Heat and Mass Transfer
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Lecture No. 06
Heat Conduction – 3

Last time when we stopped, we had just began to derive the differential equation of heat conduction and I think I told you that the reason why we need a general differential equation is that we would like to solve problems in which the temperature is not just varying in one direction or and also situations in which the temperature is varying with time. So far, we have only considered problems in which temperature is varying in one direction and we have a steady state temperature distribution. If we want to solve more general problems, problems in which temperature varies in more than one direction or problems in which there is a variation with time, then we need to derive a general differential equation for heat conduction to take care of more general situations. Now for solving the differential equation, consider that we have any arbitrarily solid like this – any arbitrary solid.

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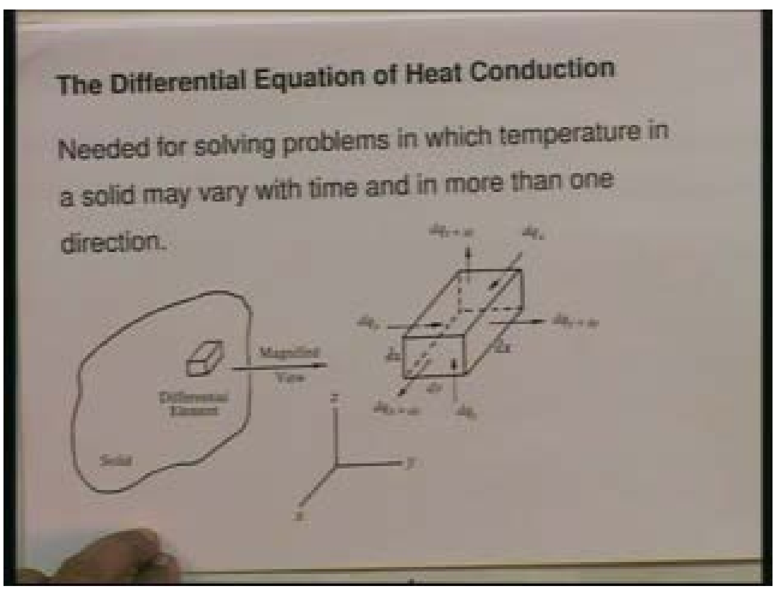

And to derive a differential equation, you always need to take an element inside this solid. Let us say we are going to do the the derivation in the cartesian coordinate system; so we will take a differential element dx dy dz anywhere, it may be anywhere in the solid arbitrarily located inside the solid and here on the right hand side you have magnified view of that differential element. Now, we will make some assumptions and with those assumptions we will proceed now to derive our differential equation.

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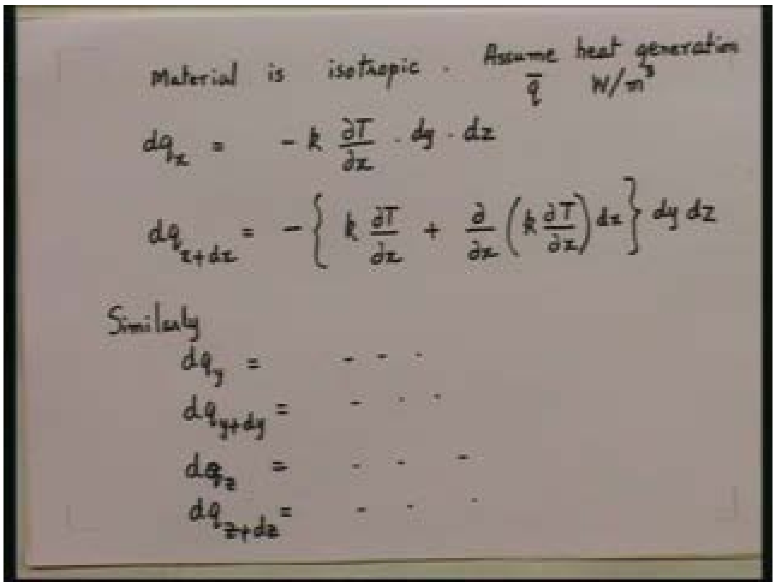

First of all, the first assumption which we are going to make is that the material is isotropic. What do we mean by this assumption? We mean that the thermal conductivity of the material is not directionally dependent at any point inside the solid, whichever direction the heat may flow, the value of k in all those directions will be the same; that is what we mean by an isotropic, the assumption that the material is isotropic. Now k may vary from point to point but at any given point it doesn't matter which direction the heat is flowing, the value of k will be the same so we are going to make that assumption. Now consider again - let us look at the element again; consider this element dx dy dz. Now in this element let us look at the 2 x faces that is faces which have, whose normals are in the x direction that is in this direction. There is one face at the back here like that and one in the front here.

Let us take this as our positive direction; so let us say the heat flowing in the positive direction in this face it will be dqx - some differential quantity because it's a differential element and the heat flowing out here will be dqx plus dx because we have moved a distance dx forward in the positive x direction. So in the, for the 2 faces which have their normal in the x direction, the heat flowing in that is the heat flowing in the positive directions are dqx and dqx plus dx. Similarly for the y direction, we will have dqy, dqy plus dy and for the z direction, we will have dqz that is the bottom face of this element and going on from the top face dqz plus dz.

Now, we will use Fourier's law of heat conduction to write down expressions for each of these quantities. What we will get from Fourier's law? We will get the following – let us look at the first one dqx; by Fourier's law of heat conduction dqx will be equal to minus k dT dx multiplied by - this is the heat flux in the positive x direction. Multiply, you want the rate at which heat flows so multiply it by the area of that face which will be dy dz; so many watts is the rate at which heat flows into this x face.

Now let me go back for a moment, we were deriving a general differential equation so let me add or make it go back and say for a moment. Let us also assume - apart from the material being isotropic – let us also assume that heat could be generated in this material and that the rate of heat generation; assume heat generation; assume that heat could be generated in the material and that the rate of heat generation is q bar so many watts per meter cubed. We will use the symbol q bar for the rate at which heat is being generated per unit volume of the solid. q bar may be a constant if heat is being generated uniformly; q bar may also vary from place to place in which case it will be a function of x, y, z. q bar could also vary with time in which case it could be a function of x, y, z and time; whatever it is, it will be some specified function or it will be specified to be a constant, so let us assume there is some heat also being generated.

Now, let us go back to where we stopped; Fourier's law tells us the rate at which heat flows in the x direction so dqx is this. If I move a distance dx forward in the x direction then I will get the expression for dqx plus dx and that will be equal to; I have moved a distance x dx forward in the x direction. Now, we ask ourselves in this earlier expression what are the quantities that may vary in the x direction? k may vary because k could be varying from point to point, dT dx may vary but dy and dz are constants; that is the area of the element that doesn't change in the x direction. So k into dT dx may vary in the x direction so we will say the following - we will say dqx plus dx. Let me write down the expression; first I will say take it outside, let us put a bracket and say k dT dx. Now this is the quantity that can vary in the x direction say plus the variation d dx - the rate at which it varies in the x direction - into k dT dx; into k dT dx multiplied by the distance that I have moved in the x direction which is dx and the whole thing multiplied by dy dz which is a constant - the area; so this is the expression for dqx plus dx.

Anytime we derive a differential equation, if I know the value at some distance x, the value at a distance x plus dx will be the increment that occurs because the function, the quanties that you are concerned with are varying in that direction. Similarly, we can write down expressions for dqy, dqy plus dy, dqz and dqz plus dz. You can write down expressions for these; I am not going to do that; I am going to leave it to you to write these down. In the same way that can be written down, these expressions can be written down using Fourier's law of heat conduction. So now we have expressions for the rate at which heat is being conducted into this element in the positive x direction, positive y direction, positive z direction and the rate at which heat is being conducted out of this element also in the positive x direction, y direction and the z direction; so we ask ourselves the question. We say what is the rate at which, what is the net rate at which heat is being conducted into the element?

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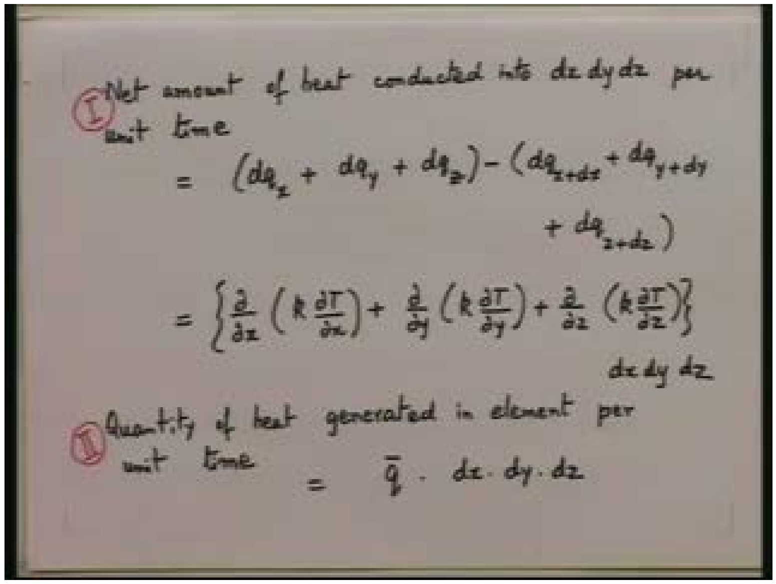

So, we say net amount of heat conducted, therefore, net amount of heat conducted into the element, into dx dy dz - that's our differential element - into dx dy dz per unit time will be equal to dqx plus dqy plus dqz. This is the flow in minus the flow out dqx plus dx plus dqy plus dy plus dqz plus dz and then we substitute the expressions we have put down earlier, we will get this is equal to d dx; putting in those expressions and canceling out, we will get d dx k dT dx plus d dy of k dT dy plus d dz of k dT dz, the whole thing multiplied by dx dy dz. This is the net amount of heat being conducted into the element per unit time.

Now the next quantity we have – let us call this, let us call this as 1. Now the rate at which heat is being generated per unit time, quantity of heat generated in element in element, the element per unit time will be equal to q bar which is the heat being generated per unit volume multiplied by the volume of the element dy, dx dy dz. Let us call this expression 2. Now, all this leading up course to applying the first law for a closed system. The third quantity we are interested in is the rate of change of energy of the element, rate of change of energy of the element.

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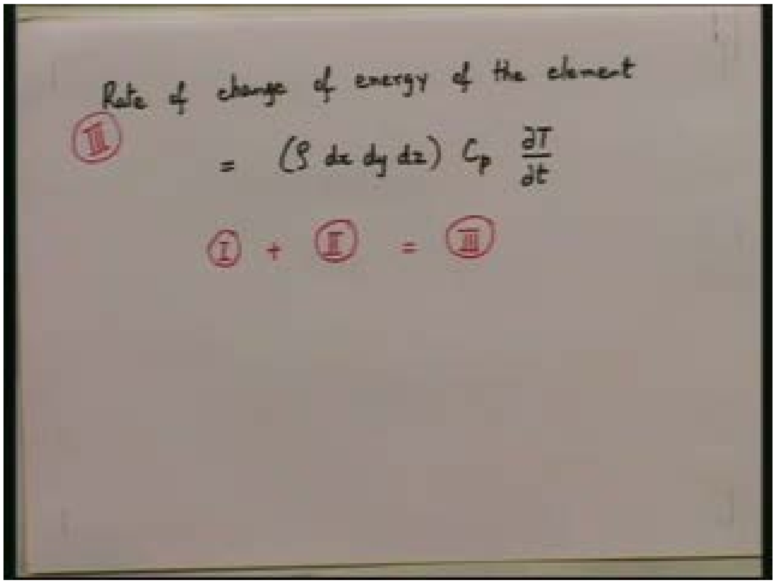

This will be equal to the mass of the element that is row dx dy dz - mass of the element - into the specific heat of the material into the rate of change of temperature dT dt; that's the rate of change of energy of the element and this we will call as 3. Apply the first law for the closed system; what is our closed system, closed system is the element dx dy dz. Apply the first law of thermodynamics to this closed system. What does the first law say? The first law says 1 plus 2 is the net rate at which heat flows in the element or is generated in the element must be equal to 3. So, let us do that; let us substitute now our expressions that we have got and we will get, we will get the differential equation once we do that and the differential equation we will get would be the following.

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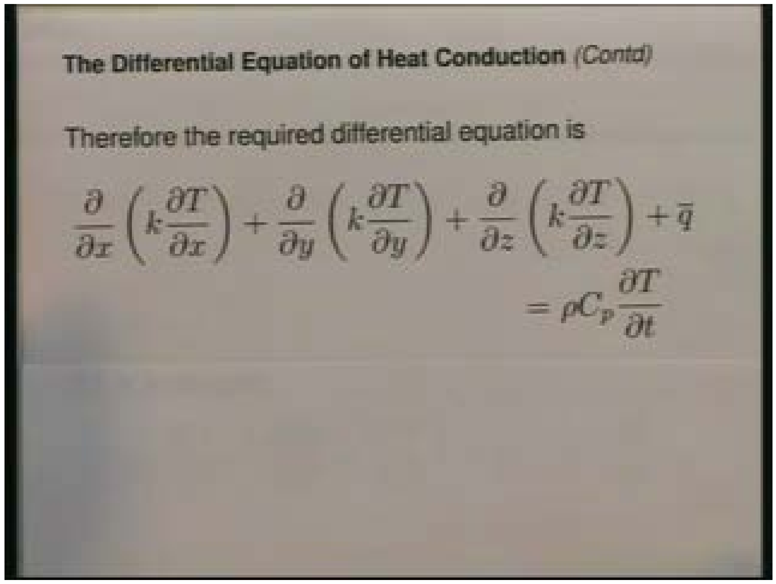

If I substitute this, I will get the following differential equation for the case that I am deriving; this is the differential equation that I will get. All I have done is put in this is the term 1, this is corresponding, q bar is corresponding to the term 2 and this is corresponding to the term 3. dx dy dz - the volume of the element - is canceled out; it is common to all of them. This is the required differential equation that we are looking for. This is a situation in which we have an isotropic material; we are working in a cartesian coordinate system. T can be a function of x, y, z and time; that is why we have partial differential equation and there may be heat generation q bar which may be varying or it may constant. Whatever it is, it will be some specified amount so this is the most general form of a differential equation that we have got. Now, let us look at some simplifications of this differential equation.

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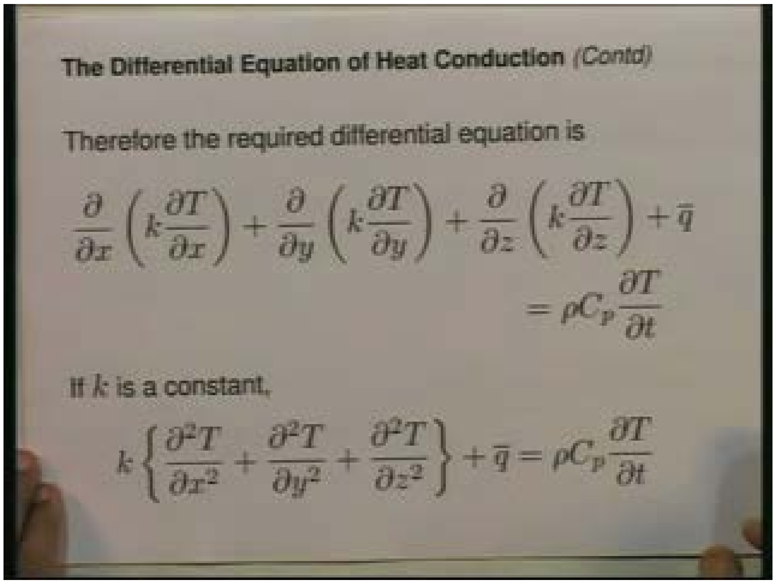

We say to ourselves, suppose the property k - thermal conductivity k of the material - is not varying from point to point inside the material; that means the material is isotropic, k is a constant; the material is what we call is homogeneous. In which case the k can be taken outside the differential here and we will get the differential equation which is written here. We will get k into d 2 T dx square plus d 2 T dy square plus d 2 T dz square plus q bar is equal to row C_p dT dt; so this will be the simpler form of the differential equation that we will get if k is assumed to be a constant. Further suppose or rather let me for a moment express it in another way symbolically.

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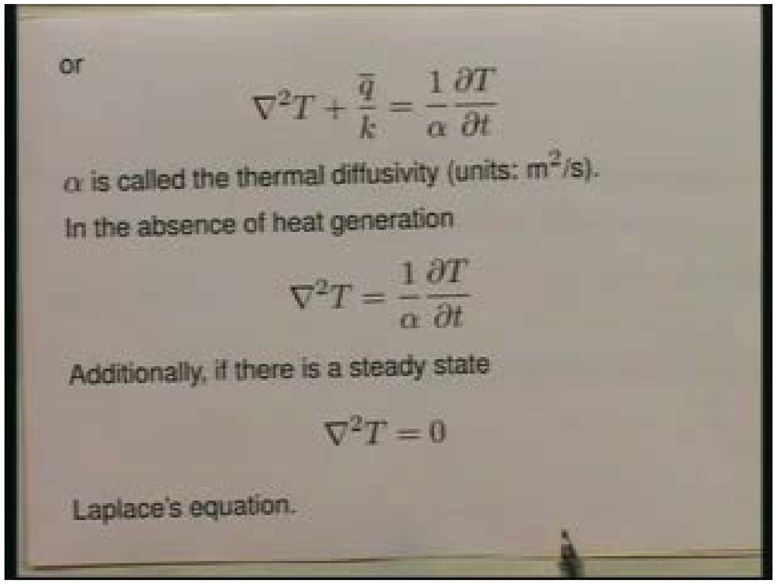

As it is put, very often the quantity d 2 T dx square plus d 2 T y square plus d 2 T dz square is written in this form - this is called the Laplacian equation. del square T plus q bar by k equal to 1 upon alpha dT dt; so this is another way of writing the equation and I would like at this stage to define that we have got instead of k by row C_p, we are writing a quantity alpha here. What is alpha? alpha is called the thermal diffusibility of the material; it is a combination of three properties - k the thermal conductivity divided by row, the density and divided by a specific heat k by row C_p. So, we define a new property k by row C_p as alpha and we put that into the differential equation. So our differential equation when k is constant can also be written in the form del square T plus q bar by k is equal to 1 by alpha dT dt.

Now, suppose no heat is being generated; there is a case of no heat generation, the q bar term will drop out in which case I will simply get del square T is equal to 1 upon alpha dT dt. Suppose in addition there is a steady state, in that case the right hand side will be 0, there will be no change of temperature with time; I will get del square T equal to 0. So this is called the Laplace's equation the last 1. So I have different versions of the differential equation of heat conduction now and I will use an appropriate differential equation depending upon what problem I am solving, what is the nature of the problem I have to solve.

Let us go back again for a moment; if it is the most general form is what we derived; first the material is isotropic but k may be varying, temperature may be varying in all directions and with time there may be heat generation. Next version k is a constant in which case I get this simpler version; third version there is no heat generation in which case the q bar term drops out. Finally, I say there is a steady state in which case the dT dt term drops out; so depending upon the nature of the problem we will have to use an appropriate differential equation.

Now this differential equation that we have got is for the Cartesian coordinate system; we may sometimes find it convenient to work with the cylindrical coordinate system or a spherical coordinate system. If suppose the object under consideration is a cylinder in which case a long cylinder or a short cylinder, it will be convenient obviously to work in a cylindrical coordinate system or let us say the object is a hollow sphere or a solid sphere in which case it will be convenient to work with the spherical coordinate system.

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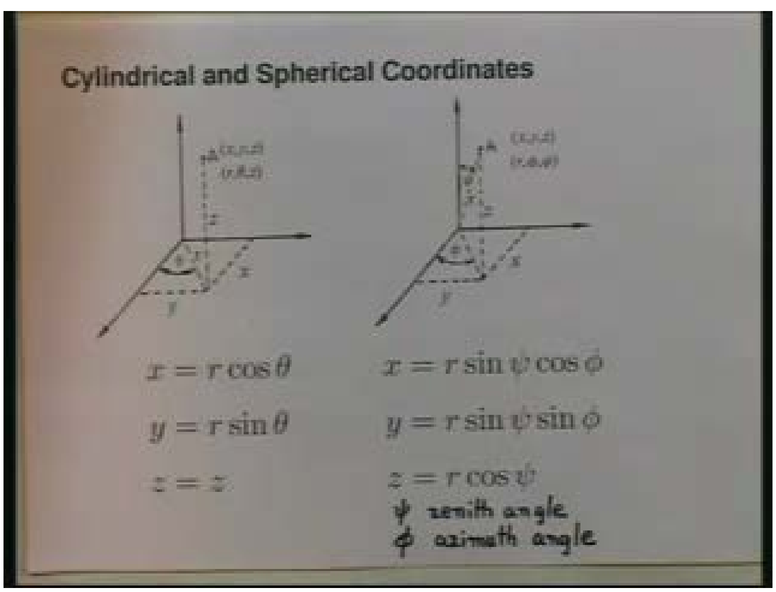

Now in the next transparency here, I am showing the 2 coordinate systems; on the left hand side here we have the cylindrical coordinate system shown here and this is the usual transformation. x equal to r cosine theta, y equal to r sin theta, z equal to z which helps us to go from Cartesian coordinates to cylindrical coordinates – r, theta and z. r is the distance in the radial direction in the xy plane, theta is the angle in the xy plane and z is the same z as in the Cartesian coordinate system. On the other hand, if I have a spherical coordinate system I have the transformation x equal to r sin psi cosine phi, y equal to r sin psi sin phi and z equal to r cosine psi. In which case I have a distance r which is the distance from the point A to the origin - that is r; phi is the angle similar to the angle theta in the cylindrical coordinate system measured in the xy plane and psi is the angle by the radius r with the vertical direction z. psi is called the zenith angle and phi is called the azimuth angle in the spherical coordinate system. I might just write that; psi is called the zenith angle and phi is called the azimuth angle. Now, you might want to remember just these names.

So, suppose now I want the differential equation in the cylindrical or in the spherical coordinate system. One way is I can use these transformation formulas for going from cartesian coordinates to cylindrical coordinate or cartesian coordinates to spherical coordinates and by doing transformations just derive it mathematically. The corresponding use whatever is the appropriate differential equation Cartesian coordinates; use these transformations and go to the appropriate equation in cylindrical or spherical coordinates. The other way is to derive again from first principles; we can derive a differential equation using an element in the cylindrical coordinate system or in the spherical coordinate system. Now let us do that.

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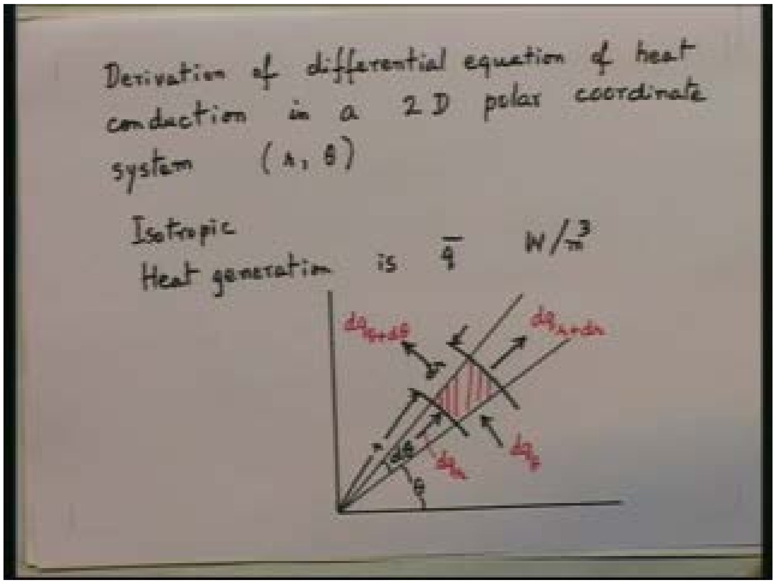

Let us for instance assume we want to derive a differential equation; let us say derivation of differential equation, derivation of differential equation of heat conduction in a 2 dimensional polar coordinate system, in a 2 dimensional polar coordinate system. So instead of deriving for a cylindrical coordinate system with r theta and z, I am going to derive for a polar coordinate system which is only r and theta; we will generalize later to r theta z. So let us say we are going to derive a differential equation from first principles like we did for a xyz in and we are going to do this derivation for an r theta coordinate system.

Assume again material is isotropic and heat generation is q bar in so many watts per meter cubed; q bar may a function of r and theta, q bar may be a constant, whatever it is. Let us look if I want to derive a differential equation in r theta, obviously I need an element in the r theta system; so let us take such an element now. Let me draw a coordinate system; this is a coordinate system, r theta I want to draw. Now this is the radius r and this is dr and this is the angle d theta. So, the element r d, the element under consideration is the one that I am shading and its area will be r d theta into dr. Again like last time, let us put down expressions for the heat being conducted in and out of this element.

So we are going to put down expressions for heat being conducted in the r direction into this element, heat being conducted in the r direction out of this element, heat being conducted into the element in the theta direction and heat being conducted out of this element in the theta plus d theta direction. This is the angle d theta and this is the angle theta so we will call these quantities, we will call these as, let me just indicate them. The first one call as dqr, then dqr plus dr then here, I will call this as dq theta and the last here, I will call as dq theta plus d theta. We want to put down expressions for each of these quantities; what will we get? We will get the following.

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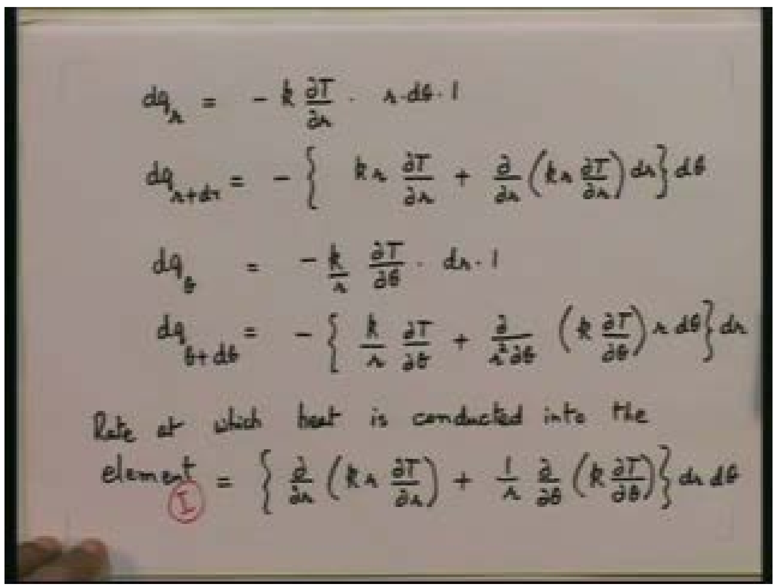

We put down expressions for these quantities and we will get dqr, the first 1. Now I am not going to explain again like last time; I will simply put down the expressions. dqr will be minus k dT dr multiplied by the area through which heat is flowing; the area through which heat is flowing will be r d theta into 1; we assume in the z direction that we have a unit width so that is the area. dqr plus dr will be equal to minus, now we will say here kr dT dr plus d dr kr dT dr into the distance I have moved – dr, the whole thing multiplied by d theta. The quantities varying in the r direction are k dT dr and obviously r; so all this k dT dr into r has to be taken inside the differential. d dr of all this multiplied by dr is the amount of variation that occurs when we move from r to r plus dr.

Similarly, we can write down expressions for dq theta and dq theta plus d theta. For dq theta, we will get minus k dT d minus k dT d theta k by r, dT d theta, because we want a variation gradient of temperature with distance so it will be dT by rd theta, the whole thing multiplied by area which will be dr into 1. And if I move a distance an angle d theta forward, that is the distance r d theta forward, I will get minus k by r dT d theta plus d by rd theta multiplied by k dT d theta multiplied by rd theta, the whole thing multiplied by dr.

Now, therefore the rate at which heat is being conducted into the element will be dqr plus dq theta minus dqr plus dr minus dq theta plus d theta and if you do all that you will get rate at which heat is conducted. In the previous term you should have an r squared here, r square d theta; rate at which heat is conducted into the element will be equal to d dr into kr dT dr plus 1 upon r d d theta multiplied by k into dT d theta, the whole thing multiplied by dr d theta; that is the rate at which heat is conducted into the element; now this is what we have called earlier as 1.

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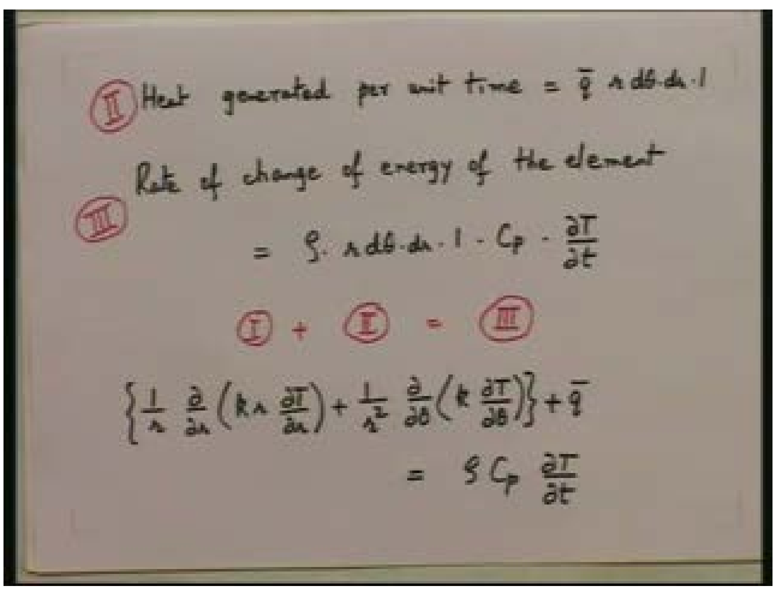

Now in the same way the quantity 2 that is rate of heat generation in the element will be, heat generated per unit time will be equal to q bar multiplied by the volume of the element rd theta dr into 1 and this would be the term 2 and finally rate of change of energy of the element, rate of change of energy of the element will be equal to the mass of the element row multiplied by rd theta dr into 1; that is the volume, mass of the element into specific heat into the rate of change of temperature with respect to time and this will be the term 3. So 1 plus 2, again by the first law, 1 plus 2 is equal to 3 so we get the differential equation. after with the clean up a little, cancel some terms, we will get the differential equation 1 by r d dr multiplied by kr dT dr plus 1 by r square d d theta multiplied by k dT d theta plus q bar will be equal to row C_p dT dt; this is our differential equation in the r theta coordinate system, in the polar coordinate system.

So, this is how we proceed to derive; anytime we have to derive a differential equation, take an element, apply the first law of thermodynamics, write down expressions from Fourier's law of heat conduction for the rate at which heat is conducted in or out of all the faces of that element. That is really what it comes to and you will get the appropriate differential equation. Now once again, we can see from this differential equation that if I have a constant k, then I will get a simplified form of this equation.

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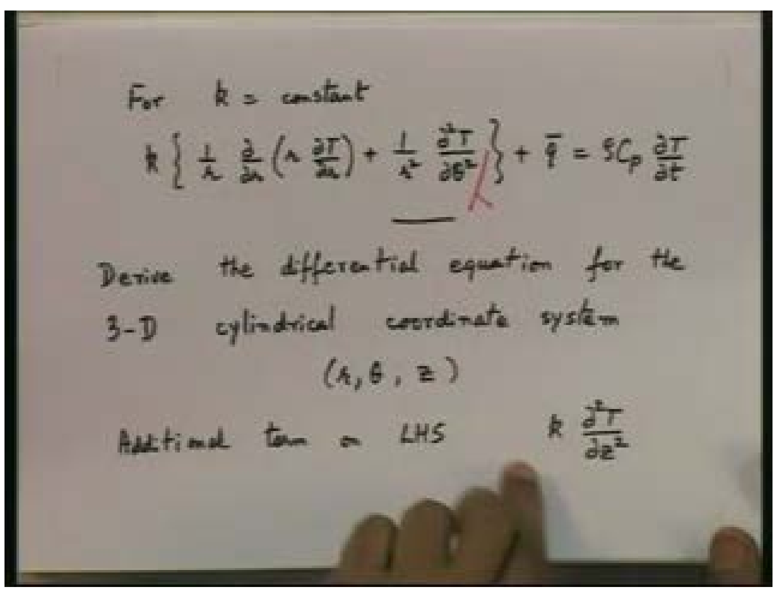

For a constant k if the material is isotropic and the k is constant; for k equal to constant, I will get a simplified form. I will simply get, k - I will take it outside the differential; so k multiplied by 1 by r d dr r dT dr plus 1 by r squared d 2 T d theta squared plus q bar will be equal to row C_p dT dt; this will be the differential equation that you will get with a constant k. And again as we did earlier, if it is a steady state problem, then the right hand side will drop out completely; the dT dt term will drop out. If there is not heat generation, the q bar term will drop and so on; you can get further simplifications.

Now, suppose I wanted to derive this differential equation for an r theta z situation; suppose I say to you – derive; this we have d1 for r theta. I say derive the differential equation; derive the differential equation for the three dimensional situation, the 3D cylindrical coordinate system. I ask you to do this; this will be for an r theta z situation. All that is going to happen if you go through it yourself is you are simply going to get one more additional term on the left hand side and what is going to be that additional term? Additional term on left hand side of the equation that we have here will be k d 2 T dz squared; this is the additional term that you are going to get in this differential equation. Out here, that is all that you are going to get out here - an additional term out here plus this quantity. This is the term which comes in because now you are in the z direction, you have to consider the difference between heat flowing by conduction in the z direction, flowing into the elements and flowing out of the element in the z direction; that is how this extra term is going to come in. So although we have derived the differential equation for r theta, you can see that extending it to r theta z presents no real problems. Now let me just put down the differential equation; I mean show them again for constant k. For, let us show the differential equation for r theta z for cylindrical coordinates.

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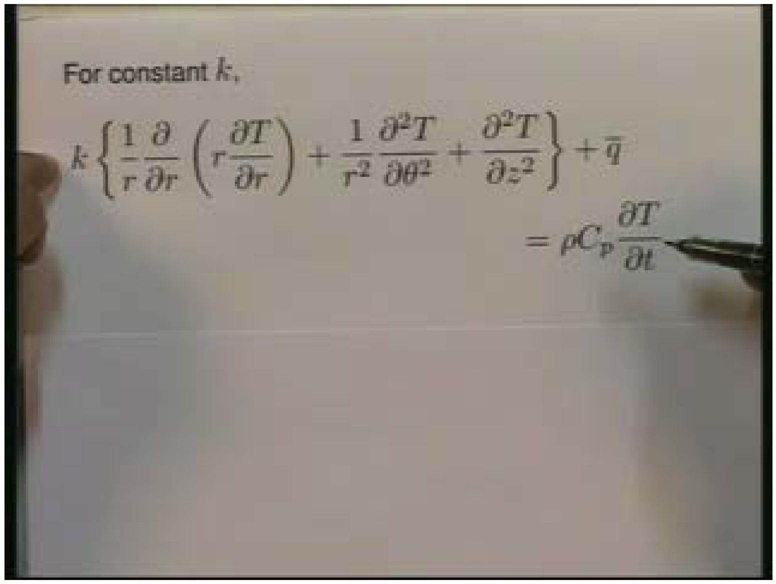

This is the differential equation that we have: k into 1 by r d dr d d d d dr of r dT dr plus 1 by r squared d 2 T d theta squared plus d 2 T dz squared; close the bracket plus q bar is equal to row C_p dT dt. This is the differential equation you will get for cylindrical coordinates and for spherical coordinates we are not deriving the differential equation but I am telling you for spherical coordinates.

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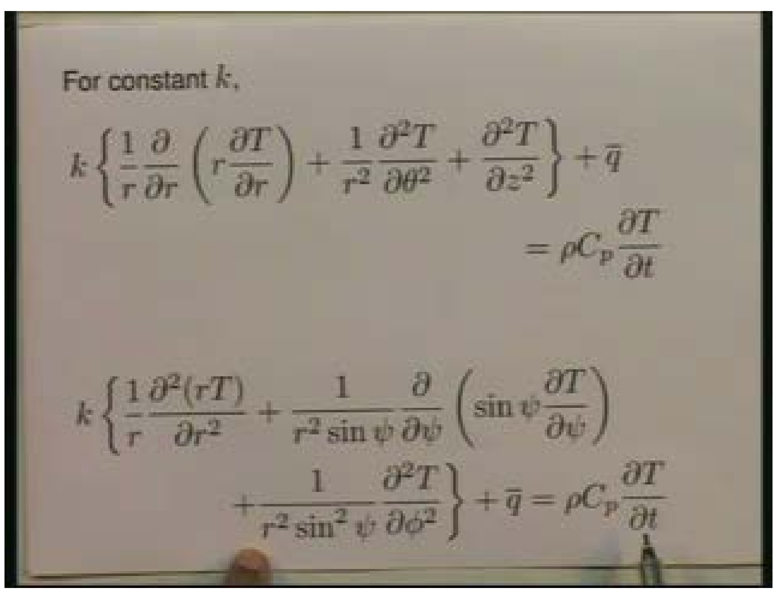

If you do the derivation you will get this second differential equation on this page; you will get k into 1 by r d 2 by dr squared of rT plus 1 by r squared sin psi d of d psi of sin psi dT d psi plus 1 by r squared sin squared psi d 2 T d phi squared, close the bracket, plus q bar is equal to rwo C_p dT dt. So, these are the 2 differential equations you will get for constant k. The first one we have derived; the second you could derive on your own if you wanted to by taking an element in the spherical coordinate system.

Now, we move on; so now we know that given a general situation either in Cartesian coordinates, cylindrical coordinates, spherical coordinates, steady state, unsteady state, variation in one direction, two directions, three directions, isotropic material constant k, not constant k, whatever it is, you are in a position to put down an appropriate differential equation given a certain solid.

Now in order to solve a differential equation and to get an explicit solution, you need to state certain conditions of the problems. What are those conditions? Those conditions are what we call as initial conditions and boundary conditions. An initial condition is the temperature; an initial condition specifies the temperature at - inside the body at - some instant of time t equal to 0 from which point onwards we are interested in knowing the temperature distribution in the solid; that is what we mean by an initial condition.

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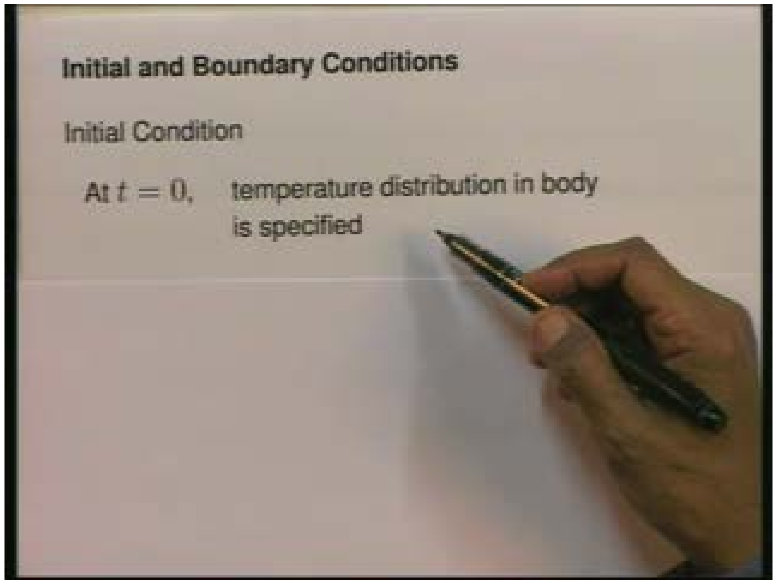

So at t equal to 0, temperature distribution in the body is specified, that is an initial condition; you have to tell your starting point and then say apply the differential equation and tell how temperature will vary with time from that point onwards – that is what you mean by an initial condition. The simplest initial condition is to say, which we use often in solving problems, is that the temperature of the body at the initial instant t equal to 0 is uniform, t equal to 0 or t equal whatever is specified; t_1, some value specified temperature right at the beginning. Now apart from that, we need to know, any solid has boundaries, so we need to know what is happening at the boundaries of the solid; those are what we call as boundary conditions.

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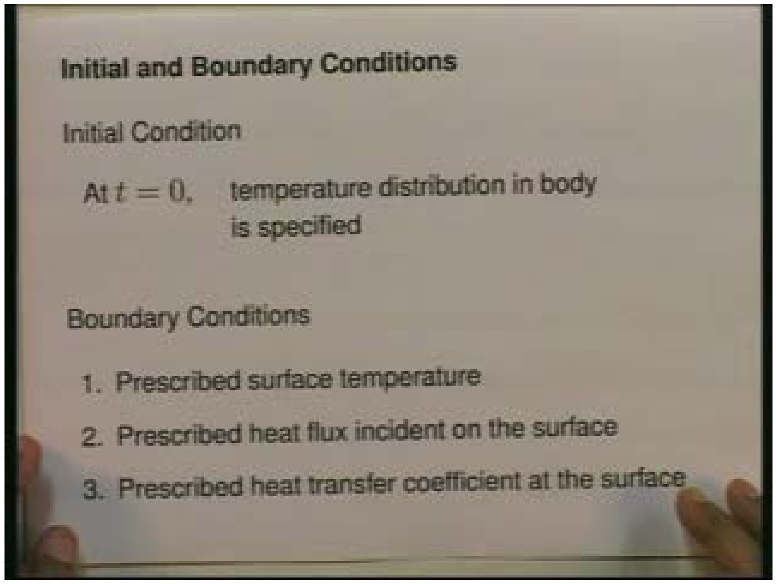

Typically there are 3 types of boundary conditions that we specify; number 1 - we specify a surface temperature at the surface of the solid. What is the temperature at that surface? This could be some which could be varying with time or it would be constant or we specify the heat flux which is falling on the surface of the body or we prescribe the heat transfer coefficient at the surface of the body. These are the three types of boundary conditions which we normally deal with - prescribed surface temperature, a prescribed heat flux incident on the surface or a prescribed heat transfer coefficient at the surface of the body. Let us now write down expressions for these situations.

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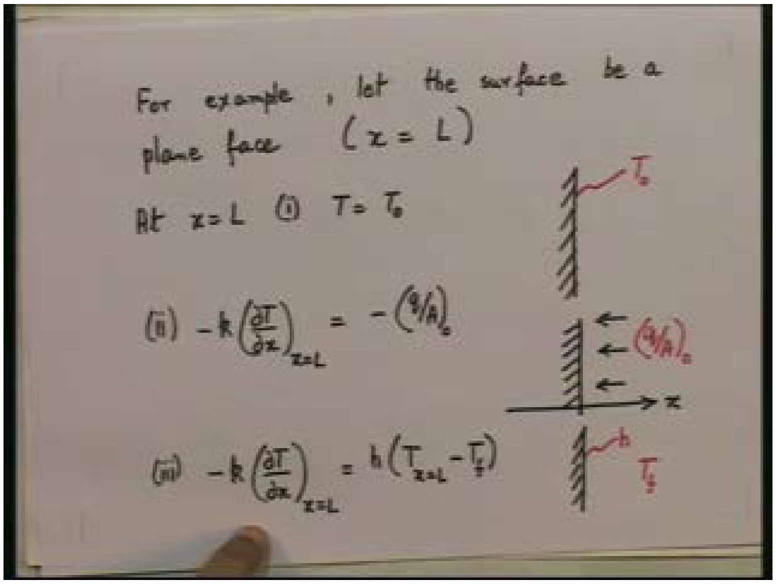

For example, suppose for example, let us start; write down expressions for these. For example, let the surface be a plain face x equal to L; let us say the surface under condition is a plane face x equal to L some surface. Now at x equal to L, we say, at x equal to L; let me draw a surface. This is the solid and we specify that at x equal to L, some prescribed temperature exists; let us say temperature is specified to some value $T_{0.}$ - So we will say at x equal to L, the first condition would be, the first type of condition would be T equal to T_0.

The second type of condition is of a prescribed flux falling on the surface of the body; so the second type of condition is I have got a surface like this and there is a prescribed flux falling on the surface of the body. Let us say that flux is q by A_0; some flux falling on the surface of the body. Let us, it is all being absorbed on the surface of this body; now if all the flux that is falling is absorbed in the surface of the body, then by conduction that flow, that flux must be flowing by conduction at the surface into the body. So, we equate this flux to the heat flux we will get from Fourier's law of heat conduction at x equal to L and we can; the second type of condition can be stated as, the prescribed heat flux can be stated as minus k dT dx at x equal to L is equal to minus q by A_0; this is how the second type of condition would be specified and mind you, I am calling this as the positive x direction; that is why I have put a minus sign on the q by A here; minus k dT dx x equal to 0; x equal to L is the heat flux by conduction in the positive x direction. This has to be equal to the q by A falling, the q by A showing falling in the other direction; therefore it is equal to minus q by A_0 so this is the second type of boundary condition. This is the first type T equal to T_0; prescribed surface temperature, this is a prescribed heat flux.

And the third is a prescribed heat transfer coefficient in which case I have again a surface like this and I specify that there is a heat transfer coefficient h at the surface and there is a fluid temperature T_f here. Now, I am going to use Newton's law of cooling and equate it to the heat flowing by conduction in the solid. I have done this earlier when I defined Newton's law of cooling you will recall. So, the third type of boundary condition would be minus k dT dx at x equal to L; this is from Fourier's law of heat conduction and this heat conducted must be equal to h T at x equal to L minus T_f - Newton's law of cooling, heat flux equated to the heat flux by Fourier's law of heat conduction at the interface x equal to L.

So, these are typically three types of boundary conditions with which we deal - a prescribed surface temperature, a prescribed heat flux which is equated to the heat flux by conduction from Fourier's law and a prescribed value of heat transfer coefficient of the surface which is again equated to the heat flux by conduction from Fourier's law at x equal to L; so this is how we put down condition. So the complete formulation of the problem, any problem in heat conduction means: put down appropriate differential equation, put down appropriate boundary conditions and state an initial condition if it is an unsteady state problem; otherwise of course, there would be no initial condition to be specified; that is the complete specification of a problem in heat conduction.

Now, we have come to end of this topic, let me introduce you to the next topic which we are going to begin. Now that we have got a general differential equation, we are in a position now to solve problems which are not just one dimensional variations in space but could be varying with time and with heat generation and so on. So we are going to next take up some problems of heat generation; we are going to take up some problems of heat generation and I just want you to introduce you to the idea that heat generation, where heat generation problems are important in solids. For instance, I may have a nuclear fuel element; heat is being generated in it because of nuclear fission. I would like to know the temperature distribution in that nuclear fuel element.

When concrete sets, heat is evolved when concrete sets so I may be interested in the temperature distribution in a concrete slab which is setting and during the setting process heat is being generated or when I have an electrical conductor carrying current we know that heat is being generated at the rate of I squared r in the conductor; so that is also a situation with heat generation. I may be interested in knowing what is the temperature distribution in the conductor. So, there are many situations involving chemical reactions, heat generation due to electrical currents or concrete hydration or nuclear fission where heat is generated inside a solid and I would like to know what is the temperature distribution in the solid. We are going to look at some problems which are one dimensional and steady state in nature because here we cannot take up more complicated problems. So, we are going to look at some one dimensional steady state problems with heat generation next.