Advanced Strength of Materials
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Lecture – 9

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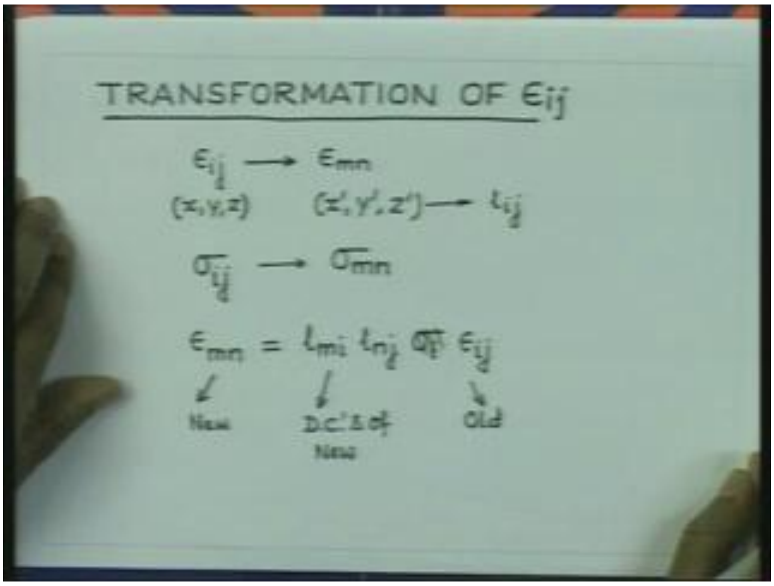

Today, we will first consider transformation of strain Epsilon i j from one system of coordinates to another system of coordinates. Let us consider that Epsilon i j is defined in the system x, y, z. And you want to define this in a new system. Let us say an Epsilon m n, which is now given by the set x z and y z and z x dash y dash and z dash. Since, Epsilon i j is a second rank tension, we can make use of the transformation rules for second rank tension.

You have already been acquainted with the transformation of stresses sigma i j given in a system x, y, z to sigma m n which is to be defined in a system x dash y dash and z dash. So, we directly adopts the transformation rules, let us just consider that the direction cosines of the three coordinates are given by Epsilon i j with respect to the old coordinates.

Then, you can write that Epsilon m n, the new system is nothing but l m i l n j sigma Epsilon i j. So, this is defined in the new coordinates. And this is given in the old coordinates and these are the direction cosines of the new coordinates. So, these are the direction cosines of new coordinates. So, it is a ((Refer Time: 03:26)) which is very similar to that of transformation of sigma i j.

This is the advantage, that it will work in tension irrespective of the field of the application. So, as long as the rank of the tension is the same, you can make use of the same rules for transformation. Now, we would like to consider definition strains in polar coordinates, you have considered the definition of strains in rectangular coordinates.

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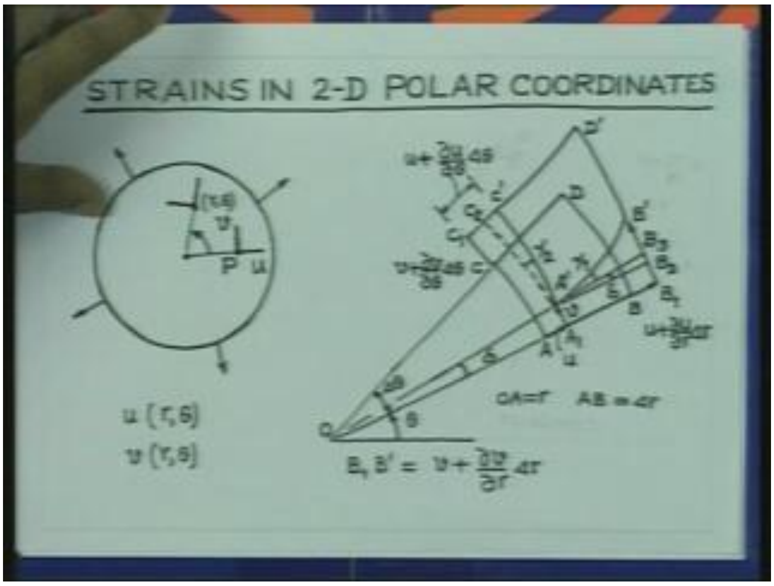

You will find that when dealing with geometries, which are have been circular consideration there in you will find advantage working in strains defined in polar coordinates. So, when a circular disc is loaded by external loading, which is undergoing deformation. And you will find that end point on the disc is going to have some displacement in the radial direction and some displacement in the tangential direction.

So, we will have to components of displacement in two dimensions, one is u which is the radial direction and v is the displacement in the tangential directions. And you must appreciate, that this u is going to be now a function of radial position and also the angular disposition of the points. So, if you consider this as the reference with respect to this reference, you can define a point by the angle theta.

So, this is going to be therefore, the point is going to be defined with respect to coordinate r theta. So therefore, u is going to be a function of r theta. Similarly, v is also going to be a function of r and theta. Now, to find out the strains let us consider the summation of a element of the body upon loading. Let us consider that we have one element, which is shown here.

Let us consider an element, which is defined by this radial dimension delta and the angular with position delta theta. So, let us consider that this particular element is spanning at an angle of delta theta. So, this is the angle delta theta, and let us take this as the reference direction. And this angle is equal to, let us say theta and this is the centre of the coordinate.

Now, let us name the element as A, B, C and D. Now, upon loading this particular element is going to take up displaced position. Let us draw the displaced position to a larger scale. So, this is the displaced position, this is the position of A. Let us see that, this is new position of A and A dash. Similarly, new position of B and B dash, new position of D and D dash and the new position of C to C dash.

Now, the displacement that A has undergone in the radial direction is this much. Let us say that, this is the displacement u and the tangential displacement, which is exactly in the orthogonal direction, this is equal to v. The point B has undergone a displacement in the radial direction of this magnitude. And it is indicated by B, B 1 here and it has undergone a tangential displacement arc of this magnitude, which is B 1 to B dash.

Now, if I consider that, this u and v are continuous functions of r and theta. So, u and v are continuous functions of r and theta, over the domain of the body. We can consider this distance A, B to be given by it is a small distance. It is given by delta r and let us also consider that the distance of point O, point A from the centre, O is equal to r.

Now, if you consider that, this point B is located at a small distance from A. You can find out the displacement in the radial direction of B by ((Refer Time: 10:36)) Expansion. So, you can write displacement of B as u plus delta u, delta r multiplied by delta. So, therefore the displacement of the radial direction, similarly the displacement B 1, B dash, which is in the theta direction. If you consider this to be theta direction, that can also be now, since this is v, you can now consider change of the B with distance A, B. And which is going to be, therefore v plus delta v delta r multiplied by delta r. You can now consider the point C, it has some movement in the tangential direction. So, let us say, this C C 1 and you can write this distance C C 1 as v plus rate of change of v with respect to theta.

So therefore, delta v, delta theta and we have moved the distance of delta theta. So therefore delta v delta theta into delta theta that will give us this distance. Similarly, it has moved by a distance in the radial direction, which is this one and C 1, C dash. You can again, write in terms of the displacement in the radial direction of the point A. And therefore, this distance is going to be u plus delta u, delta theta multiplied by delta theta.

So, this is what is the displacement in the radial direction of the point C? Now, let us do some construction, let us represent this point as A 1. And then, we try to draw an arc here, passing through the point A dash with centre O. We draw an arc passing through the point A dash, let us consider that intersection of C 1, C dash, here is C 2. And similarly, we try to draw a line joining O to A dash; that makes an angle of let us say, delta.

And now, we extend this line here, let us say the intersection of this radial line with B 1 B dash is B 3. We draw a line parallel to A 1, B 1, which is this 1. Let us say that, that let us say B 1 B dash at point B 2. So, therefore, this angle is since this is parallel to B. Therefore, this angle is equal to this angle and therefore, that is equal to delta. You also indicate the angel approximate angle between this A dash, B dash and A dash, B 3 is equal to gamma 1.

And similarly the angle made by the direction A dash C dash with this arc here is equal to gamma 2 with this we have ready to define the strains. So, if you now consider the strain in the radial direction, which is represented as Epsilon r. That is equal to limit delta r tending to 0 length of this element A B tending to 0, change in the length of A B divided by length of A B. Now, I would like you to know that point A has moved by u and point B has moved by u plus delta u delta r delta r. So therefore, the ((Refer Time: 16:14)) has been this distance minus this. So therefore, it is delta u delta r.

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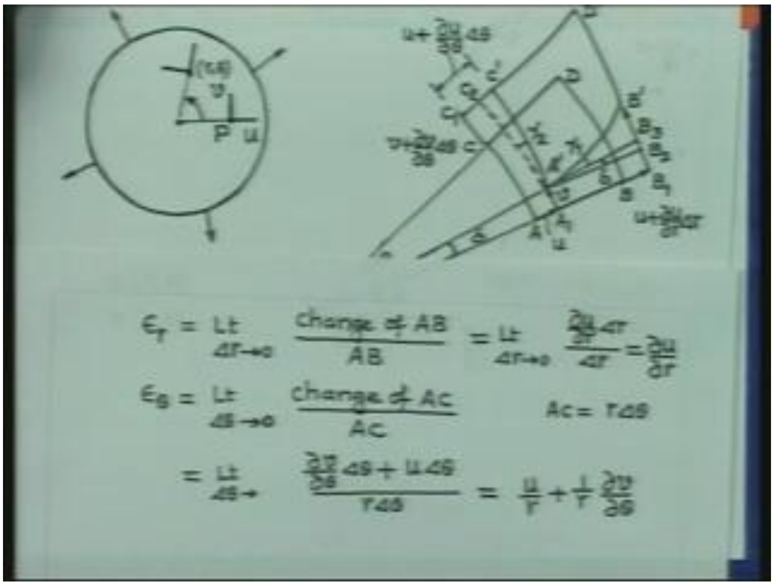

Therefore, this change in length of A B is nothing but this distance minus this u. So therefore, that will be delta u delta ((Refer Time: 16:32)). So, we can now right limit delta r tending to 0 delta u delta r delta r. And length of A B is nothing, but delta r. So therefore, this strain in the radial direction is given by delta u delta r, which is the partial derivative of radial displacement with respect to r.

Similarly, if we consider Epsilon theta, Epsilon theta is defined as the change in the length of A C divided by the original length of A C, as the angle delta theta tends to 0. So therefore, this is delta theta tending to 0 change of A C divided by length of A C. Now, if we just consider, that A C is not moving in the radial direction at all. It has remain it has remained in the same radial disposition. And it has moved in the tangential direction only.

Then you see that, point A has moved in the tangential direction by v. And the point C has moved by v plus delta v delta theta del theta. And therefore, the ((Refer Time: 18:16)) movement has been this total minus v. Therefore, that is delta v delta theta delta theta. So, that is one part which comes up from the tangential displacement of A C in the tangential direction only.

Now, if you consider, A C does not move in the tangential direction. It simply moves in the radial direction. So, it has moved from position A C to the position let us say, this to this. So, therefore, it has moved from this position move to this position. The length A C was initially, A C was that is nothing but r into delta theta. Now, if it moves to this position A 1 to this, then in that case its length is going to change by you plus r delta theta.

So, therefore, there will be a change if magnitude equal to r passive delta theta minus are delta theta. So, therefore, that will be u delta theta. So therefore, to define that the ((Refer Time: 19:39)) in this case is going to be like this. Change in length of A C will be first of all of magnitude delta v, delta theta, delta theta plus the ((Refer Time: 20:04)). Change in position in a radial direction is u delta theta. And the original length is nothing but r delta theta. So, we find now the strain is given by u by r plus 1 by r delta v delta theta. So therefore, the circumferential strain or the tangential strain is u by r plus 1 by r delta v delta theta. Now, we have to consider the shear strain.

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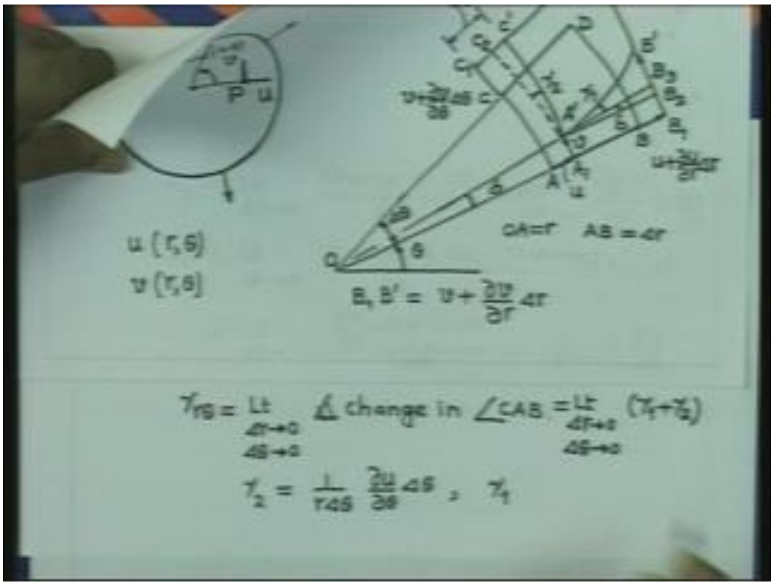

Let us consider now shear stain, which will be indicated as gamma r theta. And this is given by limit delta r tends to 0, delta theta tends to 0. Change in the angle between that two directions, orthogonal directions A C and A B. So therefore, that is nothing but change in the angle CAB. Now, C has taken a position C dash, B has taken a position B dash and A has taken a position A dash.

So therefore, the angle change is given by we are the local orthogonal direction. Please note, that the local orthogonal direction here. This is the radial direction passing through A dash and this is the tangential direction passing through A dash. So therefore, the change in angle is nothing but this angle gamma 1 plus gamma 2. So therefore, this is nothing but limit delta r tending to 0, delta theta tending to 0 gamma 1 plus gamma 2.

Now, I would like you to see this, that this angle gamma 2 is nothing but approximately this distance divided by A C approximately, because the approximate A dash C 2 by A C. Therefore, this distance is nothing but… Since this distance is equal to u 1. Therefore, this distance is nothing but u plus delta u delta theta multiplied by delta theta minus u.

So therefore, this distance is nothing but this magnitude. So therefore, this is given by only this quantity. And this distance A C is r theta. So, therefore, this gamma 2, you can write gamma 2 as 1 by r delta theta multiplied by delta u, delta theta del theta. And let us now see, what is gamma 1.

So, gamma 1 is nothing but if you consider this total angle, total angle gamma 1 plus delta. That is nothing but this B 1 B 2 minus B 1, this angle gamma plus delta is given by this distance B 2 B dash divided by this distance. Now, if you look into that this B 1 B 2 is nothing but v.

And this total distance we have got this total distance as v plus delta v delta r del r. So therefore, this total distance is nothing but given by this quantity. So, if you subtract B. Therefore, this distance here is going to be given by this magnitude. So therefore, delta v delta r delta r is this distance and this A 1, B 2 can be approximated by A B which is delta r. So therefore, this angle gamma plus gamma 1 plus delta is nothing but it is approximately equal to B 2 B dash divided by A B.

And this is equal to since B 2 B dash is equal to delta v delta r delta r, you can approximate this by delta r. So therefore, it is going to be delta v delta r. Now, angle delta is equal to this angle, if you consider that this distance O A is equal to r. Then, this angle is approximately equal to this distance divided by O A, which is going to be v by r. So therefore, delta equal to v by r. So, we have now delta is equal to v by r. So finally, we have gamma 1 is equal to delta v delta r minus v by r. So, you can now substitute the value in the expression of gamma r theta.

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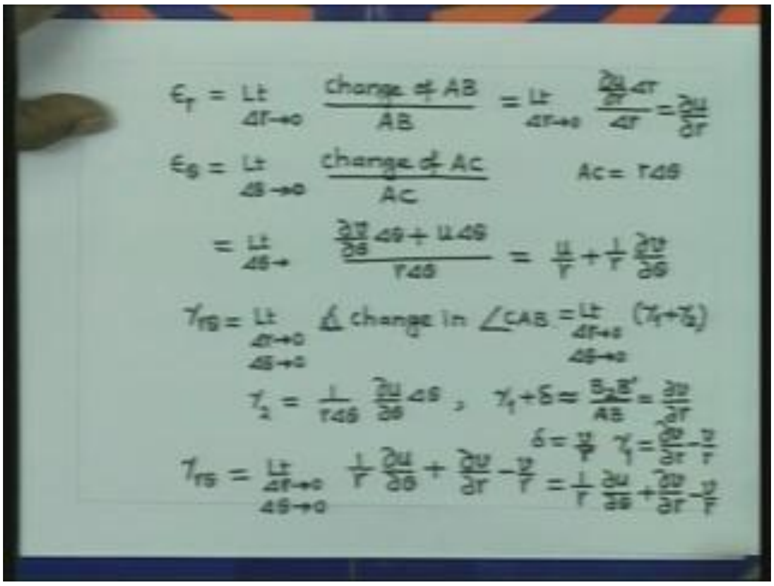

So therefore, now we have gamma r theta is equal to it is limit delta r tending to 0 delta theta tending to 0. Gamma 2 is nothing but 1 by r delta u delta theta and gamma 1 is nothing but delta v delta r minus v by r. So, that is nothing but 1 by r delta u delta theta plus delta v delta r minus v by F. So therefore, we find that the three components of strain in polar coordinates are here Epsilon r is delta u delta r. Epsilon theta is u by r plus 1 by r delta v delta theta. And gamma r theta is nothing but 1 by r delta you delta theta plus delta v delta r minus v by F. Let us now consider equilibrium equation in two dimensional rectangular coordinates.

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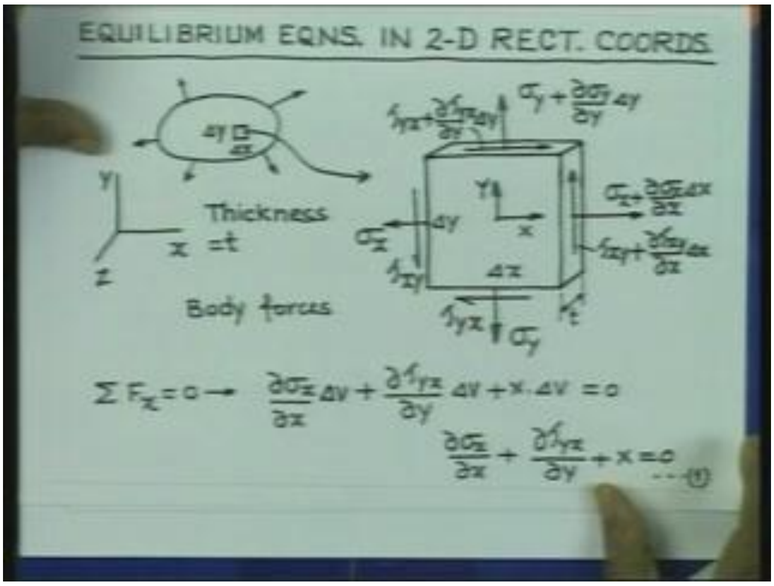

If you just consider a two dimensional object, like this subjected to loading at it is boundary, it can also have load at internal points. Then, under the action of this loading stresses are going to develop at each and every point of the body. And we will have some constant or some relation satisfied by the stresses, at each and every point of the body. We would like to find them and they are nothing but the equilibrium equations.

We just concentrate on an element, which is of very small dimension. Let us say, it is of dimension delta x by delta y. And let us consider that the thickness of this body is uniform and the thickness in the z direction, thickness let us say t that is thickness in the z direction. Now, if I try to draw this element here large scale, let us show it like this. So therefore, this is delta y, this is delta x.

Now, the stresses that are going to come up on this face. Let us say, that the stress in the this is nothing but minus x face. So therefore, this is minus face in the x direction sigma x. And the that is the normal stress and the stress, which is acting parallel to the face is nothing but tau x y. Similarly, let us say that the stress acting on the face is nothing but sigma y and the shear stress on this face is equal to tau y.

These stresses and strains are continuous function if the body is continuous. So, if you want to represent the face from the other face, which is parallel to this. But, which is at a distance of delta x. We can make use of the Taylor series expansion. And we can write the stress on this face in the normal direction to be as sigma x plus delta sigma x delta x delta x.

So, therefore, that is the normal stress. Similarly, the shear stress acting on this face is going to be tau x y plus delta tau x y delta x multiplied by delta x. Again it is the Taylor series expansion of this tau x y and we have got this stress. If we consider the normal stress on this face, that is obtainable again from the normal stress here. And it is changing the y direction.

So therefore, we can write now this is sigma y plus delta sigma y delta y into delta y. And the shear stress, which is acting here you can indicate that in terms of the tau y x and this is tau y x plus delta tow y x delta y multiplied by delta y. So, this is the shear stress.

Now, the body can be also subjected to forces at internal points, one of the type of force that is commonly placed is the body force, we indicate that as body force. Examples, like gravity forces, centrifugal forces. So, if the stresses are to be calculated under the action of the body forces. Then, we must try to bring in those types of forces at each and every point of the body.

Or if you are interested in calculating the stresses under the rotation under the action of rotation of the body, then we must bring in the inertia forces. These are all treated as body forces. So, it with depends on the volume. So therefore, we call these as body forces. So, if we consider that the body force acting at this point per unit volume in the x direction by capital X. And the body force part ((Refer Time: 34:15)) volume acting in the y direction is capital Y.

So, this is the element and the forces that are the acting on the element on it is boundary. Since, the body is in equilibrium under the action of the external forces and also the body forces. Each and every point of the body must also be in equilibrium. And therefore, this element which is subjected to forces at it is boundary like this. And at the same times it is subjected to forces X and Y.

We must have this particular element in equilibrium under the action of the volume forces. So, by the consideration of such equilibrium, whatever equations we get are nothing but equilibrium equations. Let us now consider the equilibrium in the x direction. So, some of the forces in the x direction must be equal to 0. What are the stresses that continue to the forces in the x direction, you have this component of stress, this component of stress. Then, these two shear components, this one and this one there are going to contribute to the forces in the x direction. On top of that, you are also going to have this force contributing to the force in the x direction.

Now, the force arriving out of this stress acting on the space whose area is nothing but delta y multiplied by t. So therefore, the force here is nothing but sigma x plus delta sigma x delta x delta x multiplied by sigma y into delta t. And the force which is arriving out of this stress sigma x is nothing but sigma x into delta y into t.

So therefore, the net force which is coming out from these two components, in the x direction is going to be delta sigma x delta x delta x multiplied by delta y into delta t ((Refer Time: 36:39)). Since, delta x into delta y into t is volume of the elements. So, we can write the net force in the x direction from the stresses in the normal stresses in the x direction are nothing but delta sigma x delta x into delta v, where in delta v is nothing but delta x into delta y into t.

Now, the other component which is going to come from this stress acting on this area, whose magnitude is delta x into t. And similarly, this stress y x is also acting on the similar area. So therefore, the net force is nothing but this incremental stress multiplied by the area delta x into t. So therefore, the net force is going to be again delta tau y x delta y multiplied by delta y into delta x into t.

And if you represent delta x delta y into t as delta v, then we will have it as delta tau y x delta y multiplied by delta v, then this force which is also contributing to the stress to force in the x direction, which is nothing but x into delta y. So therefore, you have x into delta v that is the force in the x direction. So therefore, sum of all these things must be equal to 0. So, delta v is a common quantities finally, we can write the equation as delta sigma x delta x plus delta tau y x delta y plus x equal to 0. So, this is the equilibrium equation in the x direction. Now, we would like to also consider the equilibrium in the y direction.

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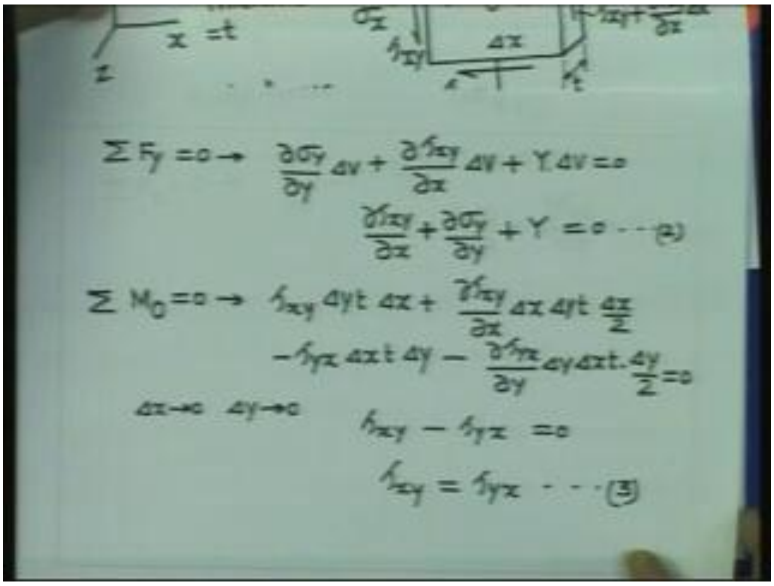

So, if you now consider in a similar manner summation of the forces in the y direction equal to 0. That will give us like this, you are the force coming out in this y direction is due to this normal stress, this normal stress. And due to this shear stress and this shear stress plus component due to the body force y.

Now, the differential force acting in the y direction due to normal stress is going to be this magnitude multiplied by the area, which is delta x into t. And therefore, again you can write delta x delta y into t as delta v. So therefore, differential force is nothing but this delta sigma y delta y multiplied by delta v. So, delta sigma y delta y multiplied by v, and from the shear stress tau x y plus delta tau x y delta x delta x. And this tau x y acting here, the area of this face is nothing but delta y into t. So therefore, the differential force in going to be simply due to this multiplied by the area is delta y t. The component would be delta tau x y delta x into delta v.

So therefore, again this is going to be delta tau x y delta x multiplied by delta v plus the component due to Y is nothing but Y into delta v. So therefore, this is Y into delta v is equal to 0. So, you can now take the common factor out and therefore, second equilibrium equation is delta tau x y delta x plus delta sigma y delta y plus Y equal to 0.

Now, we will consider let us say that the centre point here of the element is O. Now, this element must also be movement equilibrium, and therefore if you write movement of all these forces about the centre point O equal to 0. Let us what are the forces that are going to cause movement. Since, this is stress is going to be uniformly acting over this area.

Let me put it this way. That, we are considering this distance delta Y is so very small that this stress intensity can be taken to be uniform over this area. Similarly, for all other stresses you can them to be uniform over the stresses over they are acting. Therefore, the resultant of the force due to this stress will pass through this point. So, also due to this will pass through this point.

And by similar consideration, resultant force due to this stress and this stress will pass through this point. So therefore, the movement and again X and Y are going to pass though this point. So therefore, this movement caused by these two shear stresses in anticlockwise direction, these two shear stresses in the clock wise direction.

Now, if you try to consider the movement due to these two shear stresses, in the anti clockwise direction. It is nothing but we will have this delta tau x y working on an area of magnitude delta Y into t. And they are separated at a distance of delta X. So therefore, our movement magnitude is going to be nothing but that is tau x y acting on an area of delta y into t and these two stresses are at a distance of delta X. So, that is the movement due to this and this component.

Now, if I consider this it is acting on an area of delta y t. And it is acting at a distance of delta X. So therefore, we have a component like delta tau x y delta x into delta x. And it is acting over an area of delta y into t and the momentum is delta x by 2. So therefore, it is delta x by 2, that is anticlockwise movement. Now, if we go for similar reasoning for this component tau y x and tau y x plus delta tau y x delta y into delta y. This tau y x will give us movement which is nothing but tau y x multiplied by delta x into t multiplied by delta y. So therefore, that movement is going to be in clockwise direction.

So, you can write now you can write that tau y x it is acting over an area of delta x into t. And the movement turn is delta y minus the movement due to this is going to be delta tau y x delta y into delta y multiplied by delta x into t and the movement turn is delta y by 2. So therefore, we have delta tau y x delta y multiplied by delta y and the area is delta x into t and the momentum is delta y by 2.

So, these are the two constituents. So, this is in the anticlockwise direction, this movement is in the clockwise direction. So therefore, they must add up to 0. Now, we are trying to talk about a distance delta x delta y very small. So, since delta x is very small and so also delta y. Therefore, we find that these terms are going, if we just remove the delta x delta y into t term, this term is going to be left with delta x by 2 and this term is going to be left with delta y by 2. And you can neglect them. So finally, what we find is that you will have tau x y minus tau y x is equal to 0. Therefore, tau x y is equal to tau y x. So, this is the... So, let us name this as the 2nd equilibrium equation and this is the 3rd equilibrium equation.

So therefore, we have three equilibrium equations in two dimension. These are the three equilibrium equation in two dimensional rectangular polar coordinates. This one is in the equilibrium in the X direction, this is in the Y direction and this is the momentum equilibrium equation ((Refer Time: 47:16)).

Now, one important thing you must appreciate that the stresses at a point are such that the tau x y space is acting on the face, which is perpendicular to the X direction. Tau y x is the stress acting on the face perpendicular to the Y direction. These two stresses are acting on to orthogonal planes. And they are always going to be equal.

That is why some times you call, this tau y x to be complimentary to tau x y. Whenever, this is present this stress is also present. And obviously, this stress is always present along with this. So, whenever this stress is present we are also going to have this stress here. So, these complimentary stresses are equal. So, without further discussion we can always assume that tau x y to be equal to tau y x and that ensures moment equilibrium. Now, these three equilibrium equations if we make use of the tensor notation, it becomes again compact.

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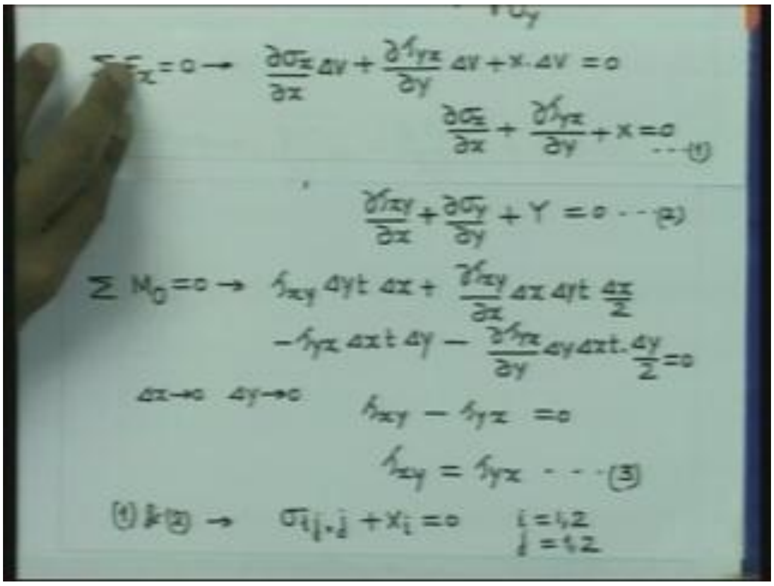

So, particularly this equation 1 and 2 we can write like this sigma i j comma j plus x i equal to 0 where takes up value 1 to 2. So, also j takes a value 1 to 2. So, when you have i equal 1 and this comma indicates derivative of this component with respect to j. So therefore, you could give us two component and this since there repeated there is summation involved.

And therefore, it gives you the equation onto. So, this is also sometimes called equilibrium equations in two dimensions. Obviously, we can have the extension some in three dimension very easily. Then, in that case we are going to have the same form. But, we are going to have values of I varying from 1, 2, 3. So, j taking up values 1 2 3.

So, I leave it as an exercise for you to determine the equilibrium equations in three dimension. So, to summarize what we have done in this lecture is that, we have considered first of all the transformation of strains. Strain is a tensor like is stress tensor. And whatever rules apply to transformation of stress can be used for the transformation of strains. So, we have written the expression for strain transformation from one system to another system.

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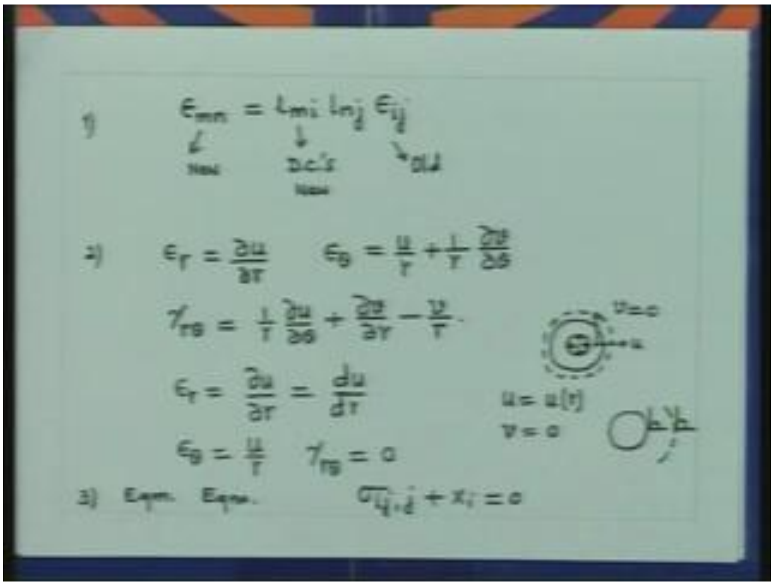

So, in the new system m n Epsilon m n is nothing but Epsilon m i Epsilon n j multiplied by Epsilon i j, where this is new and this is old set. And these are the direction cosines of the new set. Then, we have tried to consider definition of strains in polar coordinates. And we have found that Epsilon r is equal to partial derivative of the radial displacement u with respect to r.

That is the strain in the radial direction. And the strain in the circumferential direction it was nothing but u by r plus 1 by r delta v delta theta. And similarly, we had also derived gamma r theta which is nothing but 1 by r delta u delta theta plus delta v delta r minus v by r. Now, I would like you to note one situation, that there is a possibility that we may have displacements taking place only in the radial direction.

Think of the case, that you have a hollow cylinder and it is pressurized by internal pressure, under the action of this internal pressure. The points are going to move only in the radial direction. And the geometry of the body will not get distorted. And it is simply going to expand. And therefore, each point is going to just have a displacement in the radial direction, you will not find any displacement in the tangential direction here.

So, v is equal to 0 and this u is going to be only dependent on the radial distance of the point. And therefore, this u is a function of r alone. So, in this particular case u is a function of r alone and v is 0. And therefore, in this case what are going to be the strains, you will find that here Epsilon r is just going to be delta u delta r, and since u is a function of r alone. So, you can write this thing as d u d r total derivative. Rather than, partial derivative and Epsilon theta is going to be nothing but since v is not present. So, this part is 0, and the tangential strain is u by r and shear strain gamma r theta here in u is not a function of theta. So, therefore, this is going to be 0 v is not present. So, therefore, this is 0 and v is 0, so therefore, everything is 0, so therefore, there is no shear strain.

So, what we find is that in this case there is no shear deformation of the body. And this is also clear from the changed configuration of the body. In this case it is simply going to maintain the radial symmetry or symmetry about the axis. So therefore, it is not going to have any change of angle between the radial and tangential directions.

What I mean is that, if you have a circle initially like this. The angle between the radial direction and tangential direction is 90 degree. And once, it gets displaced to a position like this. The angle between the radial direction and tangential direction is again 90 degrees. So therefore, there is no shear deformation here.

So therefore, in this case; obviously, the angle 90 degree is maintained. And therefore, the shear strain is 0 there and this is a typical case, where in you have only radial displacements. Then we consider of course, the equilibrium equations in 2D rectangular coordinates. And those equilibrium equations, we have written in the form x i j comma j plus x i equal to 0.