ROBOTICS

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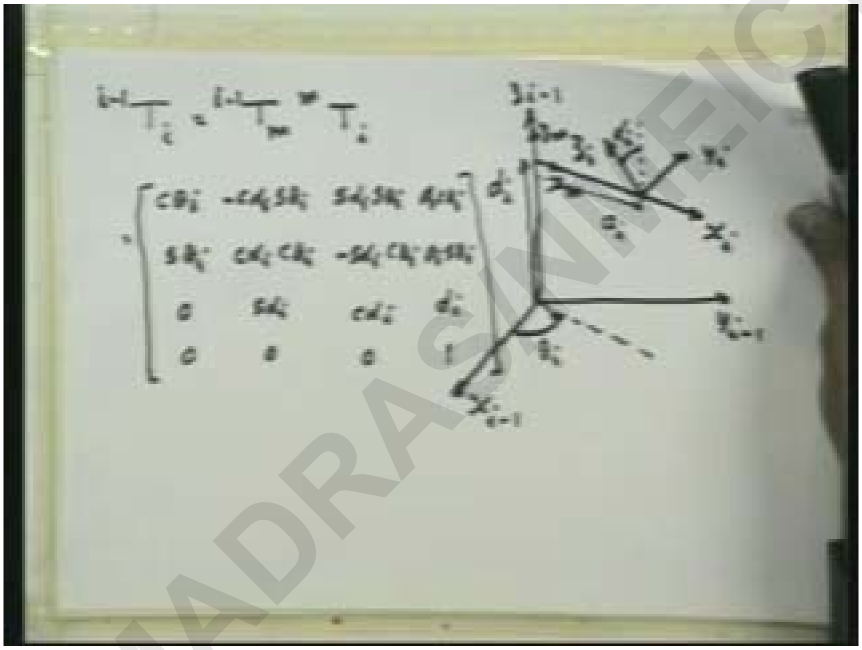
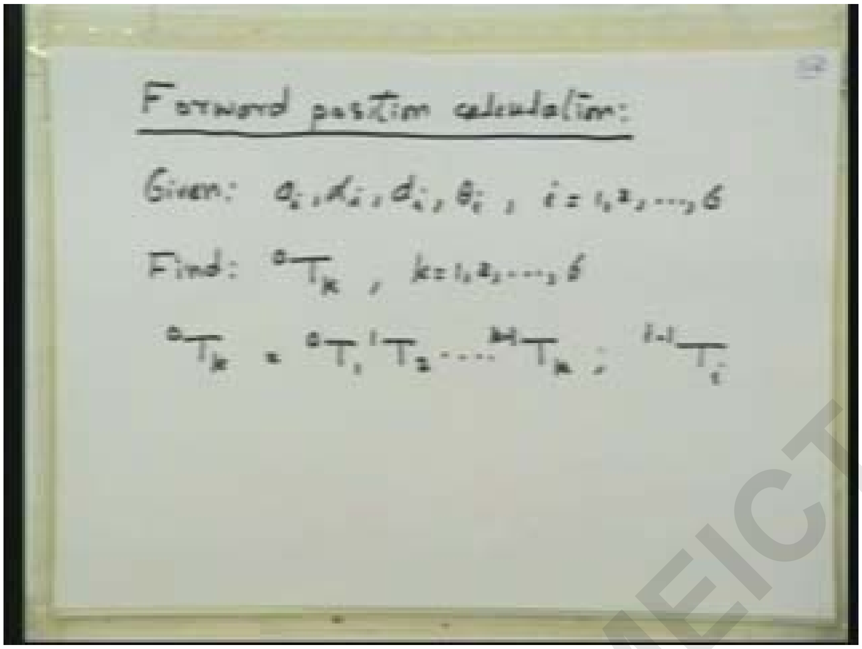
Lecture No – 21

Inverse Problem (Time:1:14)

What we did was considered the sixth revolute manipulator, set up the reference frame on each link, define the linkage parameters and the joint variables, and then we saw what the general form of the transformation matrix is (refer time: 1:45)



from one link to the previous link, this is the 4 × 4 homogenous transformation matrix, and using them, we can very easily solve the forward kinematics (refer time: 2:05)

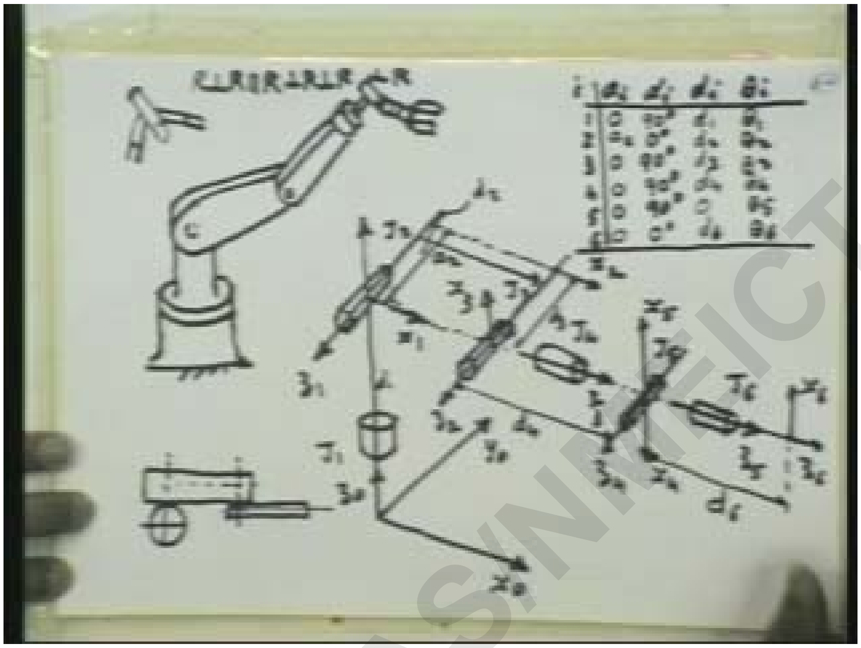


Problem, where given the joint variables and the linkage parameters, we need to determine the end effector position and orientation represented by this 4 × 4 matrix, and that is obtained very easily by multiplying these 6 transformation matrices which are known, fully known, to us, fine.

So, that problem was very easily solved. The more complex problem in this particular case for the serial manipulator is the inverse kinematics problem. Let us now define that and use the same machinery to solve that.

So the inverse position problem is as follows. (refer time : 4:32).

We are given the parameters of the mechanism. In the case of the sixth revolute manipulator we had, di's are six parameters. We are also given the end effector position and orientation. We need to determine the joint variables which would take the end effector to the given position and orientation. So this is the problem to be solved. (refer time : 4:39).



Physically, the position and orientation of the end effector is given, the parameters of this mechanism are given, we need to determine the six joined angles. They are, this rotation, the shoulder rotation, the elbow rotation, and the three rotations at the wrist, so as to take the end effector to the position and orientation given to it.

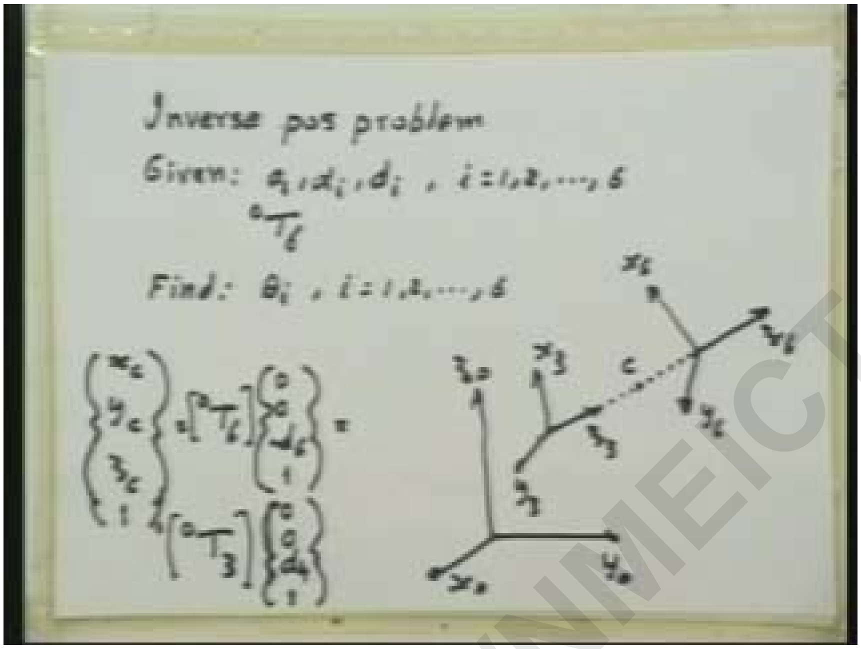
So, how do you think this can be solved? I would like you to look at the skeletal diagram of the mechanism that we have drawn earlier for setting up the reference frames.

Remember, what is given are position and orientation of this reference frame, x 6, z 6, y 6. The key concept used for solving this is very similar, is almost the same as what you had used for the prerevolute planar manipulator which we saw in the first class itself.

The thing to recognize is the following. Because the wrist is concurrent, this particular point on the wrist turns out to be a fixed point on the end effector.

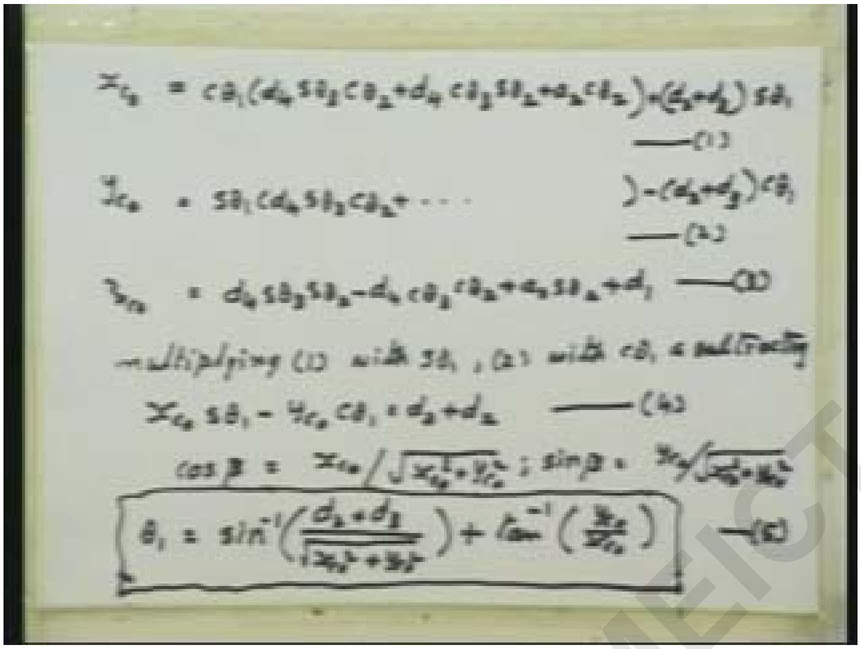
If you look at the end effector, x 6, y 6, z 6, and if we look along the z axis backwards, this particular point is at the distance, d 6 on the z axis. This is nothing but the point 00 minus d 6 on the end effector reference, right?

So then, the position of this particular point in the global reference frame can be very easily obtained because we know the end effector reference frame position and orientation, right? So the global position of this point is very easily obtained, fine, and the global position of this point can be obtained in terms of the three rotational [???] at that particular point. It had three, because, this point is also a fixed point on the third reference frame, third link's reference frame, so you get x 3, z y 3, z 3. This particular point, if you look along the z 3 axis, this point comes at the distance d 4 from the origin, so this point is nothing but 00 d 4 in the third reference, fine? So this is the key concept used for the solution. Let me note it down. (refer time 7:45).



If you take the reference frame, let's say z 6, x 6, let this be y 6, the point of concurrency, let us call it point c, is at distance d 6 in this direction, and this, the same point, is on the z 3 axis [???9:01] on the third reference frame. Find the distance from here to c is d 4, so we can write the following. The global position of this particular point is x c. This being a, we are going to use a four-dimensional transformation matrices, so we have to augment this vector, make it a four-dimensional, is equal to this particular point, expressed in this reference frame, premultiplied by this transformation matrix, so that is 00 minus d 6 one, this is minus d 6, this is equal to 3 with respect to 0. So the solution of the inverse kinematics problem because very simple because of the existence of concurrencies. If the wrist is not concurrent, we don't have this symple form.

So how do we proceed next? By the way, this is in the zeroth reference frame, so I indicate it by a 0 here. This is now fully known numerically. All the three parameters, xc, yc, and zc, are known because these this is fully known and this is fully known, right, so these three values are fully known. What about this? This contains theta, this contains the three, theta 1, theta 2, and theta 3, so we cannot find it out numerically because we don't know the values of theta 1, theta 2, and theta 3. So, when we multiply this we get a vector, four-dimensional, in which the first three elements are known as functions of theta 1, theta 2 and theta 3. If you equate those to the numerical values, we will be able to determine the unknown theta 1, theta 2, and theta 3. So, that is how we proceed. This involves some trigonometry. I will go through that, fine? So we know (refer time : 12:34)



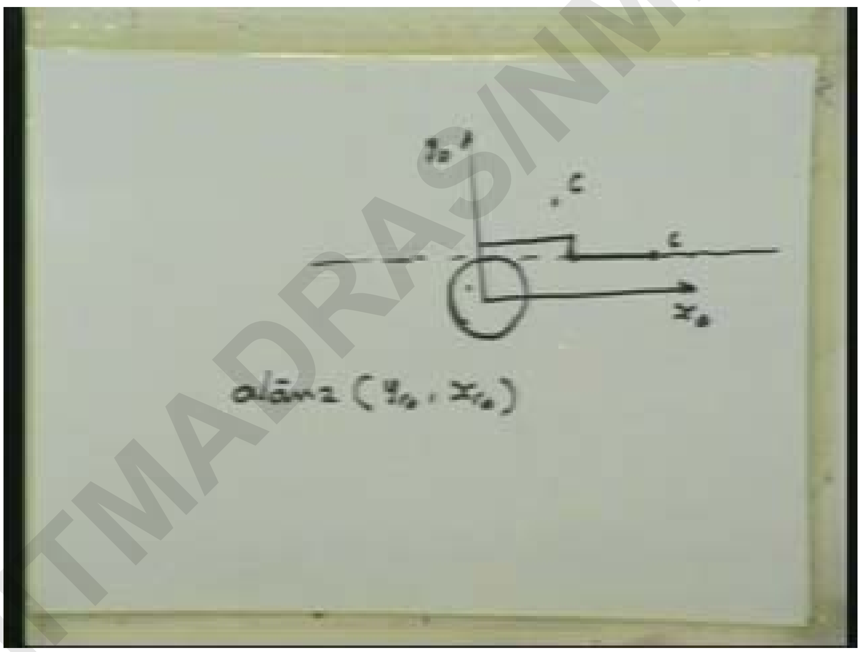
x c nought numerically. I am writing out the expression, you can derive it later. So this is expression 1. We will see that x c 0, that is the global position of the concurrency point c, the x coordinate of that is a function of all the three angles theta 1, theta 2 and theta 3. Similarly y – the same expression in the bracket, so I don't repeat it – minus is d 2 plus d 3.

To remember, d 2 and d 3 are occurring in the same direction in the sixth revolute manipulator we saw, so they occur in this clubbed form, ok? This is expression 2 and y z c nought this, what is there in brackets is exactly the same as this, fine? And this is the third expression. Now you can derive theta 1, theta 2 and theta 3 by appropriate manipulations here. I will broadly indicate what the steps are and give you the final expressions, ok? Now if you look at these two, first two equations, 1 and 2, if you multiply this by sine theta 1, and this by cos theta 1, and subtract, sorry not this one, the second equation, by cos theta 1, and subtract, this entire term disappears, ok? And then, you will get the following simple expression. I will indicate multiplying, subtracting, we get the following. This also simplifies because we get sine theta 1, sine squared theta 1, cos squared theta 1 added to 1, so you can this simple expression. No, this will be d 2 plus d 3 sine square theta 1, and this is minus d 2 plus d 3 cos squared theta 1, one minus the other,

ok? Now, using the following, we introduce an angle beta here in order to solve that, so cos beta equals x c 0 divided by root of, and similarly sine beta, y c 0, you use that substitution, use that and substitute, you get the following, you can get the following expression, theta 1, what is done here is take these two outside, I mean, take this particular term outside, so you get cos beta sine theta 1 minus um sine beta cos theta 1, fine, the whole multiplied by this term, ok?
So you get the following now. This can be done in different ways, not necessary that you have to do it this way, so this is the expression for, if I call this 4, this is 5, the expression

for theta 1. So, when you get an expression like this you are going to use this to calculate theta 1. This is, do theta 1 exists? What is the number of solutions? These are the things you have to understand, right?

So, in this particular case look at the expression. Under what conditions do you think theta 1 cannot exist? There is no answer for this. Yeah, the argument of sine inverse, if it is greater than 1 in magnitude, then there is no solution for that, right? So what about tan inverse? Yeah, but there are solutions, minus infinity to infinity, there are solutions for tan inverse, right? How many solutions are there? Look at this expression. How many solutions are there? Four solutions; two for sine inverse, other two for tan inverse. We know both the projections x and y numerically, right? So we can find out which quadrant it likes, so really, there is only one solution for this, no, two solutions for this, fine? So the argument of sine inverse not being within the allowed range, what does it indicate, basically? This is another thing that one has to understand. Yes, this is beyond the reach, but beyond the reach, where? So if you look at what we had used x 0, x c 0, and y c 0, are the projections on the x y plane of the global plane, right? So we are looking at the manipulator in that projection, ok? So, let's see the manipulation in that projection. (refer time : 22:09).

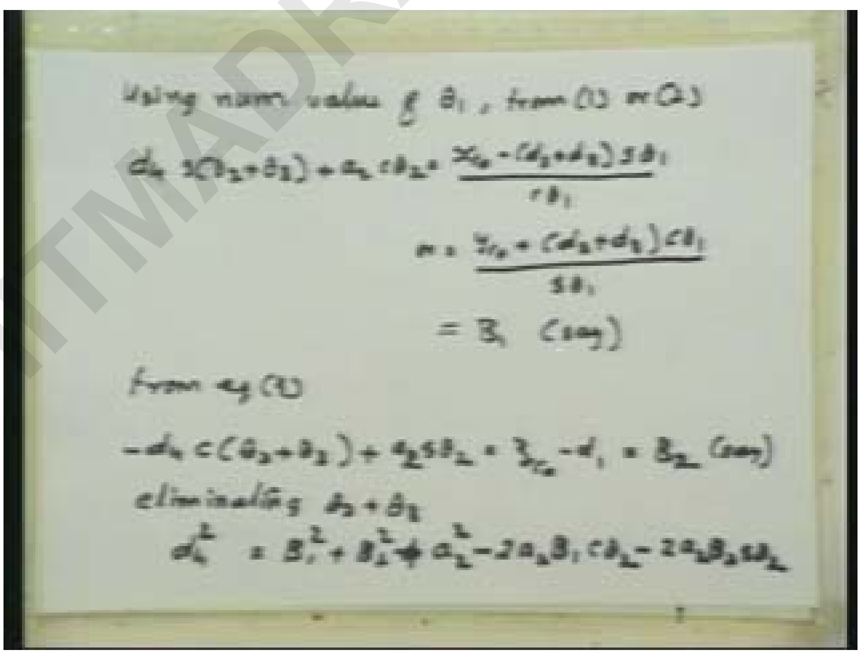


Ok, now I am going to draw it stretched out along x nought direction, the way we did. If you do that, then what you get is – I will bring back the figure again to you – this is what we see, fine? So that is the first post, this is the z nought axis, and then this link is offset like this, and this link is offsest like this; finally, this is the point c is somewhere along there, ok? So, if I draw it here, c is somewhere along this. Now, if you move theta 2 and theta 3 which is essentially rotating this about the shoulder, and rotating this about the elbow, what happens to the point c, is, it is taken to some point on this plane, ok, and what does theta 1 do? Theta 1 rotates the entire thing.

Now, so theta 1's ability, what theta 1 does is position this particular plane, ok? Suppose the point is somewhere here, the point c is actually here, the given position is here, what theta 1 does is bring this plane into coincidence with this particular point, this projection in the x y plane, right, rotate it to bring this particular line, actually, this could stretch from minus infinity to infinity because in this particular calculation you don't know the restriction whether it is reachable or not with regard to theta 2 theta 3 motion, only with regard to theta 1 are we really calculating, fine? So, this entire plane is brought into alignment with this, ok? How many ways can one do that? How many solutions for theta 1? are there for that? There are two ways you can do it. You can bring this side here or you can bring this side, ok? So, those are the two solutions. When is it that it doesn't have a solution? It doesn't have a solution when c falls within this circle.

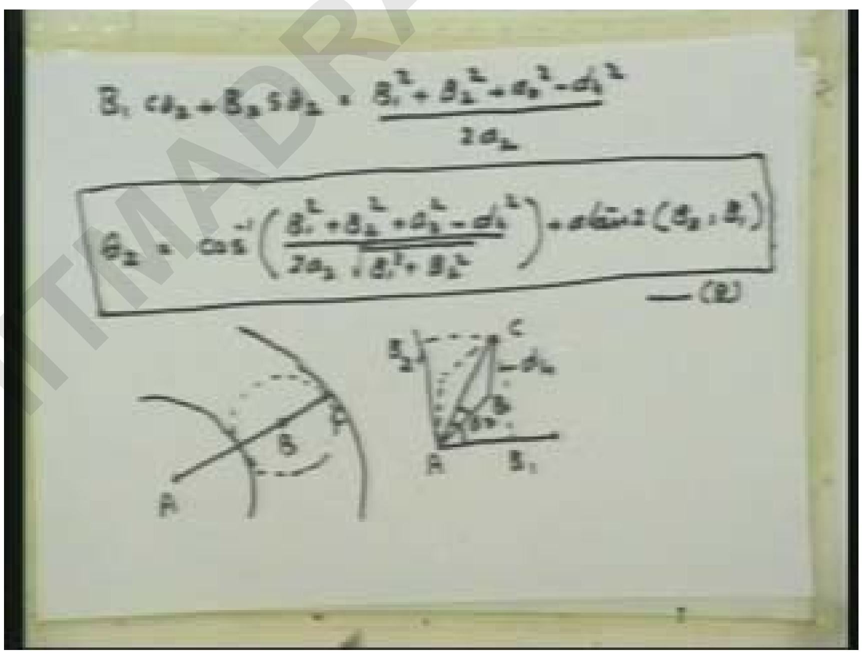
If this particular point comes back to x nought, then this plane will pass through the origin, and in all cases, theta 1 will have a solution. You are moving a plane, or this particular line in this projection, which is passing through the origin, and you are rotating it about the origin. So that line can reach all points on the plane, right, whereas, if the line is offset like this you cannot reach inside this particular circle. So, the situation of there being no solution for theta 1 corresponds to this, that c is somewhere here, fine? And the situation where there are two solutions is when this plane gets aligned with the point c, or made to pass through the point c, and you can immediately see that can be done in two ways, ok? So d 2 plus d 3 is two laws, that is, so that, and that is larger compared to this, larger than this in magnitude, right? That is, this c turns out to be inside that circle, fine? So, ok, one more thing I had used this tan inverse so you finally calculate it using a program, use this function a tan 2.

So, a tan 2 takes two arguments, the y projection first and x projection next, and it gives a solution from between 0 to 2 pi in the full ring, ok? So, it will give it in the x axis quadrant. There is only one solution for that, ok? So, that is theta 1. So similarly, we can proceed to calculate theta 2 and theta 3, (refer time : 33:38)



only, we proceed in steps, so now that we have theta 1, using numerical value of theta 1, remember one of the two solutions, that is all, and then you proceed with the calculation.

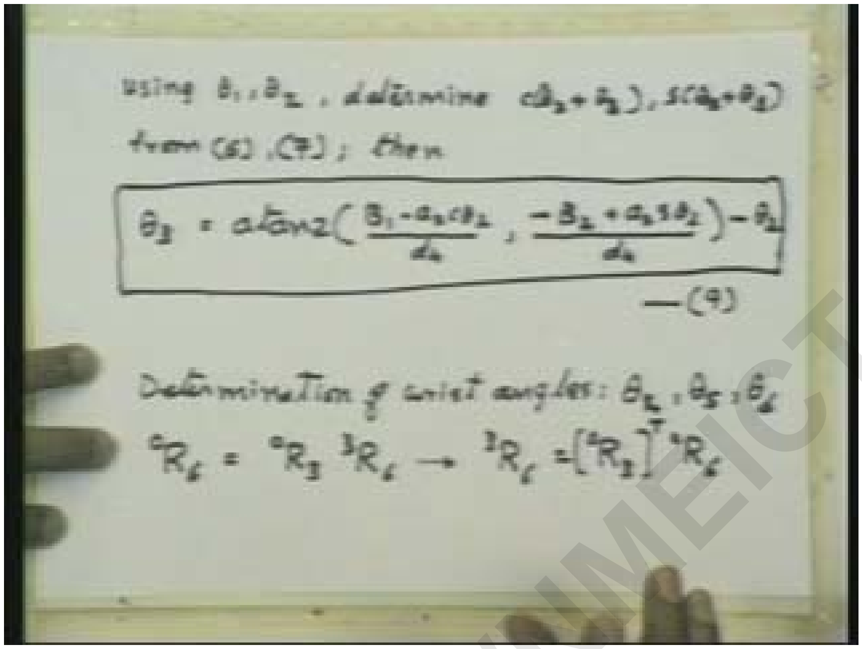
While you do it you don't use the other solution randomly, fine? You have to be consistent.
Either from 1 or 2 which is the first two equations for x 0, x c 0, and y c 0, we get the following expression: d 4 sine theta 2 plus theta 3 plus a 2 cos theta 2. I will write both the expressions. This is the reason for that, fine? I just substitute third theta 1 in the first equation to get this, or equivalently, I could have substituted in the second equation to get this. I am getting numerical value for this particular expression. Now, what is the relevance of saying you could use one or the other? You could have use just one. And you cannot use this when cos theta 1 is 0, right? In that situation you need to use this, when cos theta 1 is 0 and sine theta 1 is either 1 or minus 1, so you need to use this, fine? So calculation has to be numerically or it has to be defined, determinable. You can use one of them, ok?
So, let's call this equals b 1, let's say. So b 1 is known numerically, fine? And we write equation 3 as just writing it, rewriting it, minus d 4 cos.
So let's call this B 1, because this is fully known, this is B 2, this is fully known. So we are just giving two names for that. This is a 2, by the way, sine theta 2,
fine? Now, we can eliminate either theta 2 plus theta 3, or theta 2, one of them. What we do is, let's eliminate theta 2 plus theta 3. That can be done by transposing a 2 plus cos theta 2 to this side, a 2 cos theta 2 to this side, a 2 sine theta 2 to this side, squaring and adding. You eliminate this theta 2 plus theta 3, so I will simply say eliminating. The way it is done is with way I describe. We get d 4 square as, sorry, yeah, to this cos theta 2,
fine? When isolating this particular term, B 1 cos theta 2 and B 2 sine theta 2, we know B one and B two numerically, fine? We do that. Let this expression be 6, let this be 7, ok? So we get beta 1, B 1, (refer time : 42:15)



cos theta 2 plus B 2 sine theta 2.
So, assuming a 2 is non 0, otherwise assuming a 2 is non 0, we get the following, from this we get the following, essentially the same way you take root of B 1 square plus B 2

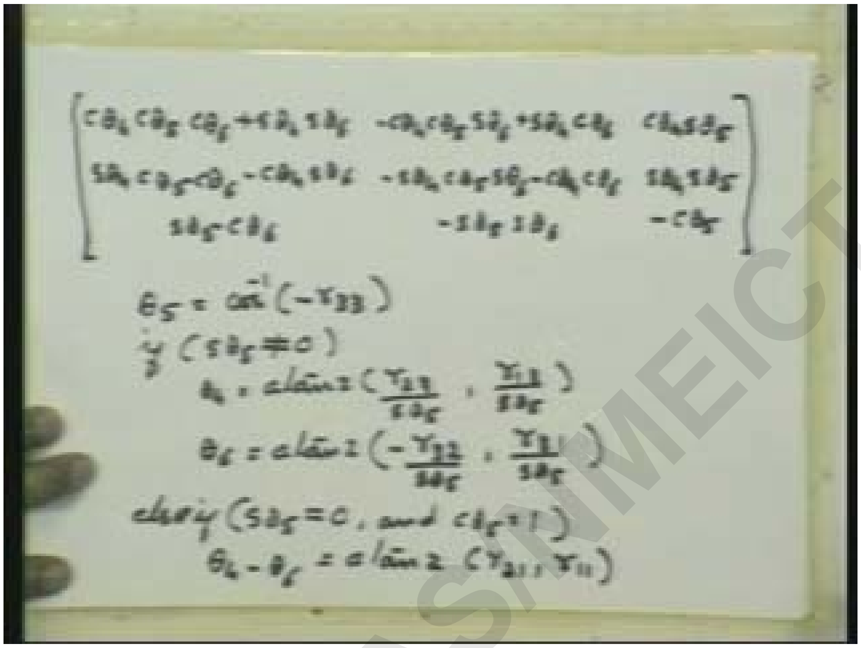
square outside, and call B 1 as root of B 1 square plus B 2 square, cos, some angle. Similarly, this is sine the same angle. Using that, we finally get theta 2 as follows.

This is the expression for theta 2, you get it essentially by this elimination, so that is the other angle. Again, let us enquire about solvability in this particular case, number of solutions. This second angle, a tan 2, is fully obtained, beta B 2 and B 1 are known numerically, so we know that uniquely, you can get it uniquely. What about this? The same problem with the argument being outside the allowed range could be there, so this could be more than 1 in magnitude, in which case, there is no solution. If it is less than 1 in magnitude, there are two solutions. If it is equal to 1 in magnitude, there is one solution. These are the various ways in which solutions [???36:25], right? And what does it mean physically? The two solutions, and now solutions. Again, it's an issue of reachability, ok? Only now we look at reachablility in another plane, So earlier, if you remember, we looked at reachability in the x y plane of the global plane, right? Now we are looking at reachability looking at a plane in this direction, along the joint two and joint three direction. If you look along that, you get a projection of the manipulator. It is in that plane that we look at the reachability for this particular calculation. If you do that, then what transpires is the following. This comes from cosine rule which is used when we know three sides of a triangle, sides that we are talking about is essentially this. We see the manipulator like this, let's say the elbow up position. This is the point c, this is some reference frame in which we are now having this projections, this is B 1, and this is B 2. This particular angle is this tan inverse that you are seeing here, and this particular angle is this cos inverse that you are seeing, fine? The solution that we have here, we said there are two solutions for this, so if you get an angle here, the negative of that also is solution, fine? So, the solution is essentially this angle plus or minus this, ok? So this is the way I have shown, this is the minus solution. The plus solution is the symmetrically reflected solution, fine? And what are the sides? The two sides are: this is a 2, and this is d 4, and this side is root of B 1 squared plus B 2 squared. So this expression corresponds to this picture. So again, there are two solutions possibly for theta 3, and when is it that there are no solutions algebraically? When this particular argument goes above 1 in magnitude, but geometrically clearly in this picture there is a limit to reachability of these two lines, right? They have to be joined always here, and the length is fixed. If you consider that, if let's call these points A and B, so if this is A and this is B, and if d 4 is this length, the set of points c can reach are obtained like this. Rotate this link, BC, with B as centre. You get this circle. This was done earlier when discussing configurations and work spaces, right? Now, rotate this circle about A, by rotating AB. Rotate AB, take the circle along with that, and you get this annular region, then, as a set of points C can reach. If C that is given is inside this or outside this annular region, it cannot be reached. That will be indicated by this, ok? There is a particular situation when that inner reachability is complete, there is no void here. When is that? When does that occur? Yes, when a 2 and d 4 are equal, then there is no problem with the inner reachability, because this will fold up to this point, right? And if you look at nature's design, our hand is more or less like that, right? We can scratch the shoulder, so that is actually the configuration which is the maximum volume index, a normalized volume index, ok? So this is the physical interpretation of the solution. So, up to now, how many solutions are there for theta 1 and theta 2? Now there are four solutions. There are two solutions for theta 1 and for each value of theta 1 there are two solutions for theta 2. Altogether, there are four solutions. Now for the final angle, theta 3. (refer time : 42:32).

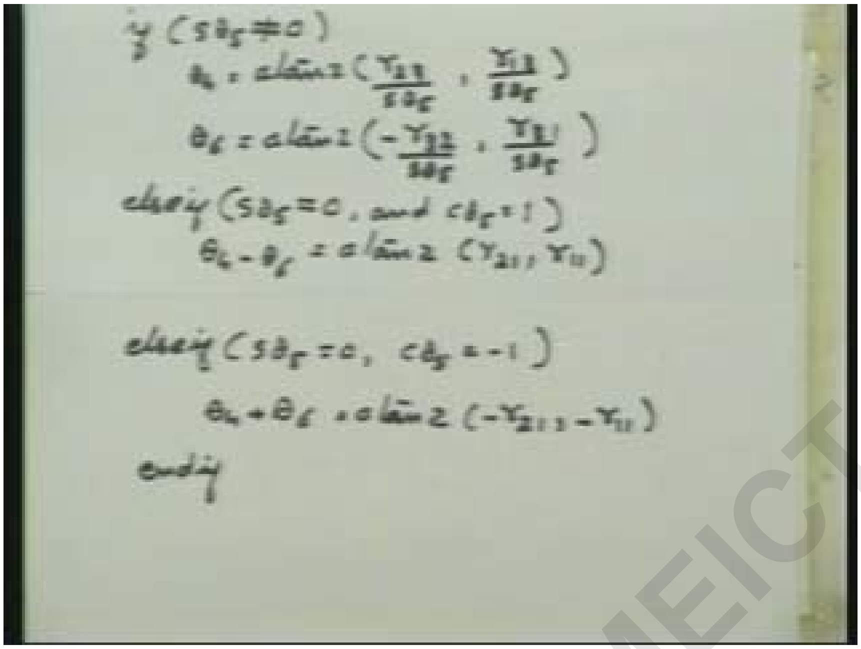


I hope I have written down equations six and seven, yeah.
And once you obtain this it turns out that theta 3, based on that, it would be, so d 4 should be non 0. For there to be a solution, d 4 has to be non 0, and you can obtain the solution for theta 3 like this. This is equation 9.
How many solutions are there for theta 3? Yeah, from this expression just one solution but if you go with the possibilities for theta 1 and theta 2, only theta 2 is occurring here, right? So how many solutions are there for theta 2? There are four, so theta 3 will have corresponding four solutions. You need to, for each theta 2 you will have theta 3 obtained from this expression, ok? So, this is the solution for the first three joints of the manipulator. This is one of the most common manipulators, and in terms of complexity, this is more complex again than scalar, cylindrical, or spherical. Plus, you can have more complex manipulators than all this, but, ok. So this is the solution for the first three angles. What about the last three?
Theta 4, theta 5, theta 6, remember that we know the orientation of the end effector. Now, we know the orientation of the third link, theta 1, theta 2, and theta 3 are known, right? So, in this, I split this as R 3 with respect to 0, and R 6 with respect to 3, because in this matrix, only theta 4, theta 5, and theta 6 occur, and in this matrix, theta 1, theta 2, and theta 3 occur, and we know this entirely numerically, and this also entirely numerically, because we have just now found theta 1, theta 2, and theta 3. So we can calculate this, fine? So then, theta R 6, with respect to 3 can be obtained numerically, ok, as R 3 with respect to 0, inverse of that which is the same as the transpose of that, into R 6 with respect to 0. You will see that this transpose is nothing but R 0 with respect to 3. That is how this begins, R 6 with respect to 3, ok? So, what we have here is that we have the right hand side numerically, we have this matrix numerically, we also have the expressions for the three matrices which get multiplied in order to give us this, so we can

express the elements of this in terms of theta 4, theta 5, and theta 6, right? So each element of this matrix can be expressed as a function of theta 4, theta 5, and theta 6, and we know each of those elements numerically, ok? So, that particular expression for the matrix you can multiply out those linkage transformation matrices and you will find the following (refer time : 48:34)



First row: each element is known numerically, and we have the expression for each element in terms of theta 4, theta 5, and theta 6, fine? Let me call the elements as r 1, 1 r 1, 2, and so on, up to r 3 3, ok? Now, this is more or less the matrix you will get to the following question. If the orientation matrix is given how do you find the Euler angles? For the 3 1 3 Euler angles the matrix will turn out to be more or less the same, fine? So that if you remember, that is one of the problems I had asked as an exercise. The solution to that also turns out to be the same, ok? Now look at the solution. Theta 5 can be obtained first. What is it? Cos inverse of minus r 3 3, right? How many solutions? There are two solutions. Now, what about theta 4 and theta 6? If you look at this element, these two elements, you will find that you can get theta 6 immediately, whereas, theta 5 has been determined now, right? So. you get cos theta 6 and sine theta 6 from these elements. And then, what about theta 4? That you can get from here, right, but we have to be sure of one thing. In these, we have to divide these elements by sine theta 5 because we have found theta 5, and sine theta 5 cannot be 0, otherwise we cannot do this calculation, right? So if sine theta 5 is non 0, then we can very easily calculate theta 4 as using this. This is r 2 3 divided by sine theta 5, and r 1 3 divided by sine theta 5. Similarly, theta 6 equals a tan 2 from here, minus r 3 2 divided by sine theta 5, r 3 1, divided by sine theta 5. This is fine. So, what if this is equal to 0? There are two possibilities. When sine theta 5 is equal to 0, this could be 1 or minus 1. Cos theta 5 could be plus 1 or minus 1. Essentially, theta is 0, the other case theta is 5, theta 5 is 5, so in this case, you can obtain using these elements. First theta 5 is now 1, substitute that here, fine? Using that, you can obtain theta 4 minus theta 6, fine? (refer time : 54:29).



So we have taken care of all the possibilities, so I will just, to make it explicit [???55:20] this equal to 0, and cos theta 5, this is minus 1. In that case, we get theta 4 plus theta 6. So this is the procedure for calculating theta 5, theta 4, theta 5, and theta 6. It has taken care of all possibilities.

How many solutions? Just two solutions. There is a possibility, that is, if theta 5 is less than, that is, r 3 3 is less than 1 in magnitude, there are two solutions. If it is equal to 1 in magnitude? No. We are calculating if that is equal to 1 in magnitude, if it is 1, then there are infinity of solutions. We can get only theta 4 minus theta 6, or if it is minus 1, again, one solution for theta 5, theta 4, and theta 6 have infinity of solutions provided this adds up to this, in case of cos theta 5 is minus 1, and thus, difference between the two is equal to this, the case of the other case. So, when theta 5 has one solution there are in infinity of solutions for the total three angles. When theta 5 has two solutions there are only two solutions for the whole set. Third thing is, is there a case where there are no solutions? Well, that will not happen if that matrix r 1 1 to r 3 3 that we had obtain numerically, is a proper orthonormal matrix it will not happen, ok? So that we have to ensure, and all these matrices are orthonormal with determinant 1, so that is ensured, ok? So this is the full solution for [???57:37] or manipulators of the same configuration, ok? So it is fairly simple in the sense once you write it down correctly you can calculate it very easily, ok? Here and there, I kept saying that d 4 shouldn't be 0, a 2 shouldn't be 0. What happens if that, those are 0, these are the things you can figure out easily. Try to do that yourself, ok? Such manipulators are not used but those are trivial cases. We'll proceed with some more discussion of inverse kinematics in the next class.