ROBOTICS

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Lecture No-17

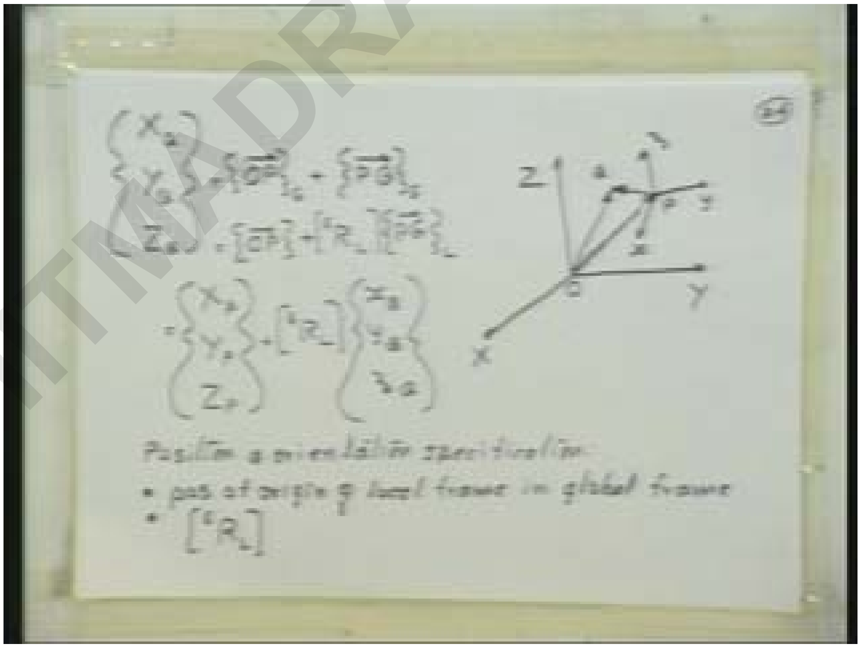
Trajectory Planning (1:20)

This is a discussion on position and orientation of rigid bodies. What we had seen in the last class was that there is the very simple way of describing position and orientation of rigid body associated with transforming position of points on the rigid body to the global frame and the orientation directions of vectors on the rigid body to the global frame.

So what we had seen in the last class was this: the position and orientation of a rigid body is described by the position and orientation of a reference frame attached to the rigid body with respect to the global frame.

So the small x, small y, small z are the reference axes, axes of the reference frame attached to the rigid body. The rigid body is not found there and global frame is this, capital X, capital Y, capital Z.

So the parameters required for transforming vectors and points from the local frame to the global frame are one the vector specifies the position, the origin of the local frame in the global frame. So let's call that particular point P and the unit vectors associated with the three axes: small x, small y, and small z are expressed in the global frame, the three unit vectors put together have some matrix because that is convenient for transformation. (refer slide time 03:57)



These four vectors, which make twelve parameters, form the position and orientation of this rigid body with respect to this. That is how we specify that.

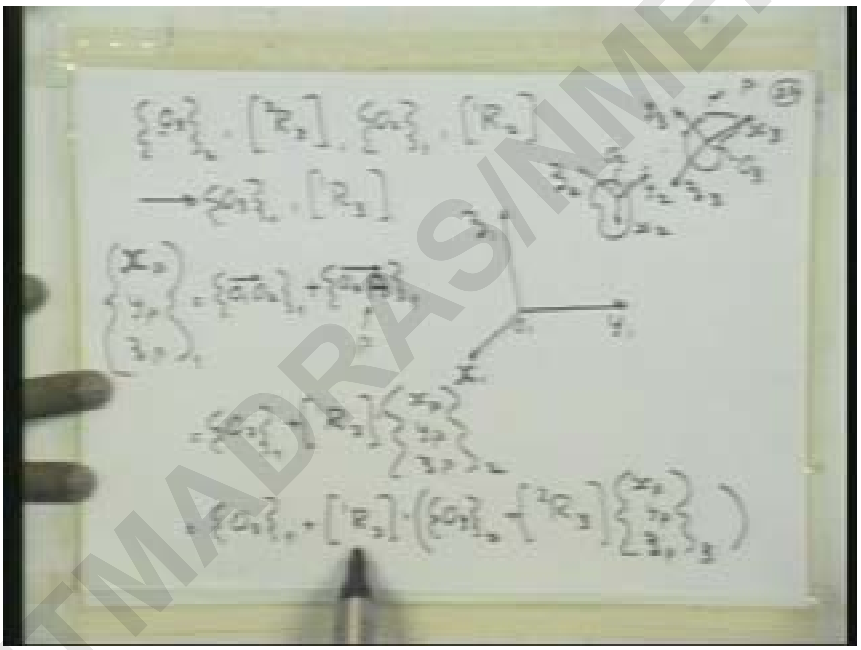
Now the transformation itself, if you have a point Q on the rigid body, and if it is described by this vector: small x Q, small y Q, and small z Q in the local frame, its coordinates in the global frame are given by premultiplying that with this rotation matrix, which takes vectors from local to global frame, and added to that is this particular vector, which gives the position of the origin of local frame and the global frame. That is this one.

So we have essentially twelve parameters for describing position and orientation of rigid bodies.

There are other ways of describing this; we'll not get in to that in detail.

What we had started out describing in the last class, at the end of the last class, was the following issue: suppose we know the position and orientation of one rigid body with respect to another, that is, second rigid body with respect to the third, third rigid body with respect to the second, and the second with respect to the first.

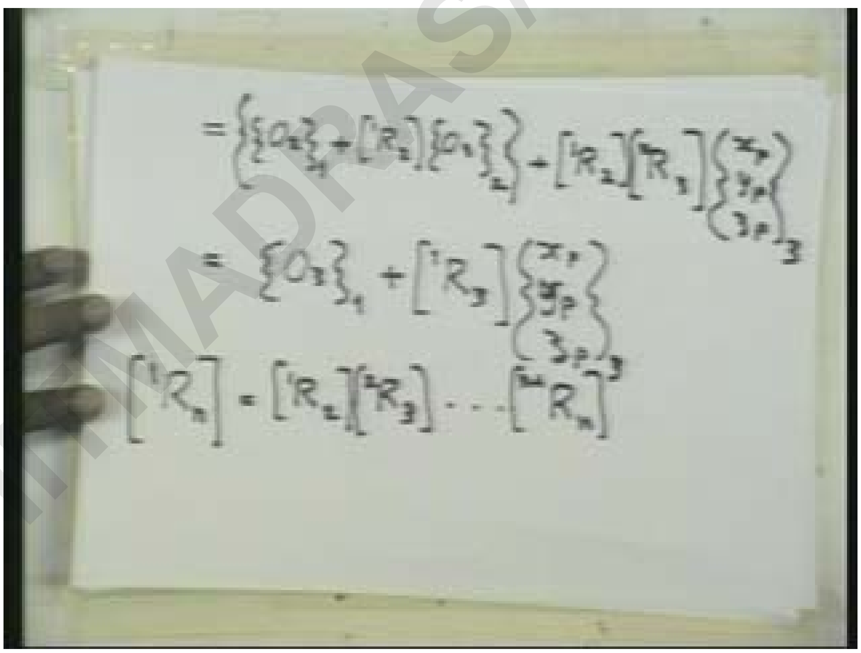
I would like to find out the position and orientation of this rigid body with respect to the first, the third with respect to the first. (refer slide time 06:30)



So then we had started working out how that can be done and if you take a point on the third rigid body, let's say the point P, we try to describe that in the first rigid body. if you do that we get the transformation.

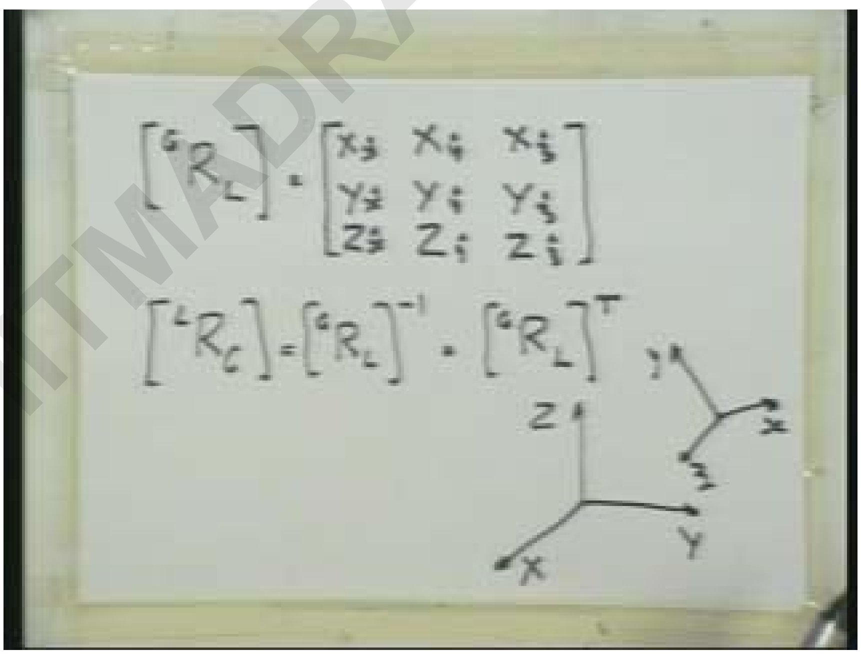
So, x p y p z p in one is obtained by the position of this point with respect to this body, that is, this vector plus this vector. We did it slightly differently: we described p with respect to the second rigid body and expressed it in the first rigid body. So, the position of the origin of the second rigid body with respect to the first, that is, this vector and the vector from O_2, the origin of the second rigid body to the point p which is on the third rigid body, is this vector. This particular vector is something we can express in terms of its local coordinates and the position of O_3 with respect to O_2. So that is what we had done here.

This is O_2 in one, that is, this particular vector plus R_2 with respect to one; two with respect to one; into the coordinates of this point in the second rigid body and then this we expanded further in this part.
There is the origin of the third rigid body with respect to the second plus rotation matrix of three with respect to two multiplied by the local coordinates of p in three.
Now if we if we expand this, we collect terms. This consists of vectors and matrices multiplying with this.
So O_2 in one plus in two; I consider this other thing to be one vector plus.
So now I have split it into two vectors. If you look at this you will realize that this consists of two vectors, that is, O_2 with respect to one, O_2 with respect to one plus O_3 with respect to two but in this coordinate frame.
So both of them put together, is the vector from O_1 to O_3. The vector is O_1 to O_3 in this reference frame or other O_3 in this reference frame.
So this vector is this and so O_3 in this reference frame plus the rotation matrix of this with respect to this, three with respect to one, multiplying the local coordinates of p is the particular way we write the transformation.
So I can call this as R3 with respect to one multiplying this local coordinates of three.
So one thing that emerges from here is the following: that is rotation matrix of three with respect to one, which is obtained out of multiplication of these two orientation matrices—two with respect to one, those multiplied by three with respect to two. (refer slide time 11:38)



I can generalize it and write R of some nth reference frame with respect to the first reference frame equals R_2 with respect to one, then R_3 with respect to two and so on, and finally R_n with respect to $n - 1$.
The fact that we can write the orientation of some reference frame with respect to something else using intermediate reference frame is very important and this helps us work out orientation rotation matrices very easily in several cases.

This is fairly simple now a couple of properties of these rotation matrices is something which I wanted to talk about during our last class.
I had pointed out that there are three columns of this: local to global. The x coordinate, the unit vector is in the local x direction in the global frame; similarly the unit vector is in the local y direction and then finally unit vector is in the local z direction.
I said that this is an auto normal matrix; that is; these columns are orthogonal, mutually orthogonal, and they are normal vectors. They are the unit vectors.
Now something very important is the following: what is the rotation matrix of G with respect to L—the global with respect to the local? Obviously that is the inverse of this earlier matrix. It turns out to be that and it is fairly easy to show that this is the same as the transpose of this earlier matrix.
So finding inverse is very simple. Just take the transpose; so, if you have the rotation matrix from local to global, local with respect to global, then global with respect to local is simply the transpose of that matrix.
All that you need to do is look at these two reference frames. Let's say global, in this particular matrix, two by three matrix, is global with respect to local. It has to be how we derived local with respect to global; the unit vectors columns should be the unit vectors along the global X Y Z axis in the local frame, expressing the local frame. And if you look at these carefully, you will see that these rows are nothing but that.
These are the local coordinates of the global x axis. These are the local coordinates of the global y axis; and these are the local coordinates of the global z axis. That's why this turns out to be the transpose of this.
So this is something very important which we should keep in mind. (refer slide time 15:52)



We have been using two terms: we have been using transformations of vectors and points and we are calling the associated matrices as orientation matrices or rotation matrices. Rotation involves a displacement; rotation means a particular type of displacement. So how are these two related? It is something that you need to see: the transformations, the meaning of transformations and displacements, and the relation between the two.

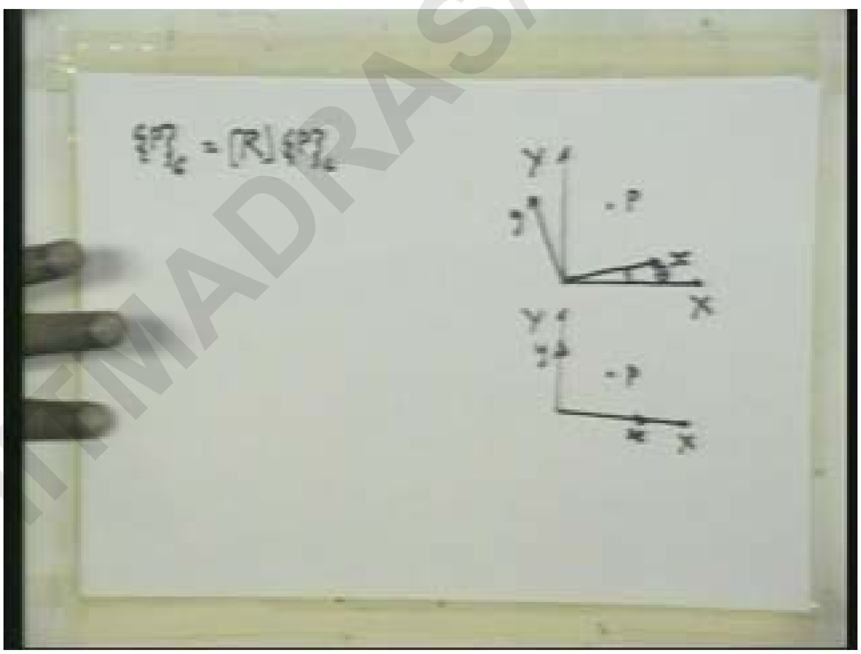
So if you look at the very simple case, in the planar case, let's say it is global x and y axis, and z axis is out of this plane. The local x y after the rotation about the z axis is this.

In transformation what we do? In transformation, if we take a point p in the local frame, we know its coordinates in the local frame; we find out its coordinates in the global frame.

So p in some local frame is transformed through a rotation matrix to give you p in the global frame. Because the origins of the two coordinate frames are coinciding, we don't have a translation here; we don't have a vector of translation here.

How is this interpreted as displacement? How do we interpret this as displacement? In displacement, a displacement operating on a body should displace the body. So this is a point in the local frame and we get a coordinates in the global frame in this fashion but this is for two fixed, two specific reference frames. Now if you look at this rotation, which involved rotation about theta, let's assume it is obtained by angle of rotation theta.

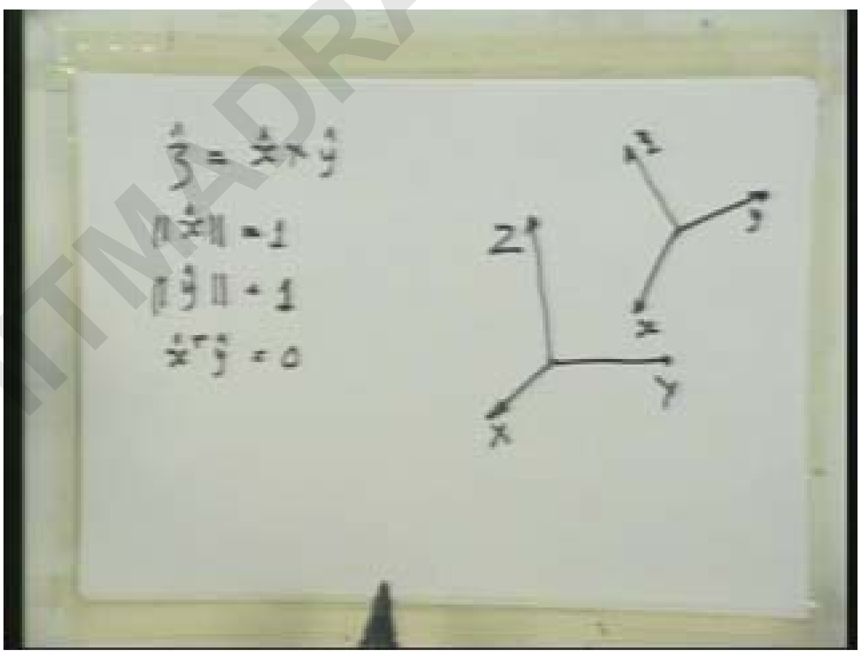
So how do you now interpret this as rotation? Rotation can be interpreted in this fashion. Imagine, to start with, that this global and local reference frames are coinciding or parallel. Take a point p in this—so initially those two bodies are coinciding. Now you rotate the local by this angle theta, where will P go to? P will go to exactly this, in particular. So that is how you look at this as the displacement in this particular case, as pure rotation. (refer slide time 19:58)



So you can interpret the orientation or rotation matrices and the translating vector, the origin the position of the origin, as displacements or specifications of displacements. So this is another point that you have to keep in mind.

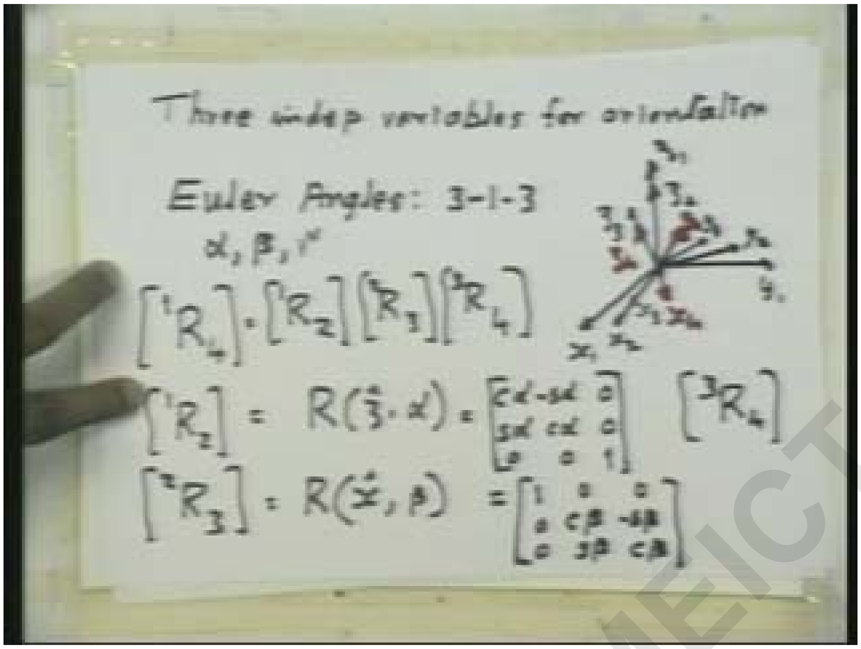
Now let me come to a very important question. In order to specify rotation, we are now using nine elements. So these are the unit vectors of the local frame. Are all of them

required—all of the nine elements? What is the minimum required—six, three, one, none?
So let us work it out. Talking only about orientation, I am talking only about the orientation matrix or the rotation matrix, which has nine elements. So if you look at it closely, you will see that only three are required. You can figure it out in the following way: suppose you know the unit vectors along x and y; z is the immediately obtained. How do you obtain it? It is a cross product of x and y. So z is not at all independent; it is simply this.
Now the question of x and y—how many independent elements are there?
So x and y are unit vectors; they are the unit vectors. So if they are unit vectors, how many independent elements are there in each unit vector? Basically two, which is not very convenient to just use, but you can do. Because the unit vector magnitude has to be unity, so definitely you can say that the squares of the components have to add up to one. So that is an equation which relates the three components; so they can only be two. Let's say x has two components; what about y? y also has two components which are independent if you don't consider x. But there is the additional fact that x and y have to be orthogonal.
So the components of x, if you square and add, have to be one. This is the root of that square; this is one for normal that vector is one; so that is one relation. Similarly, you have a normal y, which also is one; that is another relationship, and finally, x transpose y equals 0. This is one. So how many relations were we able to write? Six for nine parameters: so there are only three independent elements out of the nine. Three of them are independent; the remaining nine are related by these six equations. <u>(refer slide time 24:14)</u>



So why do we use these nine elements to represent orientation? The answer is that it is convenient for doing the transformations. These transformations are things that you like to do later.

But keeping the convenience aside, can we have three variables, which will give orientation uniquely without any ambiguity? This was answered by Euler quite a long time back and he specified various ways of doing this.
So we will see how he described orientation. There are different types of Euler angles; what we'll see are three Euler angles.
The way he obtains the rotation matrix or the new position of a body, let me call this as x_1, y_1, z_1; this is the global frame.
Initially imagine that the local frame is coinciding with the global frame. Then you rotate the local frame with respect to global frame about the local z_1 axis, local z axis. So initially, imagine that the local frame is x_1, y_1, z_1; it is coinciding with the global frame.
Now I am going to rotate it about the z_1 axis to get z_2, which is coinciding with z_1; x_1 would have shifted to x_2, y_1 would have shifted to y_2. Let these angles of rotation—I am going to specify three, a sequence of three rotations, to take the body from initial coincidence with the global frame to its final orientation: alpha, beta and gamma.
So this rotation about the z axis is alpha; so the angle you see here and here is alpha. So that is why the first rotation—this is called three one three—first rotation is about three, then about one, which is x, and then again about three.
So next what you do is, the local frame is now at x_2, y_2, z_2; now rotate the local frame about x_2 axis. If you do that, the new position x_3 will be here itself; y_3 would have gone somewhere; y_2 would have gone to y_3 like this; z_2 will go to z_3.
And finally a rotation about z_3; so z_4 is the same as this; x_3 could be somewhere like this; and y_4 is here.
So this is x_4, the final orientation of the body—x_4, y_4, z_4 with respect to the original orientation x_1 y_1 z_1.
So what we do now is we know these three angles of rotation: alpha, beta, and gamma about the local z x and x z axes.
Now we need to find out this rotation matrix. If you look at the coordinate frame, what is the rotation matrix from if the four is with respect to one?
Now we can use the same rule of using intermediate coordinate frames in order to express four with respect to one. Each of them by themselves are very simple matrices or two with respect to one for example can be this rotation matrix, which we will get by rotating about the z axis by the amount alpha.
What is that? This is something we have derived earlier; we look at the elements: this is cos alpha minus sine alpha; 0 sine alpha; cos alpha 0001.
We have seen this earlier. So the notation Rz cap from alpha is rotation about z cap by the amount alpha and the sense of alpha is in the right-hand side screw loop.
When you rotate in the positive direction alpha the screw advances: the right-hand side screw advances towards that the tip of the vector.
Now what is R_3 with respect to two in this notation? Rotation matrix is obtained by x cap y. This is the rotation about x; this is the second rotation by amount beta. For this you can write as 1 0 0 0 0 cos beta, minus sine beta, sine beta, cos beta. (refer slide time 32:49)



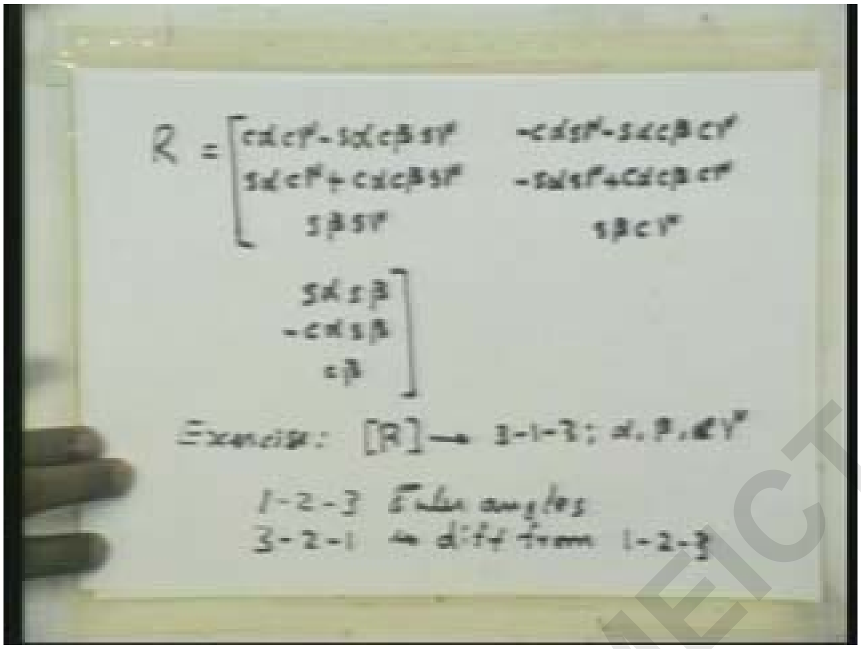
Now what about R_4 with respect to three? It is the same as R rotation about z cap by amount gamma. So on substituting this matrix alpha by gamma, you get the third rotation matrix. So multiply all of them together like this: from alpha, beta and gamma. The Euler angles one three one three Euler angles give you the rotation matrix. So if you work it out, we will get the following answer; so I will simply call the rotation matrix R. You can derive it yourself; hopefully you will get the following: c alpha, means cos alpha.

So given the Euler angles, three one three Euler angles, alpha, beta and gamma, you can obtain the rotation matrix. These are the elements of the rotation matrix is related to in terms of alpha, beta and gamma.

Now an interesting and important question is whether you can get alpha, beta and gamma from the rotation matrix. What are the Euler angles three one three Euler angles corresponding to a given rotation matrix?

So you can do it as an exercise given R determine—I will specify as three one three—alpha, beta, gamma. It is fairly easy one should assume because there are nine equations that you can use. We will get three variables, three unknowns. We have nine equations you can equate, so not too much of the problem still.

Now this is something we will do. We can try this out; it is informative to do that. How many numbers of solutions are there? There are some special cases. That we will do in detail in inverse kinematics because it turns out that precisely this has to be done in inverse kinematics. (refer slide time 39:10)



Now I mentioned three one three. Are there any other possibilities? Yes, so Euler angles really mean this idea of rotating with respect to the local coordinate frame: every time you shift the local coordinate frame you rotate with respect to the current local coordinate frame axis. That is the idea Euler used.

So we could have, say, one two three Euler angles. Can you have 1 1 1? That will not be very useful, what is quite interesting is that three one three is determined; you have to rotate about the third one, that is, y—that is quite interesting.

So you can work out the rotation matrices for one two three in whatever form they are given.

The basic idea is the same. Otherwise it's a different Euler angle; three two one is different from one two three.

Now why did the Euler really go about rotating around these local reference tags? Is it possible to do it with respect to the original global reference frame? It is possible. Only how to derive the rotation matrix has to be seen; so we will try to doing that. Now to start with again, we assume let's say same as Euler angles; let's consider one two three.

So how do we do the rotation? The idea is this: suppose this is the global reference frame. Initially the local reference frame is in coincidence with the global reference frame and oriented, aligned in the same directions. This could be started with x y z.

Then you rotate about x axis, one is x axis. So this goes to, let's say, a final position. Then what do you do? Next what you do is rotate about this body, which you have already rotated about this particular axis; that's what we have done.

So this small x y is that the new position will go somewhere else and after that you rotate about the global z axis, that is one axis, and you get the final position.

So if we draw it in a proper way, we will immediately understand in what sequence to rotate what matrices in order to get the rotation matrix. So let me show this axis getting shifted this time. So I will use different colors so that things become clear.

So to start with, the local frame is coincident with this but after rotation about the x axis, this would go to this position, what you see in blue. So let me call this x dash y dash z dash.

Now remember that we are going to rotate this whole thing about the y axis; now what you need to imagine is that y dash is fixed with respect to y. That is, the frame x dash y dash z dash is fixed with respect to x y z; imagine rotating the frame x y z about the y axis.

So what we are going to rotate is the frame x y z, which is black, about the y axis. When we do this, the blue frame is fixed with respect to the black frame as it moves.

So that is what we need to imagine. Let's assume that red is the position of the global frame; now this black frame has rotated so the black frame has rotated about the y axis. So this is the same as y double dash. Where did the z go? z went somewhere here, and x went somewhere here.

Now along with this black frame, the blue frame has rotated. So its relationship with the black frame is the same.

What I show here is this: this is x dash and this is z dash and this is y dash. So x dash z dash y dash x y z; we have to do the notations of the frames carefully.

Let me not worry about the notation; let me first show the various frames in different colors. Can you see the colors clearly? This one is black. Then I will have to give a notation; so in that shifting what I am going to do is, I am keeping the global frame as y: y, z, then z dash, z dash, x dash, and y dash. So this is a single dash, then y double dash, z double dash, and x double dash.

So the idea is this: the relationship between the dash frame and double dash frame is the same as the undash frame and dash frame output. This angle was alpha; so this angle will be alpha, this angle will be alpha, so this is also alpha, so this will also be alpha.

Now this is beta; what I do is rotate the whole stuff; rotate this particular frame x y z about the global z direction and while it rotates, it takes along all the other reference frames as if they are rigidly attached to that. That is what is important; so if I do that, I will get the following. No point in using color, so we have z y x; now we are rotating about it, so you have z dash; you have x dash.

So remember what is happening is the following. With each rotation I added dash to the reference frame. For this rotation, I added dash to the reference frame. So initially the local frame was aligned with the global frame and when I rotate, the local frame became x dash y dash z dash and because the rotation was about the x axis, x dash coincided with x.

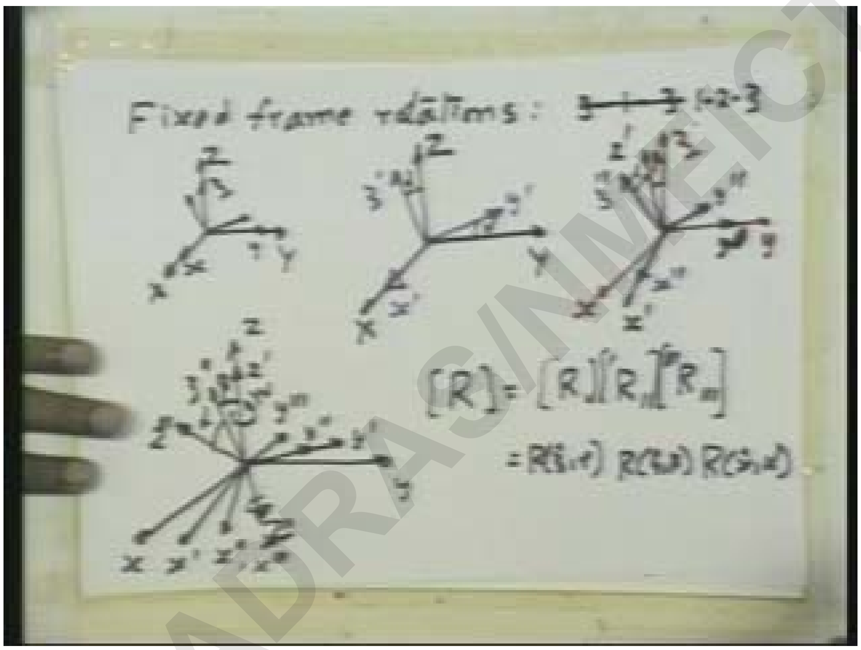
Now I rotate this whole thing with respect to the y axis; so I added dash to each of the axes.

So this becomes y dash, y double dash, z dash, z double dash, and x dash, x double dash in this frame, in this picture.

So you have the x y z in the global frame; x dash and x double dash coincide the first rotation is about the x global frame;

y and y dash coincide; y double dash is the new position; z dash and z double dash are all different; the third rotation is about z frame, the global z frame, so z dash and z are the same. You just add the dash to all the reference frames. So if we want to look at exactly what the rotations have been, this gives alpha, this gives you beta, and the last rotation, which we just now gave, is gamma. Now if you want to write the rotation matrix, you

will find that is from the dash frame to the undash frame into double dash to single dash to the triple dash to the two dash. Since we have used one two three, this is actually the rotation about x axis; this is about y axis; and this is about z axis.
So we have R with respect to x by angle alpha; then R about y by angle beta; and R about z by angle gamma. The sequence is exactly reversed compared to what you have in the Euler rotations. You can work it out as we did earlier; these matrices are fairly simple. x will have only three because one is coinciding with that. If you look at this, x double dash and x dash are coinciding; now we are going to rotate about the z frame. So x dash and x double dash remain coincident; only they become x double dash and x triple dash, whereas when I do it here, I do a new x triple dash axis, which is the same as x triple dash. (refer slide time 54:25)



So all these reference frames will have only three distinct; all these axes will have only three distinct. One, two, three are all outside of the x vector. You can probably draw it a little differently, but the basis is understood.
What we are doing is we rotate first, then keep the two reference frames rigid and rotate with respect to some global frame again; then keep the three reference frames rigid and rotate again with respect to the global frame. This is the way we proceed, and that is why this sequence turned out to be like this.
So we can obtain the rotation matrix given the one two three fix frame rotations are in any sequence that you want. Is there a question? I will just pose one more question, which is very important. Any rotation, whether it be the sequence or whether the original rotation matrix is given, the rotation can be obtained out of a single rotation about a specific axis. This particular axis will be different for different orientations.
So what I am saying is that a body can be brought from any beginning orientation to any final orientation by a single rotation about the specific axis by an appropriate angle. So, given the rotation matrix, can you find out that axis and the extent of the rotation; and,

the other way, given this axis and amount of rotation, can you find out the rotation matrix?

This is another way of describing rotation matrix. Given the rotation about an axis, can you find out the rotation matrix; and given the rotation matrix can you find out the axis and the amount of rotation? With this summary, we will briefly touch upon it in the next class before we proceed to discuss how in space mechanisms, the parameters of mechanisms, can be described. Thank you.