok. Welcome everyone to the next lecture on. So, what I plan to discuss today is what is called Transformations of Optimization Problems. How? You should relate this to what we did when we were talking of optimization over equality constraints. So, remember we were talking about this problem, where we had this ellipse and we wanted to find the inscribed rectangle with maximum area and we said we can take a point like this x star, y star on the ellipse.

Suppose that gives you the maximum area and then what we did was we have said we will use implicit function theorem to around x star, y star in a neighbourhood of x star, y star, solve for y in terms of x. For every y that we that was in this every x comma y in this neighbourhood was written in such a way that y could be obtained as a function of x. So, all so, if I gave you an x in the in this region here, I you could give me a y that lied exactly on the, right.

So, this helped you transform this was a case of basically, transforming a problem, a problem of which is in two variables x and y, the problem in x, y to a problem only in x right; so, problem in x. Now, this is a very sophisticated transformation and it was affected by the use of implicit function theorem. And moreover because of the nature of the implicit function theorem the limitations of it we can only say that it holds in this neighbourhood.

So, it was a locally valid transformation, but there are certain transformations you can make which hold on the entire region, but do not change the solution of the problem, ok. That is worthwhile I will be focusing on today. And they are as a result of them being such general purpose transformation they are also in some sense very simple.

But sometimes they are not easy to notice, you may not it is not it may not be priori obvious to you that this could this could be done. So, this, but so, I want to spend some time discussing this, ok. So, suppose we consider let us consider an optimization problem like this you I mean maximum function f of x.

The decision variable is x or you have inequality as well as equality constraints. So, you have constraint like this g i of x less than or equal to 0 for all i from 1 to m and h j of x equal to 0, for all j changing from 1 to p.

So, this will be my canonical optimization problem and I will write this in a different equivalent form. So, first thing is let us take the first form. So, do you notice that, this is actually equivalent to this optimization problem is equivalent to this problem?

So, what I have done is I have multiplied the objective by alpha, multiplied the inequality constraints by beta, ith inequality constraint by beta i and the jth equality constraint by gamma j. Now, are these two equivalent? Alpha, betas, gammas are all scalars. These are all scalars. Are these two equivalent?

Student: (Refer Time: 05:05).

Yeah. So, if the for the for equivalence you need that there they are there they you need some constraints on the signs of this, right. So, alpha beta gamma these are all scalars. I am going to take alpha is positive, all the betas also positive, ok.

And gamma's, I will make sure that I need that gammas are not 0 ok. So, if I do this what is what happens as a result of this? So, the objective function scales ok, the objective function is being scaled by alpha when I do this, ok. So, you earlier you whenever you are getting one unit of objective now you get alpha units of objective ok that is it. Now, what happens to the feasible region? Your objective function is scaling, what about the feasible region?

Student: (Refer Time: 06:11).

There is a feasible region scalar. So, the feasible region is this, the individual functions that define the feasible region are scaling, right. These functions the constraints that define the feasible region they are scaling. So, your g i of x is being scaled by beta i scaled up or down depending on whether beta is greater than 1 or less than 1. Similarly, h j of x is being scaled by gamma, gamma j. They are all being scaled, but the x's that satisfy the constraint they are still the same, right.

So, what; that means, is if there is any x if there is any x that solves these constraints that satisfies these constraints ok, let us call them let us denote these by star if there is any x that is feasible for star, then it is also feasible for the optimization problem here ok in 1. Likewise, if there is any x that is feasible for the optimization problem 1, it is also feasible for star. So, the search space the space over which you are searching right, the x's over which you are searching has not changed, ok.

So, the feasible regions of both optimization problems are the same. So, let me note this. They have the same feasible region, ok. What about the objective? The objective of 1 is is being scaled by is the objective of 1 is alpha times the objective of star, ok. So, the objective value will be scaled. So, if you send these two problems into a sub routine that solve them, what are you going to get as same for the both.

Student: (Refer Time: 08:20).

Suppose they both have one unique solution what would you get what would be the same in both? The solution would be the same. The optimal x would be the same. What about the objective that comes out of it?

Student: (Refer Time: 08:38).

Yeah. So, the objective that comes out of 1 will be alpha times the objective that comes out of star, right. So, when I say equivalence we have to be precise what we mean by equivalent. So, equivalent by broadly speaking we mean that if you solve one you should be able to solve the other if you or you should be able to recover the solution of the ok.

This is why what we mean by equivalence. Equivalence does it can mean something more specific that the solution of one is exactly the solution of the other, but sometimes it is not just that. You know, it is not just that the solutions are the same.

It you should be through some transformation you should be able to get from one solution to the other, ok. So, in this case it does happen that the solutions are actually the same, alright. So, the if you take the solution sets of both optimization problems are the same just that the objective values is a scalar, right. So, so what I so, we are I am thinking of these as equivalent these this transformation as these two optimization problems as equivalent.

Because essentially what has happened is there is a simple way of going recovering the solution of one from the other, right alright. Let us look at another problem, right. Suppose consider this problem. You suppose you have considered a problem that looks like this. Minimizing a function f of some phi of z where g and g i subject to the constraint g i of phi of z less than equal to 0.

Decision variable is x is z, sorry and h j of phi of z equal to 0. Now, what is phi here? Phi is just let us say if my x is in here x is suppose in R n, then phi is a function that from R n to R n and let us say it is one-to-one. So, phi is sum one-to-one mapping from R n to R n, ok.

So, now, you what do you see about what do you notice about problem 2? Yeah. So, what do you what can; it is a function of a function, right. So, f all, So, f has been composed with phi, all the g i's have also been composed with phi, all the h j's have been composed with phi.

So, what do you notice about this? See, what is going on here is if you think about this, yes you are searching over z's right; you are searching over the z the optimization problem is now written in the space of z. You are searching overall values of z, but the ways z enters into the problem is always through phi of z.

So, the way z enters into this is in terms of in the constraints is that what matters is not the value of z per say, but the value of phi of z right. Similar in the that is there in the inequality constraints and likewise in the equality constraint. What matters is the value of phi of z and similarly in the objective also what matters is the value of phi of z, right.

So, what what this means is effectively the although the problem 2 does not have the same feasible region as problem 1. These are two very different the space of x and the space of z are different the different spaces, ok. It does not have the same feasible region as 1, but you can , but you can still say or transform two back to 1; two back to star I mean not 1 star. You can transform problem 2 back to problem star and how do you do that?

Student: (Refer Time: 13:12).

Yeah. So, what you what we need here is a change of variables is something like what you do in calculus integration and so on you. Suppose I just denote x as phi of z then what is happening when I change my z? I am as I vary my z, I am spanning over the end all possible values of x, right. And why am I spanning over all possible values of x? It is because the function is one to one, right. Every x has a z associated with it and every z has an x associated with it. Is this clear?

So, because this is one to one as I range over the values of as I range over the values of z here, I am also search effectively searching also over the values of x, right. So, consequently I can just do this substitution, yeah do the substitution. After this substitution I am I get back to problem star. Is this clear? So, so problem 2 is actually another way of writing problem 2 is simply that I am doing something like this, f tilde of z subject to g tilde of z less than or equal to 0.

And h tilde of z equal to 0 ok, where f tilde is simply f f composed with phi, g tilde i is also g composed with phi and likewise h tilde h composed with phi. Is it clear? So, if I had shown you a problem in this sort of form without it a being made explicit that these are actually all composed on the right hand side on from the right by phi right; it may not be immediately obvious to you that there is such a function.

There is such a function lurking there, ok. But, this is one of the tricks to notice that if you know for example, changing x to minus x or changing doing some sort of a transformation to change the your coordinates or the way the x is represented; all of these are our one to one transformations, right. So, those can they do not they what they you they do not they basically give you an equivalent problem.

So, in what sense are these two equivalent? If you if I have a solution x here of star if I have a solution, let us say x star is a solution of if x star is a solution of star then, z star which is can be written, which is phi inverse of x star is a solution of 2. And likewise if z star is a solution of 2. Then, phi of z star is a solution of. Now, notice the I have meant I want to emphasize once again that phi is one to one, without that this is not going to work, ok.

So, for example, if phi is not one to one if it is many to one ok. So, there are multiple z's that give you the same x, then what would happen? How would 2 and star be related? So, when you want to establish the relation between optimization problems there are two things that you need to look at one is the feasible region second is object, ok.

So, object if well point at every value of at, every value of if I make this transformation x equal to phi of z into what I am doing is I am just evaluating f of x at some other point. So, objective is not so much of a concern here. What happens to the feasible region if phi is not one to one? So, which has the larger feasible region or are they not will are they not related at all, ok? Why does your someone saying here is that 2 has a larger feasible region, why is that?

Student: (Refer Time: 19:03).

There are many values of z that can map to x is that a correct answer? Ok. So, if phi is if phi is many to one what is happening here? Let us this is actually something that does not need much mathematics. You just need to notice what is going on. So, you are of course, a what the way z enters into the problem is through its value under phi, ok.

What matters is phi of z. So, as z varies over this range you get to a certain ranges of in the values of phi of z. Now, when you make this substitution and get rid of the connection between x and z when you substitute phi of z by x and get rid of the connection that x is equal to phi of z, right that connection is gone now. You have just substituted x equal to phi of z right and what you are what you are by the what you are saying is you have you are now solving star.

Now, you may get you may be searching over x's that do not that do not have a corresponding z, right. So, this the search space the feasible region of star includes all the values of phi of z that could have come from the feasible region of z of 2, right.

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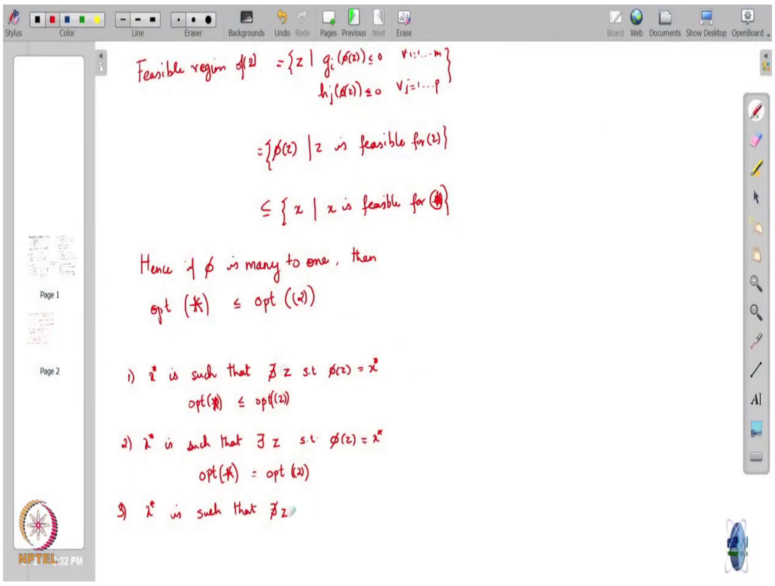

So, you take the feasible region of 2. Feasible region of 2 as z's; this is in the space of z's, z that satisfy this is the feasible region of 2. Now, look at what, but look at the look at phi of the feasible region also. Look at the values of phi of z says that z is feasible for 2. All the phi of z for which z is feasible in which z lies in the feasible region of 2, right.

And this is obviously contained in all the x's that x is feasible for star. So, when we search over when we search in star in the problem star when we are optimizing over all the x's we are including those x's that for which there is a corresponding z, but also those that for which there is no corresponding z. So, this will this feasible region is larger, ok.

So, the optimal value that you will get from here will be lower than the optimal value you will get from this from problem 2 ok, less than equal to the one from problem 2 alright. So, the if phi is not one to one if phi is many to one, optimal value of star is less than equal to the optimal value of this 2.

Now, this is a canonical mistake that many people make whereby where they make these changes of variables without checking if without checking if the if the function is one to one.

Student: (Refer Time: 23:40).

Yes because you are now, so the optimal. So, the optimization problem star is an optimization of all x's that lie in the feasible region of star, but the feasible region. So, if you look at the phi of z's that are coming in the feasible region of 2 they are included in this.

So, you are searching over a larger region in star then you are searching over in 2. So, you are get you are and you are minimizing right. So, you will get a lower value. You are looking to minimize the function over a larger region. So, you will get a you will get in general or a lower value ok. If you are maximizing then the inequality will go in the opposite direction. You will get higher value for star than you will get for 2, ok

Yeah. So, if phi is not bijective yes that is what I am saying you cannot these are not equivalent there will be an inequality and it is for. So, many things can happen here. One is that one possibility is that for by chance it can happen that the solution of star also happens to be. So, your x star this happens to be such that there is a corresponding z for it, ok.

There is a corresponding z star for it in which case in which case you will get equality here if that happens, alright. It can happen that the solution of star is such that ok. So, maybe I will list these out for you. So, x star can be such is such that so for simple cases is x star is such that there does not exist as z.

So, there is no corresponding z, in which case there is no way you can there is nothing there is no way you can get back to a solution of solution of 2. All you can say is that all you can say is that of the optimal value of star is less than equal to the optimal value of 2.

This is the this is the only thing you can say. Second case is suppose if x star is such that there exists a z; such that phi of z is equal to x star, right. So, you get to a solution and it just happens that you can actually recover the right z for it. In which case; in that case you will get an equality. These two have to be equal.

Now, the optimal values are equal. Now, it can also still happen in that x star is such that there is no; so, it can also happen that it can also happen that the optimal values turn out to be the same, but the solution of the solution of star is such that there is no corresponding is does not correspond to the solution of 2. So, I will explain what this is before I write this write this in the form of an equation, ok. So, it suppose you it can happen that.

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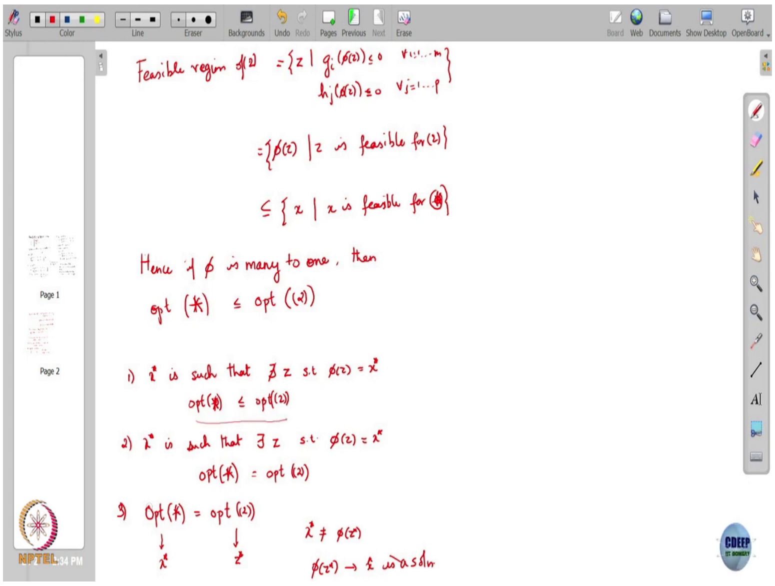

So, let me write it this way. Third case is that you can have that this is equal to two this. You solve a phi is not one to one, but yet you solve star you get and you solve 2, you get the same optimal values. You solve star this you also get from it a solution say x star.

You solve 2, you get a solution z star, but x star is not equal to phi of z star. Is this possible and why was this why would this happen? Why would this happen? So, the objective values are the same, but the solutions are not in correspondence with each other under phi

Students: (Refer Time: 28:59).

Yeah. So, the issue is that this is one solution x star. Likewise, z star is one solution there may be other solutions that are in correspondence with each other, ok. So, x star need not be the only solution there could be other solutions which you have not discovered, ok. And in particular phi of z star this point phi of z star let us call this x hat. This point is actually another solution necessarily right, this is a solution ok.

So, this guy this sort of case can also occur. So but be beware that beware of this particular possibility because I have seen this happen multiple times where you make a substitution without checking if it is one to one and then you get an what you get is an inequality alright, ok.