Advanced Strength of Materials
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Lecture - 5

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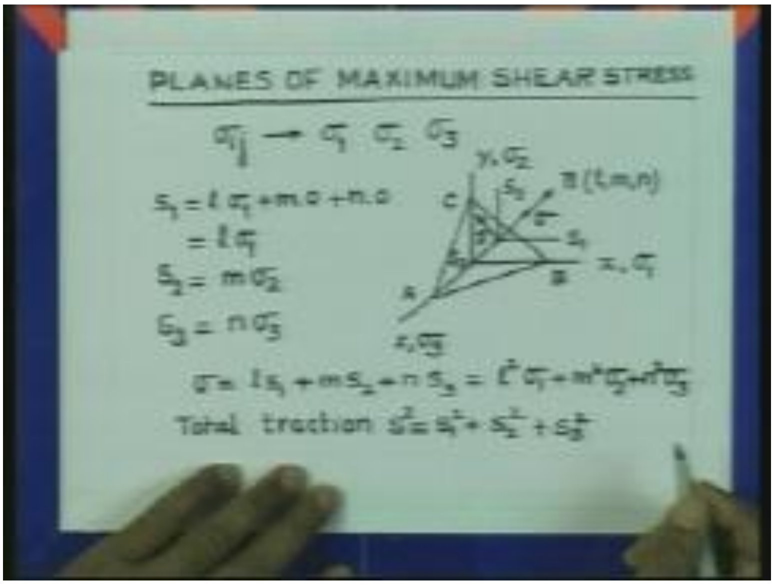

We would like to consider, determination of planes of maximum shear stresses at a point. Let us consider that, the stresses at a point are given by the tensor sigma i j. And from this, you can determine the principal stresses, sigma 1, sigma 2, sigma 3 and their directions. Now, we would like to consider reference as the three principal directions, these are the principal directions. Now, it is coinciding with sigma 1 and this y, say it is coinciding with sigma 2 and this z coinciding with sigma 3.

We consider one plane with reference to these directions, whose outer normal from the origin. Let us say, it is oriented like this, its direction cosine let us say, l, m and n. We would like to now find out a plane A, B, C. Such that, this plane will have the maximum shear stress. So, the shear stress, which is going to act on this plane. Let us say, parallel to this plane is equal to tau. And the stress, which is going to be acting normal on to this plane, is sigma. So, these are the two stresses.

And we would like to see, what would be the orientation of the plane A, B, C. Such that, this shear stress tau is going to be maximum. Now, on this plane, we can find out the tractions in the three reference direction. So, we will have tractions in this direction S 1. So, this is S 1 and similarly in the direction, we will have S 2 and in the direction of z, which is going to be this direction. So, therefore, this is the direction, which is going to be, let us say S 3.

So, we can write now S 1 to be equal to, using the Cauchy formula l into sigma 1, m into 0 plus n into 0, because tau y x and tau z x are 0. So, therefore, these tractions on this planes are going to be, simply S 1 is l sigma 1 and by the similar type of considerations. We will find that, this traction is nothing but m sigma 2 and the third component is nothing but n sigma 3.

Now, if we try to consider the normal stress. Since, these are the tractions on this plane. The forces acting on this plane would be nothing but the area of this plane multiplied by the traction. So, there will be three traction acting in the three orthogonal directions and if we calculate the resultant or if you take the component of those traction in the normal direction. That should give us the sigma stress.

So, we can now write, that sigma stress is now equal to component of this. This is acting in one direction. So, therefore, cosine of the angle is l. So, therefore, l into S 1 plus m into S 2 plus n into S 3, so therefore, this gives us l square sigma 1 plus m square sigma 2 plus n square sigma 3. So, this is nothing but the normal stress. Now, if you calculate the resultant of the three tractions. So, total traction s can be obtained by vectorially adding the three tractions. So, therefore, it should be equal to S 1 square plus S 2 square.

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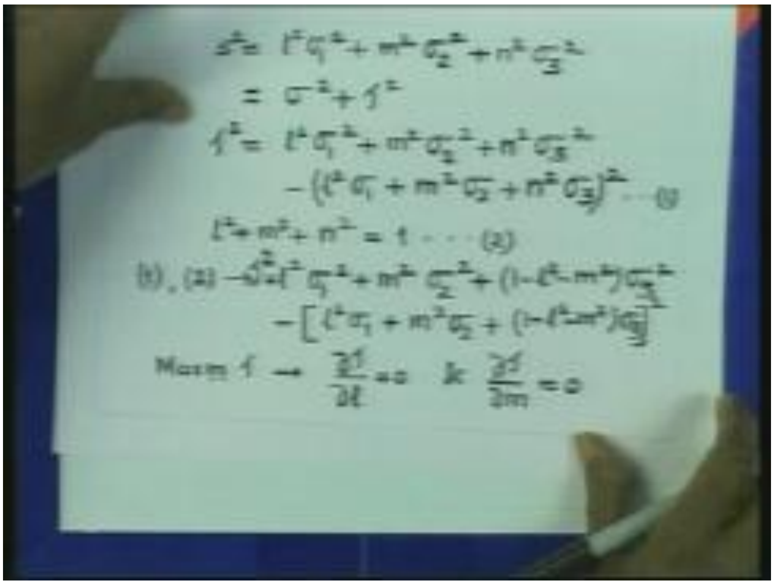

So, we can write s square equal to S 1 square plus S 2 square plus S 3 square. And therefore, we have s square is nothing but l square sigma 1 square plus m square sigma 2 square plus n square sigma 3 square. Now, if the stresses acting on these planes are sigma n tau. These are the two stresses only acting on this plane. We can also find out the traction from the consideration that, it is nothing but vector sum of the two stresses sigma square plus tau square. Note that, this is possible to add vectorially, because of the fact, that we are considering the same plane. And on this plane, the tractions are acting in the three directions of magnitude S 1, S 2, and S 3 and at the same time, we have alternatively the stresses sigma and tau.

So, we can now, get the value of the shear stress, which is going to be given by l square sigma 1 square plus m square sigma 2 square plus n square sigma 3 square. Minus, we have the value of sigma, which is l square sigma 1 plus m square sigma 2 plus n square sigma 3 and it is whole square. Now, we also have the relationship, that sum of the three direction cosines, square of the sum of the three direction cosines l square m square plus n square equal to 1.

So, if we now, eliminate n square from these, we have. So, we would like to consider that, this is relation number 1 and this is relation number 2. So, using 1 and 2, we would like to have this l square sigma 1 square, m square sigma 2 square plus 1 minus l square minus m square into sigma 3 square minus l square sigma 1 plus m square sigma 2 plus 1 minus l square minus m square into sigma 3 whole square. So, this is nothing but this is tau square.

Now, if we have to get the maximum value of tau. We can differentiate this relation and see that, delta tau delta l and delta tau delta m is equal to 0. So, if tau is maximum. So, for maximum tau, we must have, it has two variables related to it, l and m. We can think of considering delta tau delta l 0 and delta tau delta m equal to 0.

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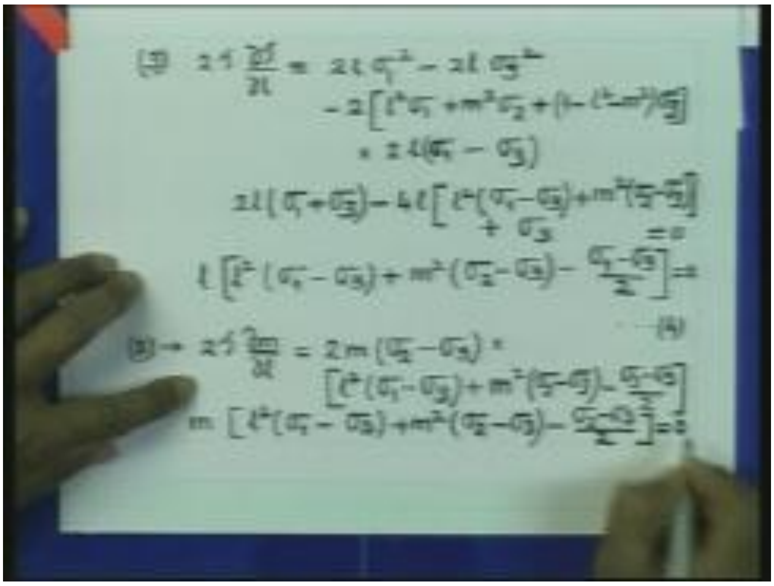

So, if we take the first, let us say that, this relationship is 3. So, from 3, will have differentiation of both sides will be thus, 2 tau, delta tau, delta l is equal to 2 l sigma 1 square minus 2 l sigma 3 square minus 2 times l square sigma 1 m square sigma 2 plus 1 minus l square minus m square sigma 3. Multiplied by differentiation of these quantity will give us, 2 times l sigma 1 minus 2 times l sigma 3. So, therefore, we will have 2 times l sigma 1 minus sigma 3.

So, now, if we take this, this is again sigma 1 square minus sigma 3 square into 2 l. So, there is sigma 1 minus sigma 3 is common factor. And if we make use of the condition delta tau delta l equal to 0, this will give us, will have 2 l sigma 1 plus sigma 3, taking sigma 1 minus sigma 3 out, minus 4 times l. This is now, l square sigma 1 minus sigma 3 plus m square sigma 2 minus sigma 3 equal to 0.

So, this is going to be m square sigma 2 minus sigma 3 plus will also have a term here, which will be left is nothing but it is sigma 3. So, the terms in the bracket will be l square sigma 1 minus sigma 3 m square sigma 2 minus sigma 3 plus sigma 3 will be there.

Now, if you try to take l common. And then you will re arrange this equation will become like this, l into sigma 1 minus sigma 3 m square sigma 2 minus sigma 3 minus sigma 1 minus sigma 3 by 2 equal to 0. So, this is the equation, which we obtained by using delta tau delta l equal to 0. So, let us say that this is equation number 4. Similarly, if we try to differentiate this relationship, if we differentiate this relationship with respect to n from both sides, we are going to get.

So, again from 3, we are going to get, twice tau delta m delta l is equal to 2 times m sigma 2 minus sigma 3 into l square sigma 1 minus sigma 3 plus m square sigma 2 minus sigma 3 minus sigma 2 minus sigma 3 by 2. So, this is the expression. Again, if you make use of delta m delta l equal to 0, this will give the relation m into l square sigma 1 minus sigma 3 plus m square sigma 2 minus sigma 3 minus sigma 2 minus sigma 3 by 2 equal to 0.

So, this relationship, if you indicate by 4, I am sorry, let us indicates it, by 5. So, this equation 4 and equation 5, we have two equations to solve for l and m. And you can make use of the relationship, that l square plus m square plus n square equal to 1 to get n. By solving these two equations, if you can see that, immediately the solutions are l equal to 0, m equal to 0.

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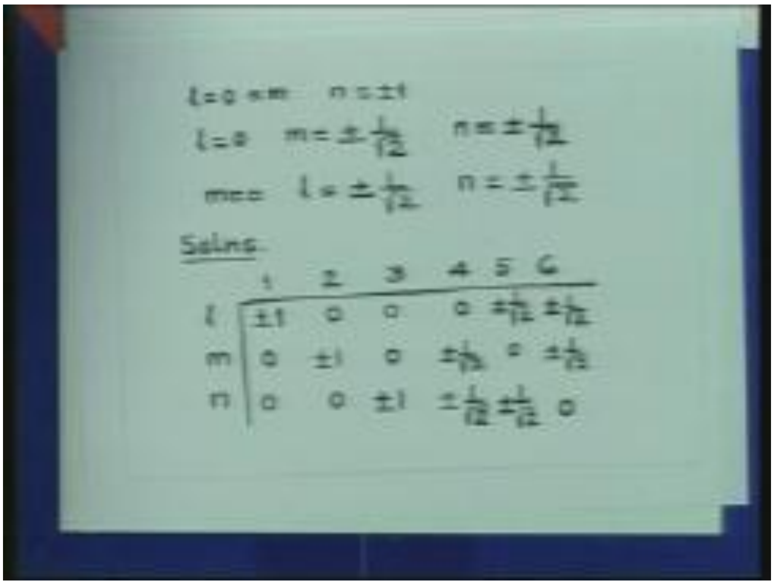

So, if we consider the first group to be l equal to 0 and m equal to 0. Then, n is going to be equal to plus minus 1. Similarly, if you consider l to be 0, then m comes out to be plus minus root off. So, if you just consider l equal to 0, then from equation 5, you get m equal to plus minus 1 by root 2. So, if you put l equal to 0 here, then from this equation m square into sigma 2 minus sigma 3 minus sigma 2 minus sigma 3 equal to 0. Therefore, this will be cancel and it will be giving this.

And again, making use of the fact, that l square plus m square plus n square equal to 1, you get n equal to plus minus 1 by root 2. Similarly, if you consider in this equation, m equal to 0. So, if we consider m equal to 0 and if we substitute m equal to 0 there. Then, in this relationship, we are going to get l square into sigma 1 minus sigma 3 minus sigma 1 minus sigma 3 by 2 equal to 0, which will give rise to plus minus 1 by root 2. And again n is equal to plus minus 1 by root 2.

So, these are the three sets of solutions. One could get the set of equations written in terms of l and m, rather than l and m. And if we have equations going through the procedure in terms of l and m will get another three sets of solutions. And again if we make use of combinations like l and m. We are going to get another three sets of solutions.

Finally, we will find that the solutions, you have really some repetitions and the distinct planes; that you are going to get; we would like to redo them. So, finally, these solutions are. So, if you write now, l, m and n. Then, you can have 1, 2, 3, 4, 5 and 6. Although, you have nine solutions, we will find that, there are four distinct combinations and these could be plus minus 1, 0, 0.

This is 0 plus minus 1, 0, 0, 0 plus minus 1. And then we have 0, plus minus 1 by root 2 plus minus 1 by root 2 and plus minus 1 by root 2, 0, plus minus 1 by root 2. And the last one is plus minus 1 by root 2 plus minus 1 by root 2, 0. These are the six planes; if you would like to find out the solutions, you find that, this particular case, it corresponds to, let us try to consider our reference.

So, if you take our reference as the principal directions sigma 1, sigma 2, sigma 3. This is the direction, which is making an angle of 90 degree with the sigma 1 directions. Therefore, what you find is that, this gives you the plane of the principal shear stress. And in this case, it really indicates that, there is no shear stress. Similarly, plane 2 would indicate this plane. If you would like to call it, this origin to be represented by O, so this is O, B and C.

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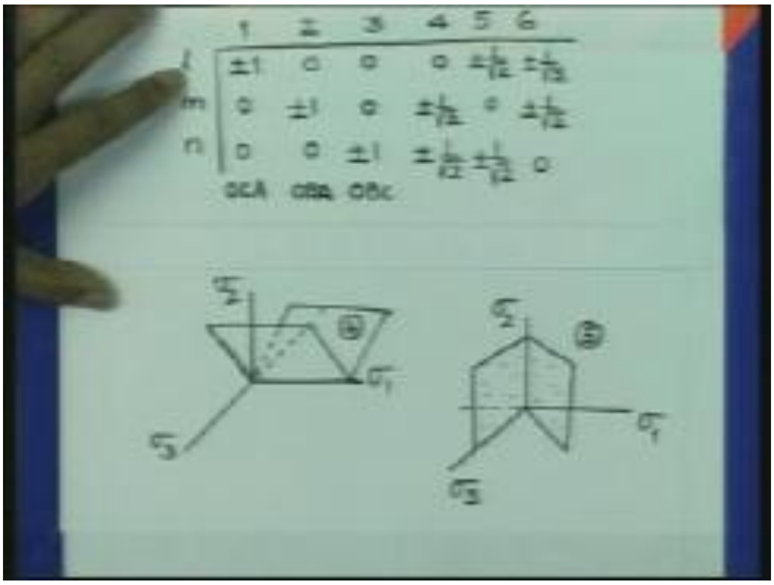

So, therefore, this is the plane the first one represent the plane O, C, A. Second one is nothing but O, B, C. And the third one, it is nothing but it should be that, this plane is O, C, A and this plane, which is making an angle of 0 with sigma 2. So, it should be O, A, B or O, B, A. So, this is the plane O, B, A and this is the plane, which is making an angle of 0 degree with the direction 3. So, therefore, this is nothing but the plane O, B, C.

Now, the plane number 4, this is making an angle of 45 degree with the direction, which is making an angle of 45 degree with the principal directions 1 and 2. So, it is making an angle of 45 degree with the direction m and n. So, therefore, which means it is making an angle of 45 degree with sigma 2 and sigma 3. So, we can show now, this plane number 4. So, if you consider your direction 1, sigma 1, sigma 2, sigma 3, then this making an angle of forty five degree with the direction 2 and 3.

So, therefore, the plane is nothing but typical plane of this type. This is the plane and you can also have the plane, which is making an angle with the minus sigma 3 and sigma 2 directions 45 degree. So, that is also the other plane, you can consider. So, this is the other plane. So, these are the two planes, which are given by combination 4. Similarly, you can consider the planes given by 5, which is making an angle of 45 degree with direction 1 and 3.

So, we can show this plane number 5, it is making an angle of 45 degree with direction 1 and 3, so therefore, 2 and 3. So, it is making an angle of 45 degree. So, therefore, this should be the direction. So, a typical plane could be this one and it can also consider to be another plane, which is like this. So, these are the two planes, which is nothing but plane 5.

You can now, do it yourself. The plane number 6 is going to make an angle of 45 degree with direction 1 and 2. So, therefore, it will be making an angle of 45 like this, 45 degree and it will be in this quadrant. So, also you can take other plane. So, I leave it to you to draw it yourself. Now, let us try to see, what is the value of the maximum shear stress? That we get on these planes. So, we had the expression for the maximum shear stress is given here.

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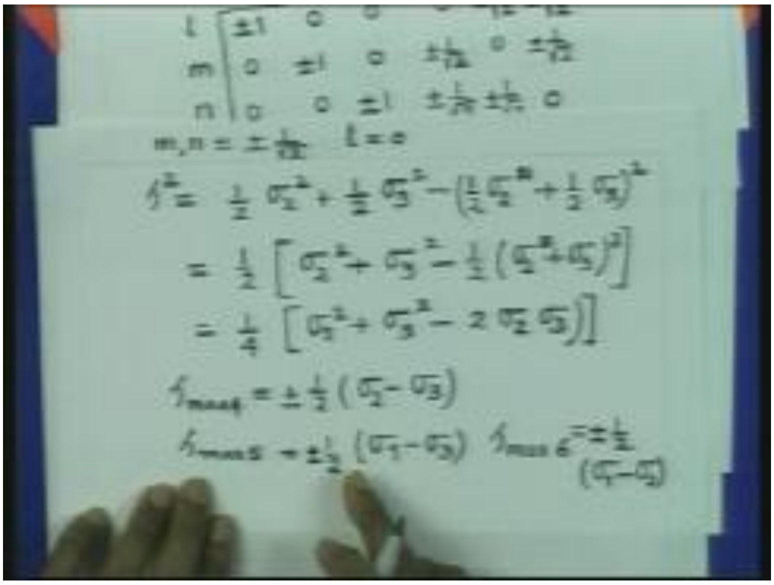

So, this tau square is equal to, if you consider the plane number 4; where in you have m and n equal to plus minus half. So let us consider the case, m and n plus minus 1 by root 2 and l is equal to 0. Now, from this expression, you find that, tau square is nothing but 1 by 2, sigma 2 square. And here, when you substitute this, this come out be, this is going to be plus half sigma 3 square minus here it is half sigma 2 square minus. No, this will be simply half sigma 2 and this is going to plus half sigma 3 square.

So, once you simplify these, this is going to equal to half. It can take comma half and it will like this. It will be sigma 2 square plus sigma 3 square, minus, it will be half sigma 2 square plus sigma 3 square. So this gives us 1 by 4, sigma 2 square plus sigma 3 square minus, this is again wrong, so this is sigma 2, sigma 3. So, tau maximum is nothing but equal to half sigma 2 minus sigma 3. So, the maximum shear stress is going to be plus minus half sigma 2 minus sigma 3.

So, the directions are 2 and 3 involved and the principal stresses involved are sigma 2 and sigma 3 and the maximum shear stress is going to be like this. So, you can write now for the other two planes, the shear stresses to be given by. So, you have the other, if you call it 1, then or let us say, this is plane number 4. Then, we have tau max plane number 5 is going to be given by plus minus half, this will be involved with sigma 1 and sigma 3.

So, sigma 1 minus sigma 3 and the plane number 6 will be involved with plus minus half into in this case, directions 1 and 2 are involved. And then the stresses are sigma 1 minus sigma 2. So, these are the principal stresses. It is nothing but the difference between the principal stresses divided by 2. And they act at an angle or on a plane at an angle of 45 degree with these two directions.

So, same is the case for plane number 5, which is at an angle of 45 degree with directions 1 and 3. Plane 6, it is at an angle of 45 degree with direction 1 and 2. We will now consider determination of octahedral shear stress and octahedral planes; rather we will try to define octahedral planes. And we would like to calculate the shear stress on such octahedral planes. Again, at a point, we will have the three principal directions.

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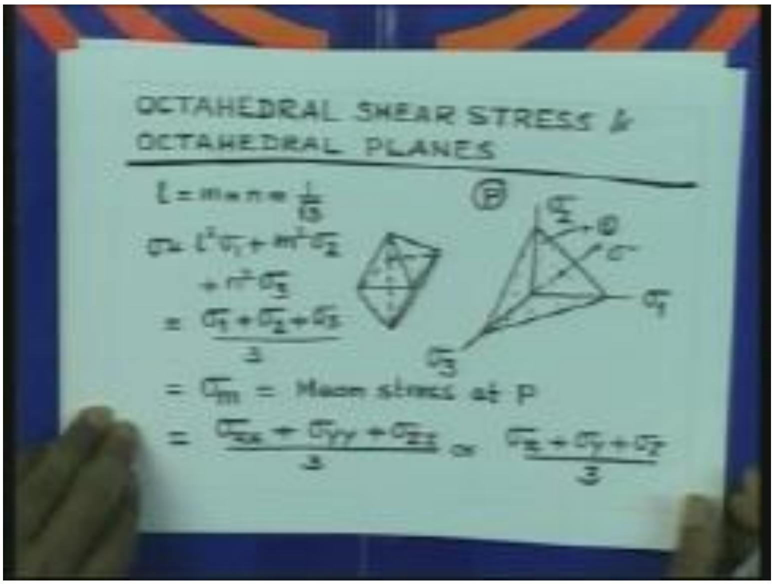

So, with those as the reference directions, if we consider a plane, which is equally inclined with the three principal directions. So, we consider a plane, which is equally inclined to all the three principal direction, such a plane, it is a plane 1, it is known as an octahedral plane. Obviously, you can understand that, the direction cosine of this plane, would be l equal to m, equal to n and it is going to be equal to 1 by root 3.

How, many such planes will be there? In fact, in the positive quadrant, we have such a plane. We can consider the quadrant form by the direction, sigma 1, sigma 2 and minus sigma 3, we will have another octahedral plane. So, therefore, in the upper half, we will have 4, such octahedral planes. Similarly, in the bottom half, we are also going to have four octahedral planes. So, there are eight octahedral planes and they will form geometry like this.

So, you are going to have such eight planes arising out of the eight quadrants. The normal stress on such a plane, sigma is going to be given by, just we have derived in the last lecture. That sigma on this plane can be written in terms of l square sigma 1 plus m square sigma 2 plus n square sigma 3.

So, this is the normal stress and therefore, if you substitute the value, it will be sigma 1 plus sigma 2 plus sigma 3 by 3. So, it is the average of the three principal stresses and it is known as mean stress. And it is indicated as sigma m. So, this is the mean stress at the point p, if you would like to consider. So, if you consider that your typical point is p, this is the mean stress at the point p.

Since, sum of the three principal stresses are also equal to the sum of the three normal stresses in Cartesian coordinates. One can immediately write, that this is nothing but sigma x x plus sigma y y plus sigma z z by 3 or sigma x plus sigma y plus sigma z by 3. So, these are the various forms of this mean stress.

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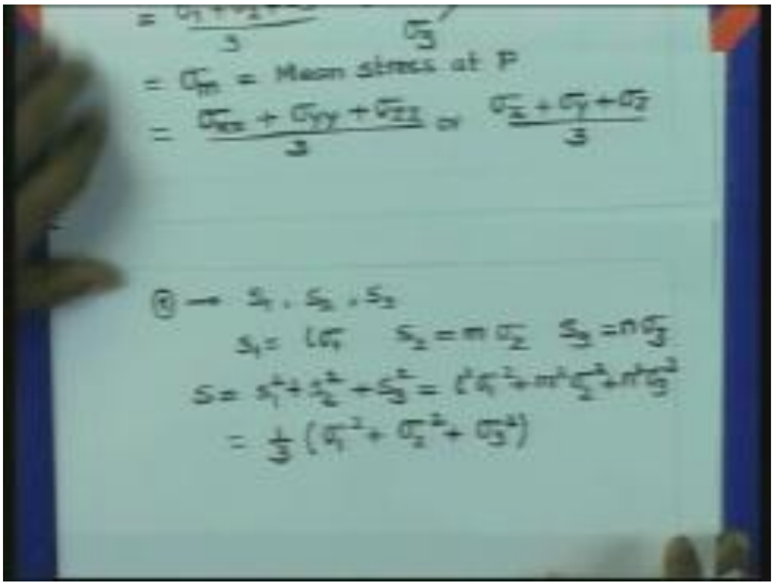

If we consider the resultant, if we consider the tractions on this plane 1, so this plane 1 would have tractions S 1, S 2, S 3. We can write S 1 to be equal to l sigma 1, S 2 to be m sigma 2 and S 3 equal to n sigma 3. The resultant traction s is equal to S 1 square plus S 2 square plus S 3 square and this is equal to nothing but l square sigma 1 square plus m square sigma 2 square plus n square sigma 3 square.

Now, once we substitute the value of the direction cosines 1 by root 3, 1 by root 3. We get the following; this will be 1 by 3 into sigma 1 square plus sigma 2 square plus sigma 3 square. We can also find out, if you consider that the shear stress on this plane is given by tau.

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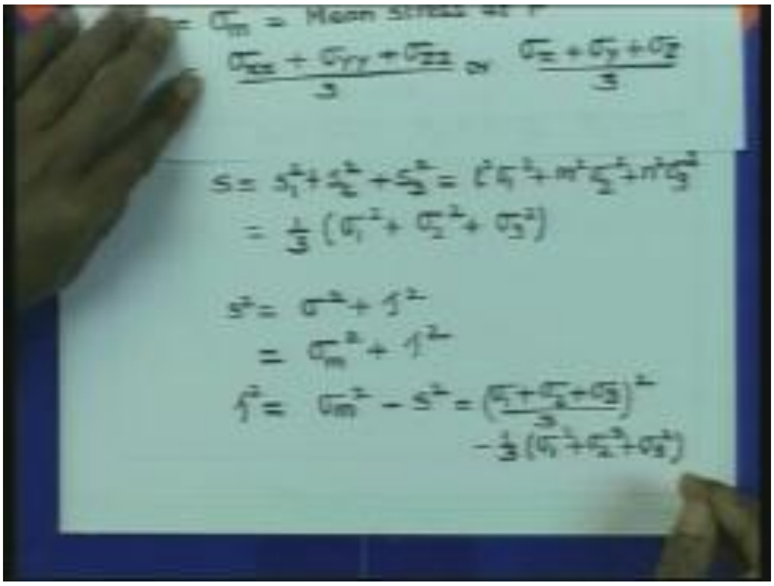

Then, the resultant tractions can also be obtained by considering the fact that, resultant tractions is nothing but sigma square plus tau square and sigma is nothing but sigma m on this plane plus tau square. So, we can now get tau square is equal to sigma m square minus s square. And if we substitute the value, we have gotten earlier. It is nothing but sigma 1 plus sigma 2 plus sigma 3 by 3, whole square minus one-third sigma 1 square plus sigma 2 square plus sigma 3 square.

So, this is the expression of the shear stress and if you now simplify this expression, tau square is equal to 1 by 3. It should be just opposite, this should be this way and therefore, we will have the expressions slightly different, so this is it. And therefore, we will have it like this, sigma 1 square plus sigma 2 square plus sigma 3 square minus 1 by 9 sigma 1 plus sigma 2 plus sigma 3, whole square.

And this can be simplified to the form, which is nothing but 1 by 3 sigma 1 minus sigma m square plus sigma 2 minus sigma m square plus sigma 3 minus sigma m square. Therefore, this stress, which is consider to be octahedral shear stress. It is indicated as tau subscript or octahedral. It is nothing but plus minus 1 by root 3 into sigma 1 minus sigma m square plus sigma 2 minus sigma m square plus sigma 3 minus sigma m square exponent with exponent 1 by 2.

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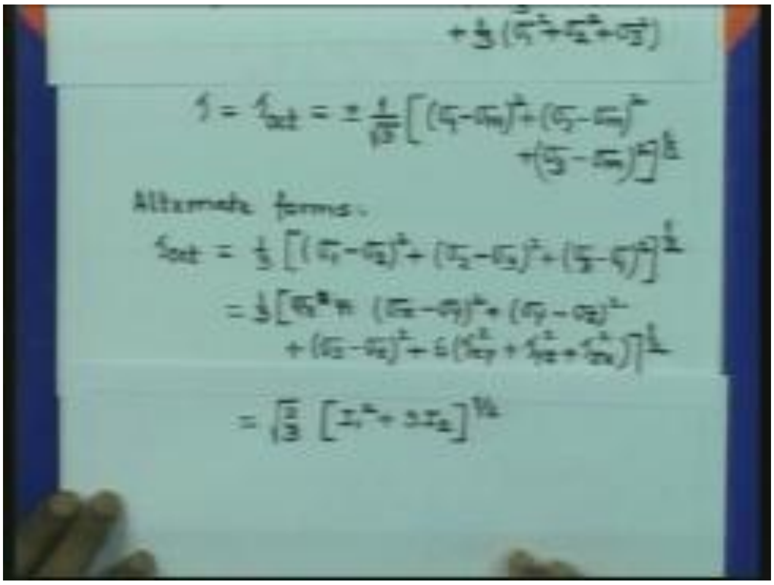

The alternate forms, you can have alternate forms of tau octahedral. So, we can write now, tau octahedral to be given in this form one-third sigma 1 minus sigma 2 whole square sigma 2 minus sigma 3 whole square plus sigma 3 minus sigma 1 whole square raise to the power half.

And it can be also written in terms of the Cartesian components, which is nothing but one-third. This is sigma x sigma x minus sigma y, whole square sigma y minus sigma z whole square plus sigma z minus sigma x, whole square plus 6 times tau x y square tau y z square tau z x square raise to the power half. It can be also written in terms of the stress invariants. This is also given in this form root 2 by 3, I 1 square plus 3 I 2 raise to the power half.

So, the octahedral shear stress is something has some significance attached. We know that this I 1, I 2 are stress invariants. They are not dependent on the selection of coordinates. And this relation between the two stress invariants and the octahedral shear stress is indicative of the fact, that the tau octahedral shear stress is also an invariant. So, it is an invariant quantity. We look into the breaking up of total stress tensor into components like Deviatoric and Hydastatic stress tensors.

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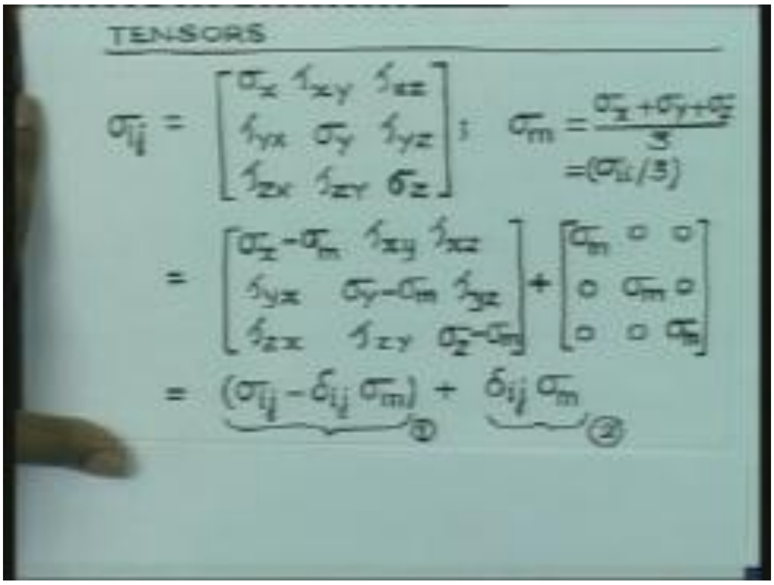

If you consider the stress tensor, at a point sigma i j, which is given by this matrix. Now, the mean stress at the point is given by sigma x plus sigma y plus sigma z by 3, which can be also retained by this tensor notation sigma i i by 3. Since, this indices i r repeated, it implies a summation, this means that, this is equal to sigma x x plus sigma y y plus sigma z z by 3.

Now, if we try to break up this stress tensor into 2 parts, like these. We have here sigma x minus sigma m, this remains as it is, this also remains as it is. Then, in the second row we are simply amending this stress in a y direction. And in the z direction, we are amending the stress in z direction, plus we have this sigma m, sigma m, sigma m in the three directions. These can be written in tensor notation like this. This first part is sigma i i minus delta i j into sigma m plus delta i j into sigma m delta i j is nothing but the Kronecker delta.

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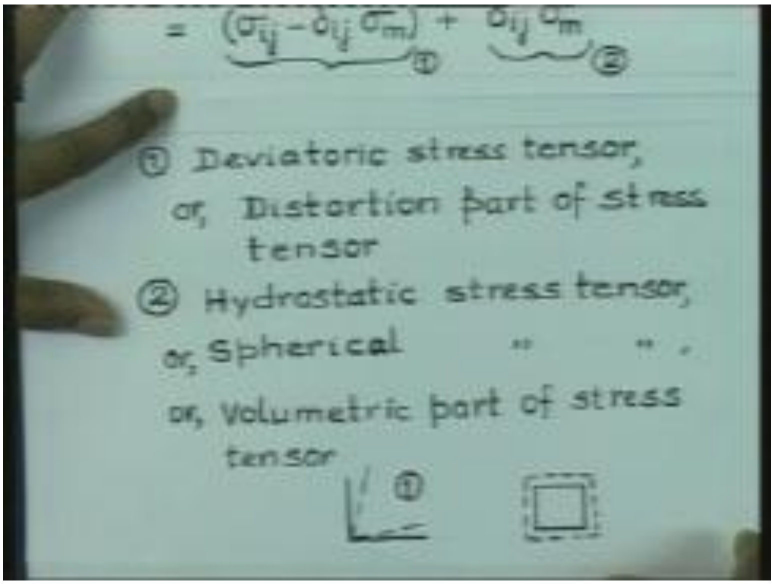

This first part is known as the Deviatoric stress tensor. It is also known as distortion part of the stress tensor. And the second part is the Hydrastatic stress tensor. It is also called spherical stress tensor or volumetric part of the stress tensor. These two parts have distinct roles. The Deviatoric part is responsible for the deformation of the element and Hydrastatic part is responsible for the change of volume of the element. This hydrastatic part does not give raise to any distortion of the element. Similarly, this Deviatoric part is responsible only for the deformation, but it does not give raise to any change in volume.

And you will find that, under the action of this Deviatoric part, any two orthogonal directions will change orientation. That means, if you think of the angle between the x and y direction, before deformation, after deformation they will undergo change. So, there is shear deformation. So, this is the effect of part 1. On the other hand, if you consider the Hydrastatic part or Hydrastatic stress tensor, this is going to give raise to change in volume of the elements, which can be represented in two dimensions like this.

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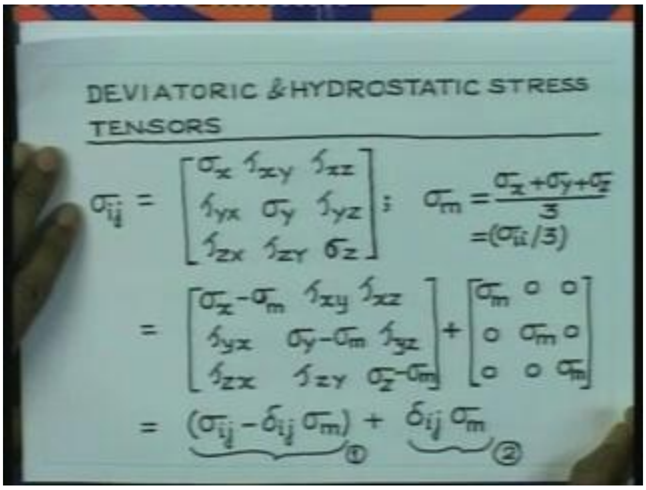

It is possible to find out the direction of the principal stress, just considering this Deviatoric part of the stress tensor. And the total principal stress can be again obtained from the principal stresses arising out of the Hydrastatic part. And after adding the mean stress, I would like to explain this point.

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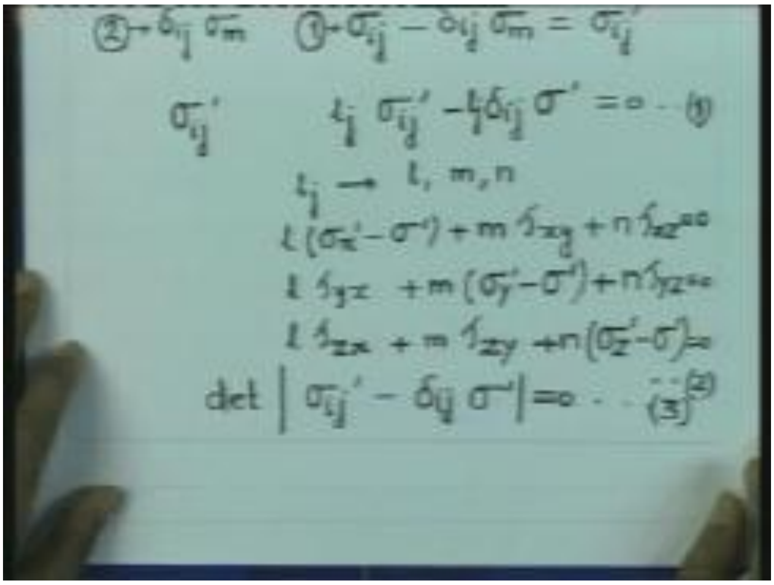

So, if we consider the Hydrastatic part to be delta i j sigma m and the Deviatoric part sigma i j minus delta i j sigma m. So, let us represent this thing by sigma i j dash. So, this is really in our earliest notation, this is part 2 and this is nothing but part 1. Now, we can go for considering only this part 1. So, this is sigma i j dash. If we write the equations, involving the principal directions, then it will be in short form like this l j sigma i j dash minus delta i j sigma dash.

Following the step as you have done in the case of the total stress, we will get the equations involving the direction cosines of the principal directions given by this equation. So, this is, let us say equation number 1. To expand this, it will look, if I consider that l j has components like l, m and n. Then, we will get the equations like this, sigma x dash minus sigma dash plus m into tau x y plus n into tau x z equal to 0.

Similarly, will have l into tau y x plus m into sigma y dash minus sigma dash plus n into tau y z equal to 0. And the third equation is, tau z x plus m into tau z y plus n into sigma z dash minus sigma dash equal to 0. So, this set of equations will have solutions for the direction cosines l, m and n provided, we have the determinant, we have the determinant of the coefficient matrix. That is sigma i j dash minus delta i j sigma dash equal to 0. So, this is the characteristic equation, given an equation involving this sigma.