ROBOTICS

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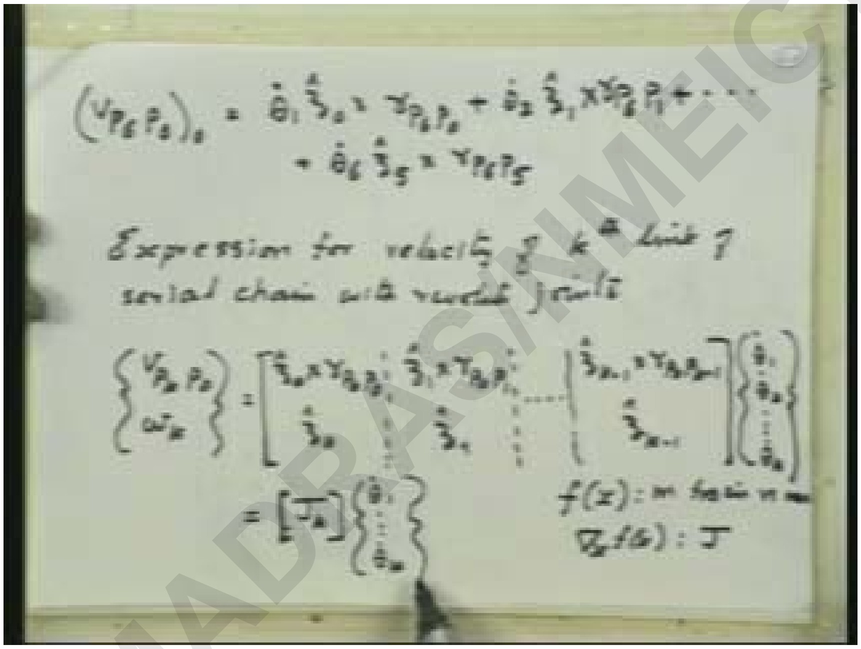
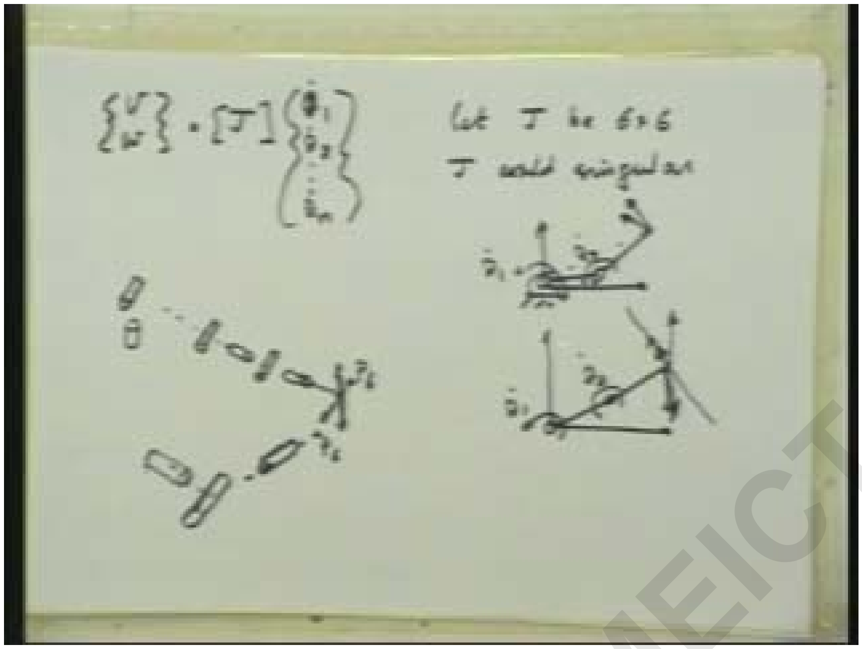
Lecture No – 24
Dynamic Analysis (Time: 1:21)

We derived the velocity relationship for a serial manipulator. The relationship between the velocity of the end effector, or a point k, the kth link of the serial manipulator, which consists of velocity of a point p k. This is the origin of the reference plane fixed to that link in our notation and the angular velocity of the link is related to the joint velocities. (refer time : 01:52).



First-order derivatives, time derivatives of the joint variables, through this relationship where these joint velocities get multiplied by a matrix to give you the end effector or the last-link velocities. k could be any of the intermediate velocities. So, this is true for a serial manipulator, Here, j k, the Jacobean corresponding to the kth link, consists of columns which are given here. The columns are six-dimensional vectors. The two parts of the column are (1) the unit vector along the axis of the joint, and the cross-product of the joint axis with the vector from the previous link's reference frame to the current link's reference frame, Pk with respect to P nought, Pk with respect to P 1, fine? So this part corresponds to linear velocity imparted by a rotation, these are all rotations, so we have considered a serial manipulator with revolute joints. We will consider the case where there are prismatic joints very soon. Now, I mentioned when we started this velocity analysis (refer time : 26:01)



that one of the reasons for doing velocity analysis is to understand some of the properties of the mechanism. Apart from being able to find out the velocities of links and points of links, this is one of the important reasons why we do velocity analysis.

So, in this context, this particular relationship, that is the, let's take the end effector now, the last link, so, velocity of the end effector and the linear velocity of a point on the end effector and the angular velocity of the end effector are obtained from the Jacobean corresponding to the end effector, multiplying. Let's now consider the revolute joint manipulator, so, theta 1 dot, theta 2 dot, up to some theta n dot, where n is the number of links, number of joints.

So typically, in the case of a six-jointed manipulator, which is a typical case, we have J being … What is the dimension of J, the size of J? Six by n. And in the case of a six-revolute manipulator? 6 by 6, fine?

So naturally, the question arises, under what conditions can we really find out this given this, or find out this given this. These are the forward and inverse problems in velocity analysis, given the joint rates, find out the end effector velocity; or given the end effector velocity find out the joint rates. So, the way we have derived the relationships there is no division by 0 in any of this except to get some unit vector which is fairly straight-forward, so, this matrix exists and so the calculation in the forward direction, from here to here, is always positive, right? That is always positive. So, given the joint rates finding out the end effector velocity is positive, but what about the reverse? That is, given the end effector velocity, can we find out joint rates which will give you that end effector velocity? Yeah, for that to have a solution, this system of linear equations, I mean, this is something you have to note, that this is the system of linear equations if the unknowns are the velocities. So the velocities occur in the linear form, right? The position variables occur in the nonlinear form in the elements of the Jacobean, but our unknown is not the position; position is given in the case of velocity analysis. So the unknowns are either this or this, which occur in the linear form. So this is the system of linear equations, fine?

So this is in the case of a six-revolute manipulator for the end effector. This is a 6 by 6 matrix. So given the left hand side here, under what conditions can we find out the thetas, theta dots? Only if the system of equations are consistent, right? So given any arbitrary value for this, solution for this will occur only if this is nonsingular, fine? What is meant by nonsingular or singular in the case of manipulator? It turns out that in certain positions, this Jacobean can be singular or rank deficient, and then you cannot produce any end effector velocity you want. Or given any end effector velocity, you may not be able to find out the joint rates which will produce that. So let me show a case.

Yeah, I will come to that, fine? So, let J be 6 by 6 or let me, so let J be 6 by 6, so J could be singular, fine? There are certain positions in which J could be singular. Let me show a simpler case of a 2 by 2 system and then come to the 6 by 6 system. So consider this two-revolute manipulator with just two joints, with the end point linear velocity, let's consider, be the only velocity we are interested in, they are independent, typically, so we don't consider the angular velocity of the end effector, we don't include it in the total setup velocities of the end effector. Let's consider that trivial [??09:38], fine? Now, consider, this is one position in which I have drawn this, consider another position like this, so this is 180 degrees, this angle is 180 degrees. Now, suppose you give a theta 1 dot here, and a theta 2 dot here, what can be the velocity of this point?

In order to determine that, let us go back to this and consider the same question. I gave the velocity theta 1 dot here, and a velocity theta 2 dot here, so when I give velocity theta 1 dot, I assume theta 1 theta 2 dot are fixed, the linear velocity imparted to this point due to the rotation theta 1 dot is obtained by drawing this particular line, and drawing a line perpendicular to that. It gets a linear velocity corresponding to that. Depending on the magnitude of theta 1 dot this will be small or big. Suppose I lock this and rotate this, what is the velocity that I get at the tip out of that? It will be a velocity which will be perpendicular to this line. These two linear velocities being in two different directions, you can see that I can get velocity in any direction about this point by appropriately choosing theta 2 dot and theta 1 dot. I can get it as a linear combination of these two independent vectors on the plane.

In these two vectors, how may independent [??11:37], right? What about here? If I had, corresponding to theta 1 dot, I will get something which is perpendicular to the line from this point to this point, and corresponding to theta 2 dot I will get a vector which is perpendicular to the line from here to here. Since this is 180 degrees, these two vectors are in the same direction, so now, can I get a tip velocity in any direction I want to? I cannot. It can only lie along this line perpendicular to the line from here to here, or from here to here, through that point. What it means is that when this Jacobean becomes rank deficient, [??12:28] even be singular, when it is rank deficient, the set of velocities you can impart to the end effector gets limited, right? In the general case, you could have produced two independent velocities here, so any velocity span by these two independent velocities of the linear combination of them, plus here, the linear combination takes me only along this that, so this is a typical case of singularity. So now comes the important question. Given any velocity can I generate, can I find out the joint rates in order to generate that? That is the question. You can immediately see that if I had proposed a velocity, let's say, in this direction,

can I generate it with any velocity I want? I mean, given any velocities here, can I generate that? I cannot. It has a component which I cannot generate at all. So this is the singular position. Any end effector position cannot be generated by an appropriate joint, ok?

So, this is physically the case of a singular position. So this happens at a singular position. There could be other positions of the manipulator, but it doesn't happen, where you don't have singularity, ok? So now, let me take the case of our 6 r manipulator.

If you remember, in the case of the 6 r manipulator, the joints, there was a vertical joint, and then, there was something which was, this direction. I am showing it in the stretched out singular position, fine? There was another joint like this, again horizontal; there was the wrist which had three joints, one in this direction, another in this direction in a particular position of the wrist, and the third again in this direction. And then came the last point, again stretched out from here.

So, this is the point P 6 that we are talking about. So, with the joints like this, this joint being vertical, these three being parallel to each other, these two being parallel to each other, what are the velocities that you can generate for the end effector, consists of the linear velocity of the point P 6 and an angular velocity of the link 6, the last link, right?

So look at the linear velocity of this particular point. When I rotate this, it can have a velocity in this direction, right? When I rotate this, this, or this, stretched out, I can have a velocity in the vertical direction. When I rotate this or this, I don't impart any velocity to that point, so that lies along the axis of those two joints, these two. It is in the particular position in which I have held the mechanism, right? So in that particular position turns out that this particular point can be given only velocities in a two-dimensional plane, not in a three-dimension, which is what is required in order to give the full velocity for this point. So this is a singular position. With regard to generating the linear velocity of the point 6, this is not a good position.

What about angular velocity? If you remember angular velocity, it consists of the sum of the angular velocities imparted by each revolute joint. So, if these three, if there are at least three revolute joints in three independent directions, we are done. We are able to give whatever angular velocity we want. So by rotating about this I give vertical angular velocity in the vertical direction, by about this, in the horizontal direction, and by rotating about this, in another perpendicular horizontal direction. So I am able to give all angular velocities I want at this particular, on this link, so with regard to angular velocity there is no problem. But if I decide to lock these first three joints, the capability of the wrist, if you remember, the concurrent wrist is to able to give any angular position for the end effector. If you extend it further, we can see that we can also give any angular velocity to the end effector using the concurrenct wrist. Using the concurrent wrist alone you are able to orient the end effector anyway you want, right? So I should be able to give any angular velocity also. I needn't really use the first three joints. Let's accept that assumption. So I lock these three. The question is, can I give any angular velocity to this joint, in this position? In this position I cannot, because now I use only these three joints, two of them are parallel to each other, so I can give angular velocity only which are confined to only a plane. My freedom is restricted in a sense.

On the other hand, in this following position, when the point P 6 is here, so I have actually rotated this. These two vectors are no longer parallel, fine? I have rotated this particular joint, so now there are three axis which are in three different directions. They don't form an independent, they form an independent set in the three-dimensional plane, into a three-dimensional space, so I can now impart any angular velocity I want to at the end effector, but not in this position with the wrist. So wrist has a singularity which if you look at a little more care carefully you will remember that in the algorithm for finding out the joint angles given the end effector orientation we found that the case of this angle being 180 degrees or 0 degrees is a special case, the singularity corresponds to it. So there are singularities in the workspace, and if the end effector, if the manipulator is in that singularity, then our ability to impart any velocity we want gets restricted, and typically, these singularities turn out to be at the boundary of the workspace. In the case of the linear velocity, that turns out to be on the boundary of the workspace. We can see that this is stretched out.

In this particular case of the planar manipulator this is stretched out. When it is holded in also, when the tip is at the boundary, at the boundary of its workspace, you will see that the same singularity is there, right? So, singularities are associated with often the boundaries of the workspace, but not in all the cases. Singularities can occur within the workspace, also.

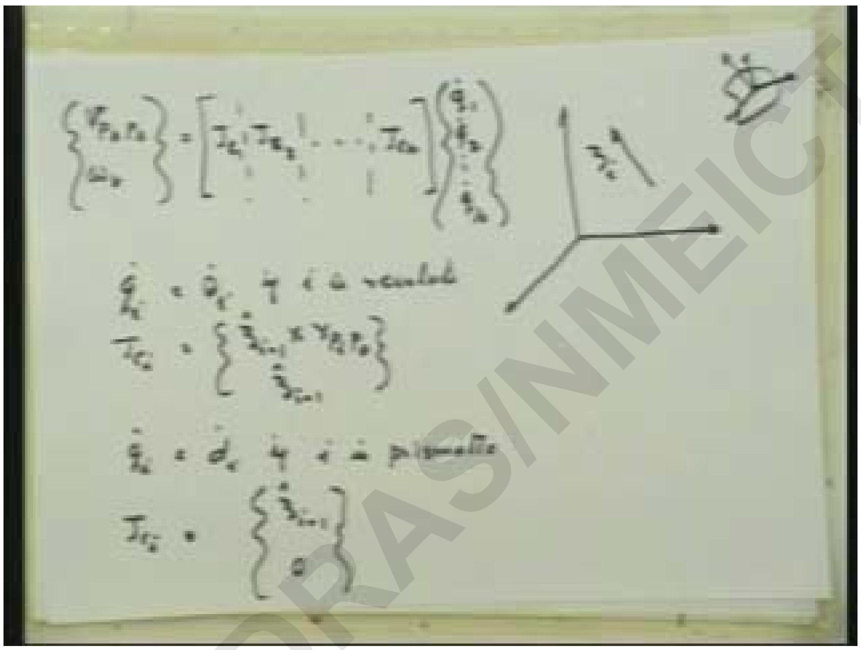
Now let me come to one of the questions which was raised. It is not necessary that the number of joints in a manipulator is only six; could be seven, could be five, could be more than six or less than six. So consider the case where it is more than six.

So, n is more than six; so, in that case what about the inverse problem that is given v and w, how do you find out this? Rather than how do you find out this, how many solutions do you think there are? There are typically infinite solutions, right? If there is a solution there are infinite solutions. And suppose this is less than six? Then there may not be a solution. Yeah, there may not be a solution. There may be one solution, or there may be infinite solutions also. It depends on, again whether this becomes a rank deficient or not, right? So the issue to look at is whether this is rank solution or not. That is one issue. The other, whether this system is consistent or not. This can be rank deficient and the system can still be consistent. So let me explain what I mean by this. These are conception linear solution of linear equation. You must have seen it earlier, or use of linear transformations. When this is rank deficient, what it means is that the columns of this Jacobean do not form an independent side, right? So, for example, if this is a six-dimensional case, there are any six, then these columns may, when it is rank deficient, it actually spans, the columns may span a five-dimensional space or a four-dimensional space depending on what the rank deficiency is. If it is rank deficient by 1, it's a five-dimensional space. What it means is that whatever this is this can only be a linear combination of these columns, right? So, when it is rank deficient by 1, there are five independent columns here, the subspace, this is confined to a five-dimensional subspace of the six-dimensional space.

In this particular case, this gets confined to a one-dimensional subspace of the two-dimensional space. This gets confined to a five-dimensional subspace of the six-dimensional space, fine? So that is what occurs in rank deficiency, but you can still specify this to be within that subspace, which is spanned by these columns, in which case the system is still consistent, and you can get solutions, ok? So if we choose this appropriately, you can still get solutions.

Another very important point is that which you have to note. At this singular point I said this is at the boundary. Can I move outwards? I cannot move outwards. Infact, I cannot have a finite velocity which is out of this line, right? What about inwards? Can I move inwards? Yes / No. The question is, actually, I should now ask the question in a more specific way. Can there be an inward finite velocity? It cannot. It cannot be any velocity which has a component which is in this direction, right? There cannot be any velocity in that, so there cannot be any finite velocity in that, but in order to move inwards I can start with 0 velocity and accelerate, so that is possible. Just because a manipulator is at the end of its, is stressed out and is at the end of the boundary, workspace at the boundary doesn't mean that you cannot move it. You can. Only, you need to move with, start with, 0 velocity inwards, fine?

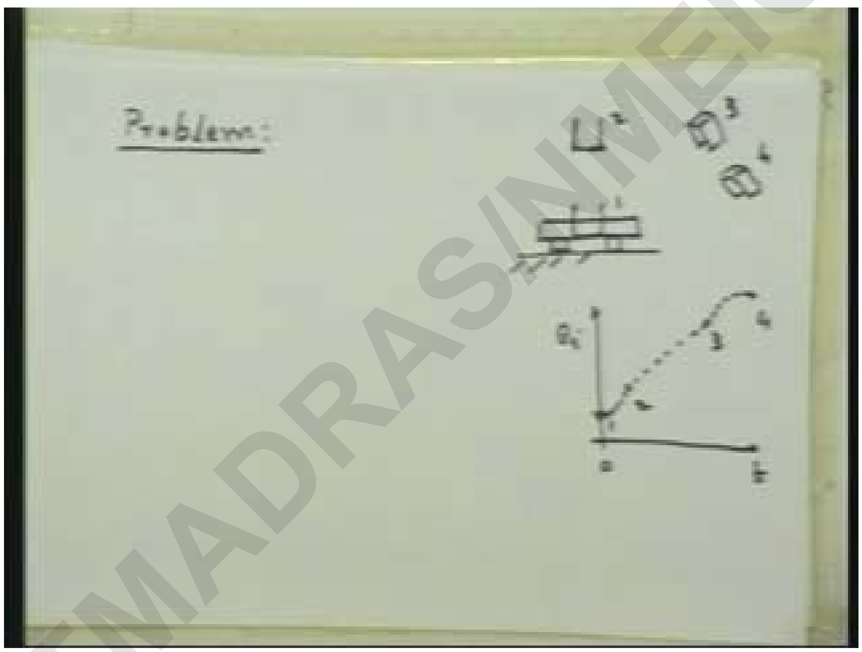
Now for the case of prismatic joints also we can derive the Jacobean. It turns out to be fairly simple. (refer time : 31:30).



So, if there is a serial manipulator, and let's assume one of these joints is prismatic, then the end effector, the point where, which we are trying to find linear velocity, suppose we lock, we have found out the linear velocity of this point can be obtained by locking all joints except the one you are now considering, and just moving that joint with the appropriate velocity, so velocity imparted to this point through that. Consider that. Add up all such velocities, you get the velocity of this particular point, right? If you go by that, if this is a prismatic joint, you lock everything else and you move the mechanism at the prismatic joint only. What happens is that this whole thing translates in this direction, which gives a velocity in this direction for the, for this particular point, fine? So the linear velocity imparted by motion of a prismatic joint is in the direction of the prismatic joint at that particular position. So it is as simple as that. What about the angular velocity imparted to this by this motion? There is no angular velocity imparted to it because this is a translation. A prismatic joint allows only translation. You have locked all the other joints; only this joint is moving. So right now, this will be translating with respect to the global reference frame when I move only this. So there is no angular velocity imparted by the prismatic joint. So I can generalize for a serial manipulator the Jacobean in the following way. So let me consider that these are the, I will call it as column 1, column 2. So, Jacobean column 1, Jacobean column 2, and so on up to Jacobeans column k.

Multiplying now, I will call this as q 1 dot, q 2 dot, and so on, up to q k dot, where q i dot is theta dot, is theta i dot if i is revolute, and q i dot is d i dot if i is prismatic, that is, the ith joint is prismatic. You will remember that di is the offset of the joint, and theta 8 is the angle of the joint. Now what about j c i? It consists of, in the case of the revolute joint and prismatic joint, it consists of two halves, upper and lower. The upper half is z i minus 1, z i minus 1 cap, the axis along the revolute joint, cross r p i p nought, that is the top half; and the bottom half is simply this. This we had already derived. In the case of the prismatic joint, J c i will be z i minus 1 cap. That is the top half 3 by 1 vector, and angular velocity cannot be generated by this, so this will be a 3 by 1 matrix of vector 0's. So this is the general expression for a serial manipulator with prismatic and revolute joints.

This can be derived in detail, but the revolute case is more difficult. This is fairly simpler, fine? Now, let me pose a problem which you can try to solve. Go back to that block with a few three pegs that we tried to pick up using a manipulator and place somewhere, ok? Consider that (refer time : 36: 45)



We used a gripper to pick it up and then raised it to some other position, right? So we had specified the first position, the second position, then the third position, before we actually brought it in to touch the pegs, aligned with the hole, and then brought it down to make the pegs just touch the holes, or be on top of the holes. So some third position and a fourth position, fine? So, 3 and 4. So, we now move from 1 to 2, straight up, so as to ensure that these pegs don't actually go into the table on which they are kept, correct? There is no interference. That is the reason for doing this. Now, if you try to do this with a manipulator of the sort that we have considered what will happen is that when it goes from 1 to 2 to 3 to 4 we can calculate the various theta i's as functions of time. These four positions are given, so we know the starting position at time t is equal to 0. We know the starting position theta i. And finally, it has to go to some particular position, 2, then 3, and then maybe four. So, it has to go from 1 to 2 to 3 to 4. It will obviously start with 0 velocity, go like this, and come to 0 velocity. Now how are we sure that when it goes from here to here, that this is not going to dig into this particular table. Just because it starts with 0 velocity? Or because we have placed 2 above 1? None of these things actually fully guaranteed that it will not dig in. It gives you a broad control over interference, not a very precise control. In order to have a precise control over interference in the particular case we should have specified that from 1 to 2 the end effector should only translate. That's a very strong demand, but if we had said that, then we can ensure that it does not actually dig in. So, what we need to do is at any point of time, from here to here, we have to check at fairly close intervals, whether any of these pegs are touching the table or not, right? Suppose if it is touching at some position; at the next position whether it will go in or not can be obtained by looking at the velocity of that point [???36:31], ok?

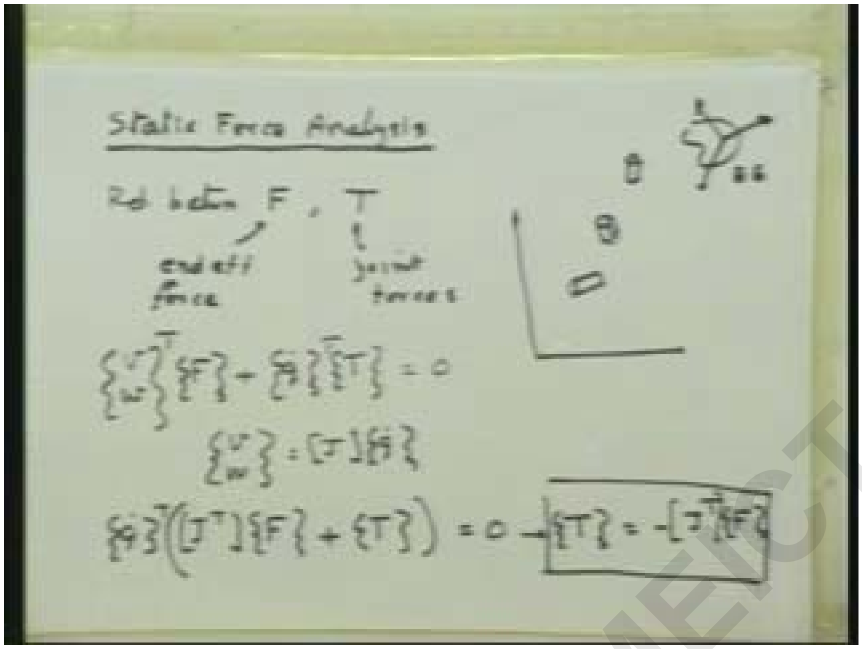
So these are things you can check out in detail once you do the inverse kinematics and you get these theta i's, you do the joint trajectory interpolation, you get the theta i's, then do a forward calculation to find out the positions of the thing, and you can check using velocity also, whether it can, it will really indicate. Velocity can be used in certain ways like this. So what we had done mainly

so far was address two important kinematic problems, the more important being possession analysis, both the forward and inverse, and also the next step there, which is the velocity analysis. Going one step further, we can do acceleration analysis. I will not do the derivation, I will just indicate what the final form of the equations are and we usually do acceleration analysis in order to do dynamic analysis. So, again I will indicate what the form of the equations there are, fine, without deriving the whole thing. But there is one thing we can do rightaway. Having done velocity analysis we can do static force analysis of this particular mechanism. It turns out that the velocity analysis problem and the static analysis problem are dual [???38:14]. The connection is through principle of virtual work, ok? So, let's pose the problem. So we have a manipulator whose end effector is this, which has the reference frame attached to that, and there are joints, prismatic or revolute, at which we apply joint forces are torque. Now, the question is, what is the relationship between the end effector forces and the joint forces?

So by force I use the, I mean generalize force, so it could be a torque or linear force, fine?

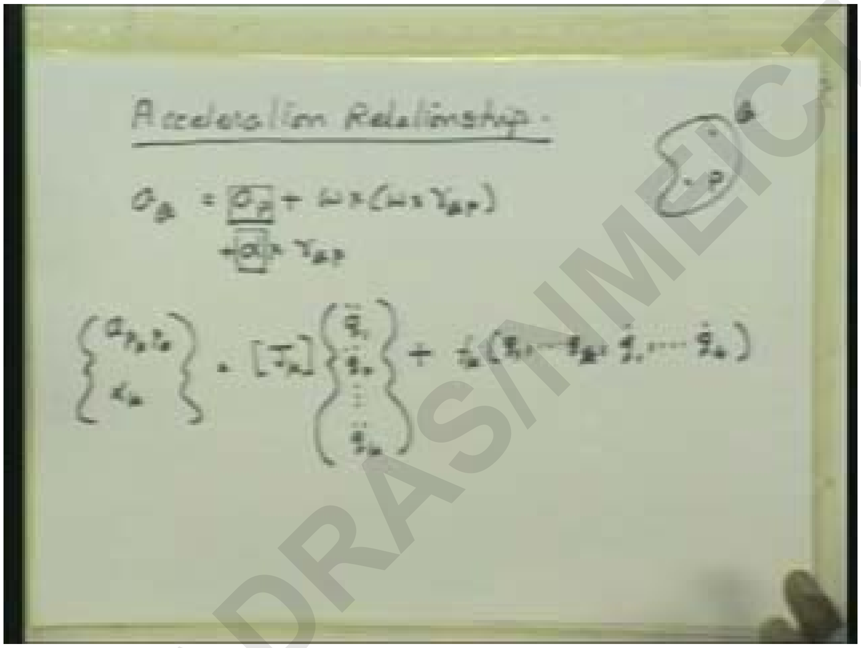
So, what is the relationship between F and T, where these are the end effector, end effector force? So this is the end effector force vector, fine, any system of forces can be represented by an equivalent system of a single force and a couple, right? Which requires six parameters for definition.

This is the set of joint forces, so assume that F is in the direction of the angular and linear velocities of this end effector, and these torques are in the direction, or the joint forces are in the direction of the velocities of the joints, ok? (refer time : 46:16)



Then the principle of virtual work tells us that for admissible motions, the virtual work done by this system of forces is 0, right? So if you use that then we have the angular velocities and linear velocities which are in the direction of those virtual displacements which are admissible, transpose this force F, which is the end effector force giving as a virtual work corresponding to t hese forces, plus the joint velocities, transpose each element multiplying the corresponding element in the joint force is a virtual work corresponding to the joint forces. These have to add up to 0, right? And we have an expression for this in terms of q dot, that is in the Jacobean, multiplying q dot gives you this, right? So if I substitute this into this, what I get is q dot transpose, Jacobean transpose, multiplying F, sorry, yeah, multiplying F, plus q dot transpose T. So I put this whole thing, take q dot transpose outside, so I get the joint torques here. Now, this is true for any q dot that I apply, I have, right? Which means that the two sides of this are the same, element by element, so I can write the equation: the joint torque is equal to minus of J transpose F.
This is the static force equation.
So the end effector forces and the joint forces are related through the same Jacobean, as the end effector velocity and joint velocities,
Fine, but now the singularity actually is a little, it has to be interpreted differently, that is, given any F end effector force you can find a joint torque which will balance that, because there is no singularity associated with the calculation of this. This is fully defined. Only its inverse could have a problem, right, so the problem is that in singular positions when this turns out to be singular J and J transpose are singular if they are full matrices, square matrices, they are singular together, fine? With the singular it turns out that for any given T, what happens? You cannot find an F. Well, the statement is not really that.
For any give T, you cannot find an F which will keep the system in equilibrium, fine, it may turn out the system may accelerate. You cannot have equilibrium with any T. Singularity amounts to that, ok?

So this is something you can work out with that two-revolute manipulator which I had shown. Try to work this out for that, and you will immediately see what it means physically,
Ok? It will turn out that in order to keep that particular system in equilibrium in the stretched out position, which we had seen, let me go back to that,
in the stretched out position, in order to keep this in equilibrium, it turns out that the torque I apply here, and the torque I apply here, have to have a certain relationship, fine? I cannot apply any torque I want and keep it in equilibrium. So this is very important, this is very useful. It turns out that if we do velocity analysis we have already done static analysis, ok? Static force analysis. (refer time : 50:11)



Now, so if you take a rigid body the linear acceleration of any point Q can be obtained from the linear acceleration of the point P in the following way. Acceleration of the point Q is acceleration of the point P plus two terms. One comes from the angular velocity and the other comes from angular acceleration. So the acceleration of the body is parameterized by these two three-dimensional vectors, the linear acceleration of the point, and the angular acceleration of the body.
So if you look at the kth link, the linear acceleration of the point p k with respect to p nought, that is, taking derivative of this particular position vector twice basically and the angular acceleration of that link, k, if you can derive this, turns out to have the following form, exactly the same Jacobean which you saw earlier, the kth 1, multiplying the angular accelerations, I'll call it q 1 double dot, q 2 double dot, up to q k double dot, all this, by the way this is true only for serial manipulators, ok? The way I am writing it, is true only for serial manipulators. Basically it comes from the fact that the kth links acceleration of velocity is affected only by the joints from k to 1, 1 to k, not from the joints after that. That is because it is a serial manipulator.
This is not all here. To this we have to add some terms. In general, let's say, some f k, a k-dimensional vector, f k is a k-dimensional vector, let's say, which is the function of all

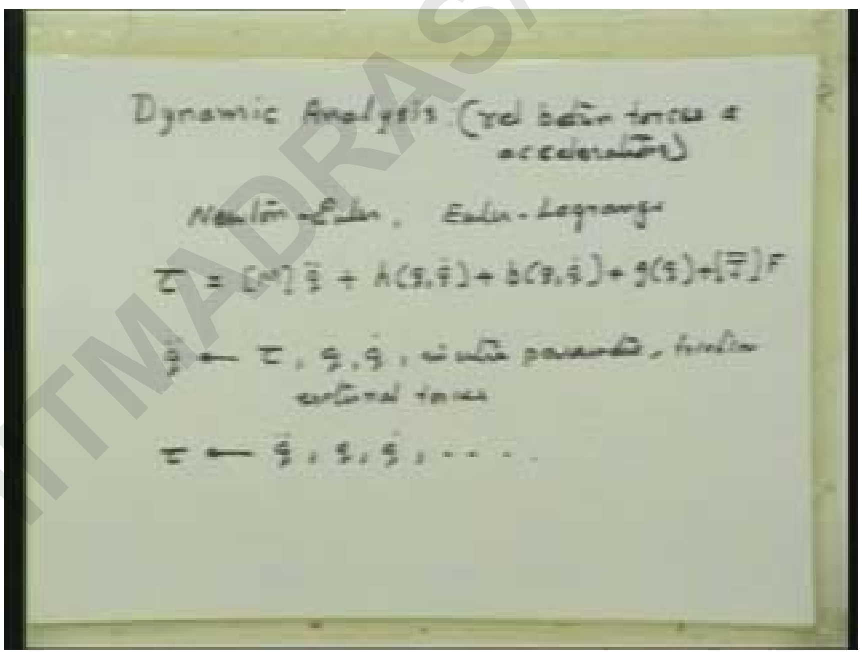
these joint variables, q 1, q 2, q k and q 1 dot, up to q k dot, where q 1 dots and q k dots occur in quadratic form, q 1 to q k, may occur in the nonlinear form, fine? These occur in the quadratic form. That is the way if you take derivatives twice it turns out to be like that. So, in this, you have the Coriolis acceleration,[???50:00] acceleration and all those things involved here. This is the form, fine? It is this part which is exactly the same as the velocity relationship, and you have this part which consists of quadratic velocity terms, ok? You are not required to derive this. Maybe you can try your hand and see how it is for simple manipulators, ok?

So, having come this far, let us just look at the form of the relationship for the dynamic equations because you may require it in control.

I will just give you the form of the relationship, so, but before I come to that let me ask the following question. Suppose the manipulator is not a serial manipulator, but a parallel manipulator, how do you get this velocity relationship?

What needs to be done there conceptually is, there are several loops in the manipulator. You need to write each of these loops, consider each of these loops, as a serial manipulator which is actually gripping itself, which is a closed loop, ok? The end effector grips the base. It's a loop. And then you need to write the velocity equations for each of them, and then you get the total set of equations using which you can solve the total system, but whatever that be, the problem of finding out the velocity is given, some velocities turns out to be a linear solution of a system of linear equations always, fine?

Now, let us look at dynamic analysis. (refer time : 58:07)



What is meant by dynamic analysis? Well, in very simple terms given the system and the forces acting on it, find out acceleration, or in the reverse way, given the system and an acceleration to be generated, find out the forces which will generate that, ok?

Ok, in very simple terms, you can define acceleration analysis as that. So you basically have to get the relationship between forces and accelerations. That's what we need, ok? This relationship in the case of the manipulator or general mechanism can be obtained using typically two approaches are used, typically there are many approaches, many formulations. One is Newton Euler, which is our idea of considering rigid bodies, the free bodies, and apply the Newton Euler equations in three dimensions to each of them. So it includes the reaction forces also which can be eliminated later, and the Euler Lagrangian, which is based on an energy formulation and systematically using the Euler Lagrangian condition for solution of a particular variational problem, and which uses generalized coordinates, and reaction forces don't appear exclusively in this. Finally if you remove the all reaction forces and joints an equation of the following form the joint τ

that apply generate acceleration, there's some inertia matrix, so you get q double dot, these are vectors, by the way, I don't put curly brackets over vectors right now, plus the Coriolis, the quadratic terms in velocities coming from Coriolis, and send the equal terms, so these are quadratic in q dot. There are masses and inertias involved here also, plus if you have viscous friction at the joints there may be a term corresponding to that. So this will be linear in q dot due to viscous friction being linear in the velocities. Plus [???55:22] may be a term coming from the weights of the links, which have to be supported, functions of q, function of q, right, plus

we have a term which some matrix, which I will call J bar, which is not exactly the single Jacobean that we have, it is a composite set of Jacobeans. This is a matrix, multiplying all the forces on each of the links.

So external forces on each of the links, all put together into F, let's say. So the τ has to balance that, has to balance the gravity, has to drive it against the viscous friction, has to overcome the centrifugal and Coriolis forces, and still generate some acceleration, ok?

So mass is M, is the inertia matrix. M is typically non-singular, is positive indefinite so you can find the q double dot given a τ , right?

So, given finding q double dot, given τ, and q, and q dot, and inertia parameters, friction, external forces, etcetera, is the forward problem of calculating acceleration. And given τ, finding out the forces to be applied or the joint forces to be applied, τ by the way, is the joint forces, so could be linear forces or τ's, depending on whether they are prismatic or revolute joints.

So, given a desired acceleration, and of course, the position, velocity, and the rest of it, is called the inverse dynamics problem. The inverse dynamics problem is a very important problem to be solved in certain types, strategies of control, that are used for manipulators. The forward dynamics problems is not directly solved, but this particular equation, the form of the equation, its properties, they are important in control, fine? So, dynamic analysis ends up being something like this, although it is easy to state arriving in this is not that simple, it requires fairly lengthy calculations to arrive at these various terms here, fine? So with that, I will conclude positions kinematics of the manipulators.