Advanced Strength of Materials
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Lecture – 4

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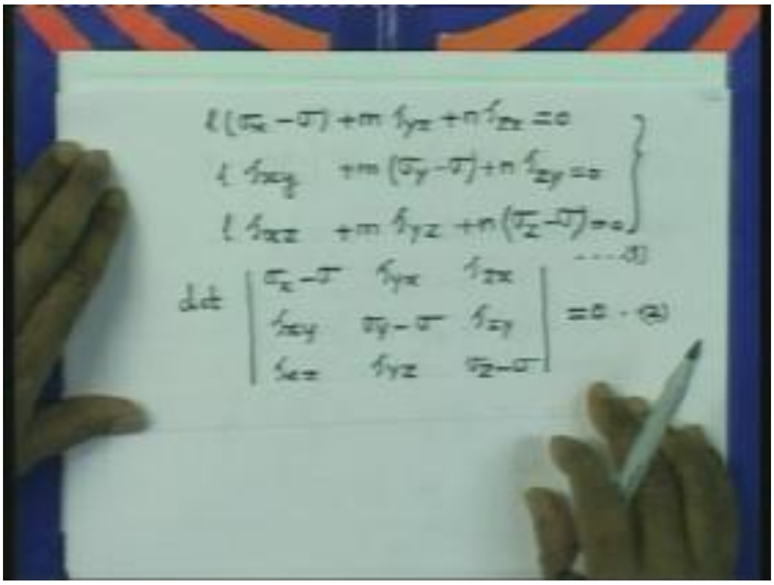

Last time we have seen that if the principal plane has direction cosines l, m and n. Then, they will satisfy these set of equations and a magnitude of the principal stress is given by sigma. To determine the principal stresses, we take the characteristic equation determinant of this expression equal to 0. You could get the principal stresses by solving this set of equations.

Now, the question is, how do you get the direction cosines corresponding to a particular value of sigma? You take the value of sigma. Let us say sigma 1, substitute in these three equations, thereby, you will get again three equations involving l, m and n. Solving these equations; you are going to get the direction cosines corresponding to the stress sigma. So, you will have three values of sigma out of the characteristic equation. And therefore, you can solve these set 1, thrice and you can get all the direction cosines.

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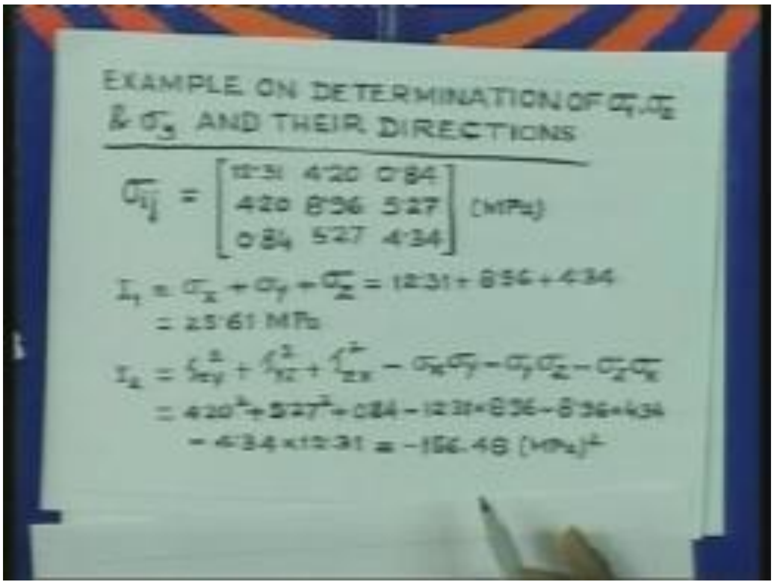

We would like to illustrate these by taking some example today. It is given the stresses at a point are given by these tensor. These magnitudes are all in Mega Pascal. So, sigma x x is 12.31, tau x y is 4.20, tau x z is 0.84. Similarly, you have tau y x, tau sigma y y, tau y z and this is tau x z, tau y z and sigma z z. Now, we calculate first, the stress invariants, using the data I 1 is sigma x plus sigma y plus sigma z.

So, if you substitute the value, these are the three diagonal coefficients. It gives you 25.61 Mega Pascal. Similarly, if you consider the second stress invariant, I 2. It is given by tau x y square tau y z square plus tau z x square minus sigma x sigma y minus sigma y sigma z minus sigma z sigma x. So, this, when I substitute the values, this is 4.20, this stress. Then, this is 5.27. This is 0.84. Then, we have 12.31 into 8.96, 8.96 into 4.34 and then 4.34 multiplied by 12.31. So, on simplification, this will become 156.48 Mega Pascal square.

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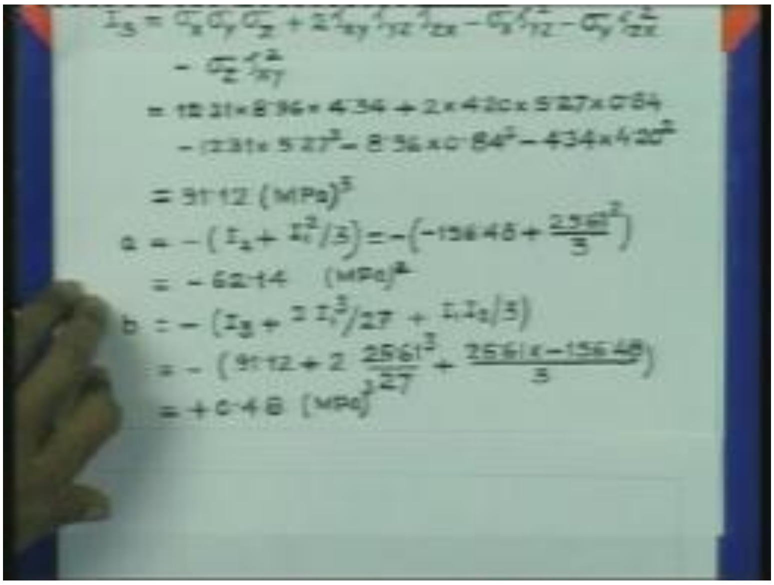

Similarly, if you try to consider the 3rd stress invariant I 3 which is given by sigma x into sigma y into sigma z, plus 2 tau x y tau y z tau z x minus sigma x tau y z square minus sigma y tau z x square minus sigma z tau x y square. So, again, if you substitute the values of the respective stresses, you get these. And on simplification, this will become 91.12 Mega Pascal square.

Then, we try to calculate the constants, a is minus I 2 plus I 1 square by 3. And so if you substitute the value, we have got earlier. This is minus 156.48, that is I 2, and then I 1 is 25.61. So, therefore, 25.61 square by 3 and on simplification, this will become 6 minus 62.14 MPa square. Similarly, if we calculate the constant b, which is related to minus I 3 plus 2 I 1 cube by 27 plus I 1 into I 2 by 3. And again, if we substitute the values, respective values, we get the simplification to be plus 0.48.

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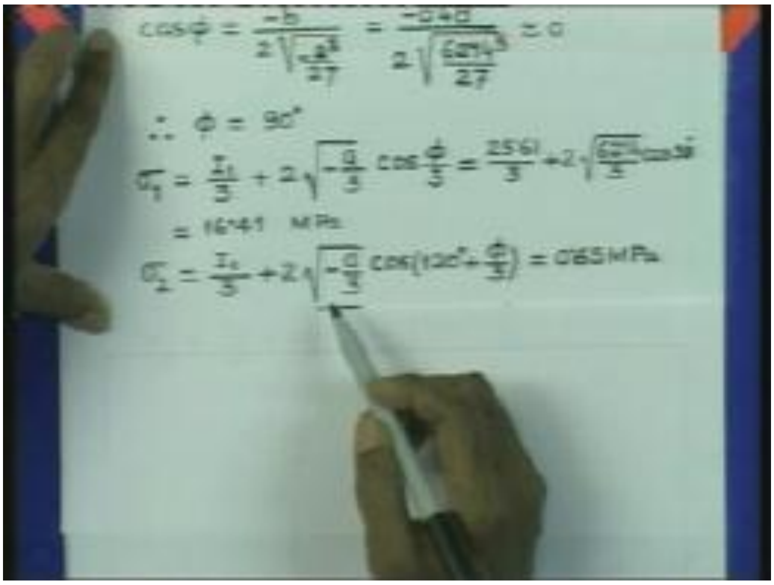

Then, we calculate the angle phi, cos phi is equal to minus b by 2 into square root minus a cube by 27. And if we substitute the values, this comes out to be very small quantity, which is almost equal to 0. Therefore, it means that phi angle is equal to 90 degree. Now, the value of the principal stresses given by sigma 1 equal to I 1 by 3 plus 2 square root minus a by 3 cosine phi by 3.

So, substitute the value 25.61 by 3 plus 2 square root 62.14 by 3 into cosine phi is 90. So, therefore, this becomes 30 degree. And if you simplify this comes out to be 16.41 Mega Pascal. Similarly, if we consider the second principal stress, it is given by sigma 2 equal to I 1 by 3 plus 2 square root minus a by 3 cosine 120 degree plus phi by 3. Again, once you simplify, you get the sigma 2 stress as 0.65 MPa. If we now take the last one, it is going to be I 1 by 3 plus 2 square root minus a by 3 cosine 240 degree plus phi by 3. Again, on some simplification, this is equal to 8.53 MPa.

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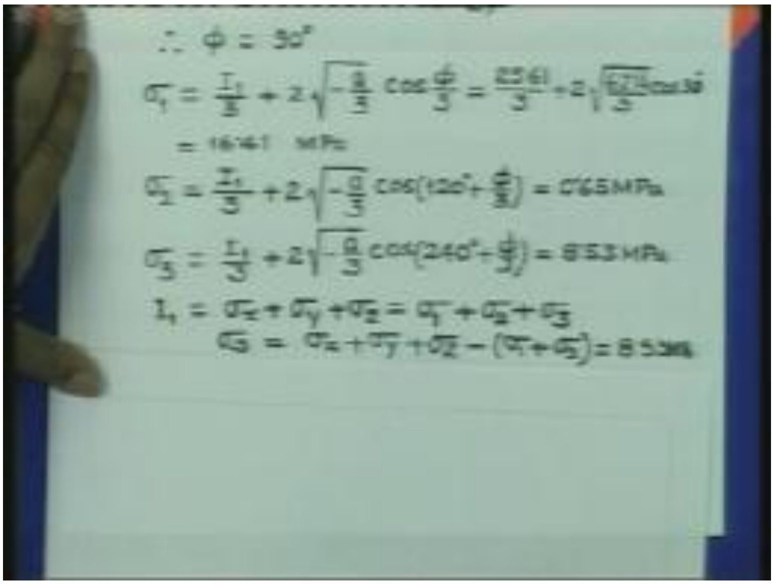

Now, at this point itself, I would like to tell you, that sigma 3 can be calculated in a slightly different way. Since, we have the set stress invariant I 1. This I 1 is equal to, it is given by sigma x plus sigma y plus sigma z. And it can also be equal to the sum of the three principal stresses. If we consider our Cartesian system to coincide with the directions of the principal stresses. So, this must also be given by sigma 1 plus sigma 2 plus sigma 3.

Now, we have already got the values of the two stresses sigma 1 and sigma 2. And therefore, this sigma 3 is equal to sigma x plus sigma y plus sigma z minus sigma 1 plus sigma 2. So, this can be a simpler way of going about and if you calculate, this will also become 8.53 MPa. So, once, we have got the principal stresses, now we have to determine the direction cosines. That means, we are interested in finding out, what is the direction for the stress sigma 1. In other words, if you consider the direction cosines of the plane only sigma 1 is acting is given by l 1, m 1 and n 1. Then what are the magnitudes of these three direction cosines.

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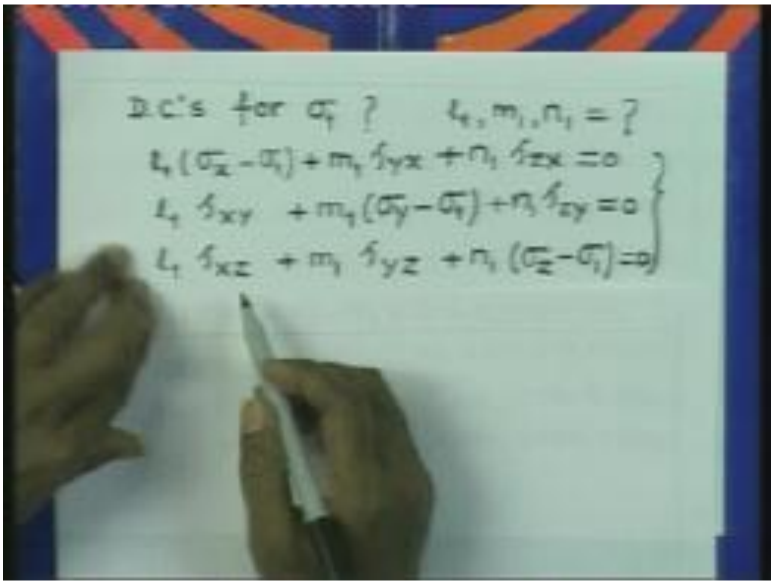

Now, these direction cosines are going to satisfy, equation number 1; that we wrote last time. And this will give us that l 1 into sigma x minus sigma 1. So, we have substituted sigma by sigma 1 plus m 1 into tau y x plus n 1 tau z x equal to 0. Similarly, l 1, tau x y plus m 1 sigma y minus sigma 1 plus n 1, tau z y equal to 0. Similarly, l 1, tau x z, m 1 tau y z plus n 1 into sigma z minus sigma 1 equal to 0.

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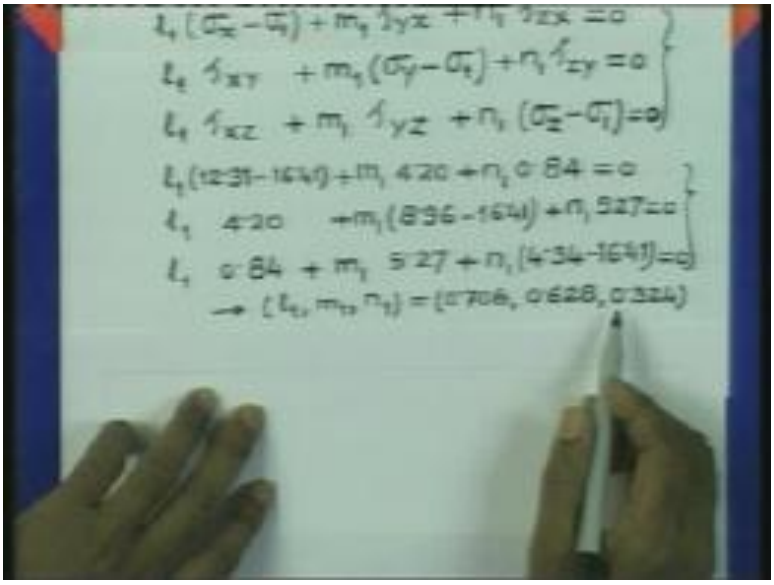

These three equations, once you substitute the value of the stresses Cartesian components of the stresses. Then, in that case, you get sigma x substituted by this, 4.20 is the value of tau y x and 0.84 is the value of tau z x. Similarly, you have the second equation here, this is tau x y plus m 1 into sigma y y minus 16.41 is the value of the first principal stress plus n 1 into 5.27. And last equation is l 1 into this is the value of tau x z and m 1 into 5.27. That is the value of tau y z plus n 1 into 4.34 will be the value of sigma z z minus 16.41. So, these three equations are the equations to solve for the constants l 1, m 1 and n 1.

Now, it is important to know, that the three direction cosines are going to satisfy the equation, which is nothing but l 1 square plus m 1 square plus n 1 square equal to 1. So, you can make use of that. And after solving this set of equations, you get the direction cosines l 1, m 1, n 1 and 0.706, 0.628, 0.324. So, this is the direction a cosine of the plane on each sigma 1 stress is acting.

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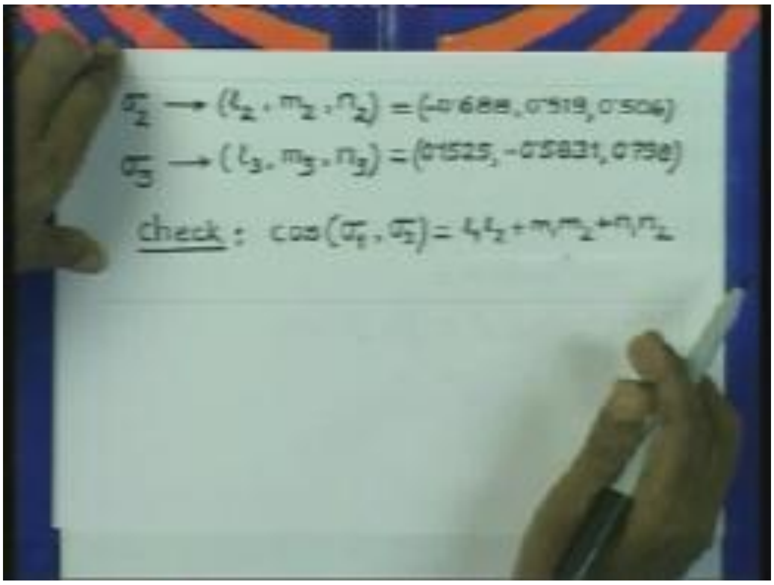

Similarly, for the second stress, second principal stress sigma 2. If you go by the same procedure, we will find that the direction cosines l 2, m 2, n 2 will come out as minus 0.688, 0.519 and 0.506. In the similar way, sigma 3, the direction cosines of the plane on the sigma 3 is acting are l 3, m 3, n 3 is going to be given by 0.1525 minus 0.5831, 0.798. Now, you have got the results, you can now check it. You can check whether the direction cosines are alright. We know that the directions sigma 1 and sigma 2, they are orthogonal. And the angle between the two directions, cosine of the angle between the 2 direction sigma 1 and sigma 2 is going to be given by l 1, l 2 plus m 1 m 2 plus n 1 n 2.

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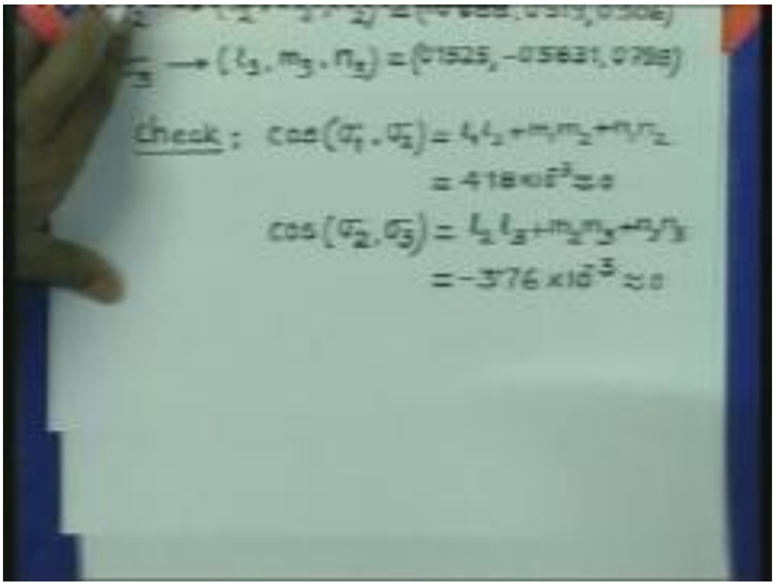

So, if we substitute the values that we have obtained earlier. We find that l 1, l 2 plus m 1, m 2 plus n 1, n 2 is equal to 4.18 into 10 to the power minus 3, which is close to 0, corresponding to the values that we have got for the individual direction cosines. Similarly, you will find that for the directions sigma 2 and sigma 3, cosine of the angle is given by l 2 into l 3 plus m 2 into m 3 plus n 2 into n 3. This comes out to be minus 3.76 into 10 to the power minus 3, which is again close to 0.

So, you can now go on considering the other two stresses sigma 3 and sigma 1. And we will find that, the angles between the two directions are again going to be 90 degree. So, that gives you a final check on the calculations that you have done. Now, I would like to show you that, it is possible to consider. It is possible to prove that, the principal directions are mutually orthogonal. You have just now seen that, the principal directions direction cosines are such in the angle between the two directions are 90 degree.

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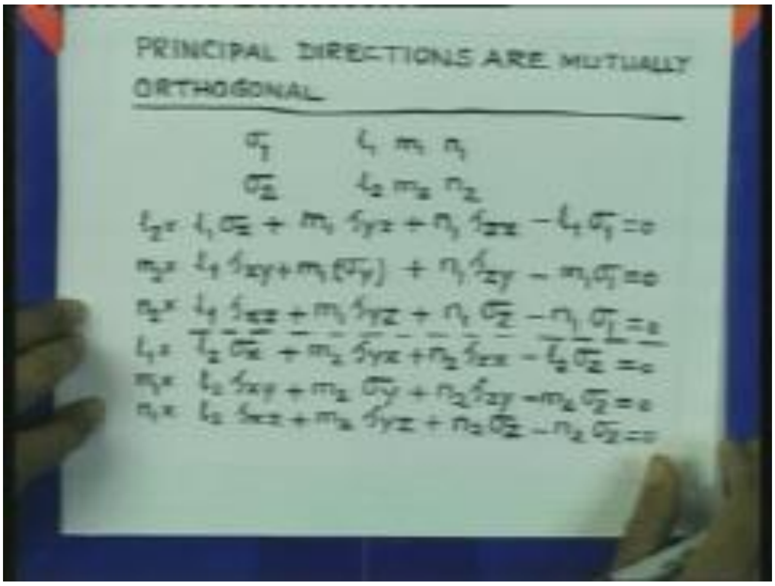

Now, we would like to show it as a general case. So, let us consider the proof for principal directions are mutually orthogonal. Let us consider that sigma 1 is the principal stress 1 and the direction cosines are l 1, m 1, n 1. Similarly, sigma 2 is the second principal stress and the direction cosines are l 2, m 2, n 2. Now, we have the equations, which we have written little while ago, that l 1 into sigma x plus m 1 into tau y x plus n 1 into tau z x minus l 1 into sigma 1 is equal to 0. So, that will be first equation.

Similarly, will have l 1 into tau x y plus m 1 into sigma y sigma y plus n 1 into tau z y minus m 1 into sigma 1 equal to 0. Similarly, will have the third equation l 1 into tau x z plus m 1 into tau y z plus n 1 into sigma z minus n 1 into sigma 1 equal to 0. So, for the first principal stress will get these set of equations. Similarly, if we write the set for the second principal stress, it is going to be like this, l 2 sigma x plus m 2 tau y x plus n 2 tau z x minus l 2 sigma 2 equal to 0.

Again, l 2 tau x y plus m 2 sigma y plus n 2 tau z y minus m 2 sigma 2 equal to 0. And the last one, l 2 tau x z plus m 2 tau y z plus n 2 sigma z minus n 2 sigma 2 equal to 0. So, now we have two sets, we like to multiply this equation by l 2, this equation by m 2 and this equation by n 2. And we would like to add these three equations. Then, we multiply this equation by l 1, this second equation by m 1 and this equation by n 1.

Again, we add up these three relations, and then the resultant out of this sum is subtracted from the sum that we have got from this. So, you will find that, we are going to have terms like l 2, l 1 sigma x, here also it will go to be l 1, l 2 sigma x. So, therefore, this term will cancel this 1. Similarly, l 2 m 1 tau y x and you will find here, you are going to have l 2, m 1. So, therefore, you will now find l 2, m 1 tau x y, because tau y x is equal to tau x y. So, all these terms are going to cancel.

However, you will find that, the term that you are going to get l 2, l 1 sigma 1, it is not going to cancel. So, also m 2, m 1 into sigma 1, they are not going to cancel. So, the terms, which are going to come out of this and also of this, they are not going to cancel. And you will find that, the term here is l 1, l 2 sigma 1. Here, also l 1, l 2 sigma 2.

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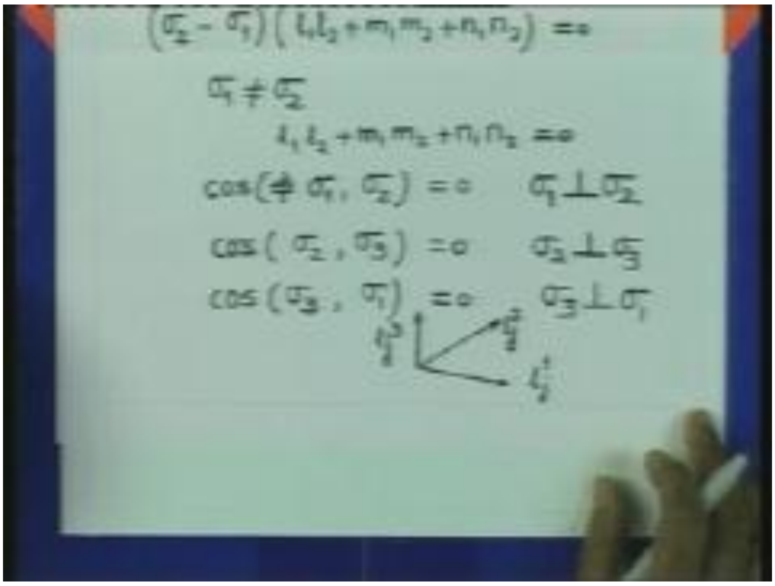

So, you can immediately appreciate that. By this process, we are going to get relation like this. It is sigma 2 minus sigma 1 multiplied by l 1, l 2 plus m 1 m 2 plus n 1, n 2. This is equal to 0. We know that, this two principal stresses are distinct and they are different. Therefore, sigma 1 is not equal to sigma 2 and this equation or rather this expression has got to be 0. So, this gives that l 1, l 2 plus m 1 m 2 plus n 1, n 2 is equal to 0.

This means that, cosine of the angle between the two directions, sigma 1 and sigma 2 is equal to 0. So, sigma 1 and sigma 2 directions are mutually orthogonal. By similar approach, you can again show that cosine of the angle between the direction sigma 2 and sigma 3 is equal to 0. Therefore, sigma 2 is orthogonal to sigma 3 and again cosine of the angle between the directions sigma 3 and sigma 1 equal to 0. And therefore, sigma 3 is orthogonal to sigma 1. Thus the three directions are mutually orthogonal.

So, what it means is that, you are two directions, any two directions, if you consider that the direction cosines are l j 1 and the other direction is l j 2 and the third direction is l j 3. Then, these three directions are going to be already mutually orthogonal. Now, these expressions that we have written, it has taken more space to write. We can write in tensor notation and then it will be much more compact.

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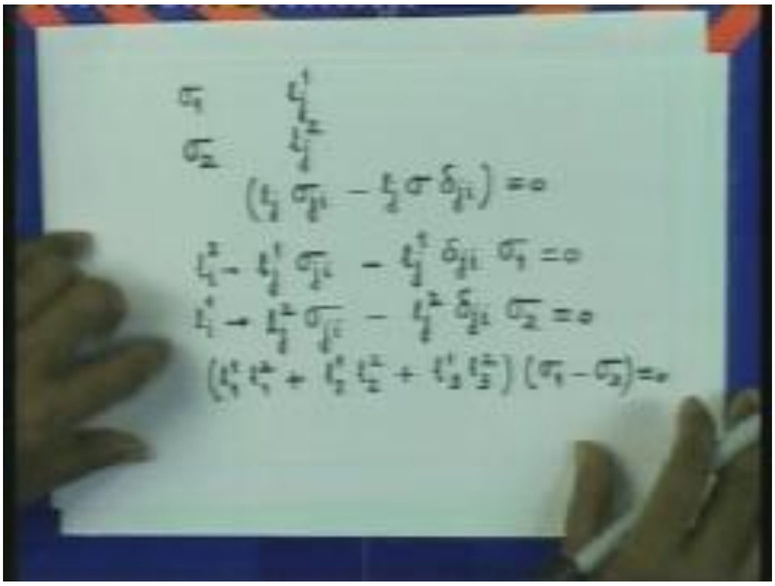

Now, let us consider, again that the sigma 1 is the principal stress and the corresponding direction cosines going to be given by l j super script 1. Again, we have sigma 2 and the corresponding direction cosines are l j super script 2. Now, the equations that we had obtained involving the principal stresses, they were of the form we are written earlier. That l j sigma j i minus l j sigma delta j i equal to 0.

So, now for the stress sigma 1, we can write that this is nothing but l j 1 sigma j i minus l j 1 sigma j i sigma 1 equal to 0. So, this is the set, we get from the principal stress sigma 1. Similarly, for the principal stress sigma 2, without any difficulty we can write, that l j 2 sigma j i, l j 2 delta j i, sigma 2 equal to 0. Now, we will like to multiply this thing by l i 2, this equation. And this equation by l i 1, with the understanding that, I is also varying from 1 to 3.

Then, having multiplied, if you subtract this resultant equation from these, you will find that, you are going to get this terms canceling, only this terms are going to be not canceling. And at the same time, only when we have I equal to j, we will get some value. And therefore, by simplifying this, what you are going to find is this, l 1 into super script 1, l 1 into super script 2 plus l 2 into super script 1, l 2 into super script 2 plus l 3 super script 1, l 3 super script 2.

So, this is the term, which we are going to get out of this. This multiplied by sigma 1 minus sigma 2 equal to 0. Again, you find that, this is nothing but according to our earlier expression it is l 1, l 2 plus m 1, m 2 plus n 1, n 2. So, this is the advantage of using tensor notation, you get very compact relationship.

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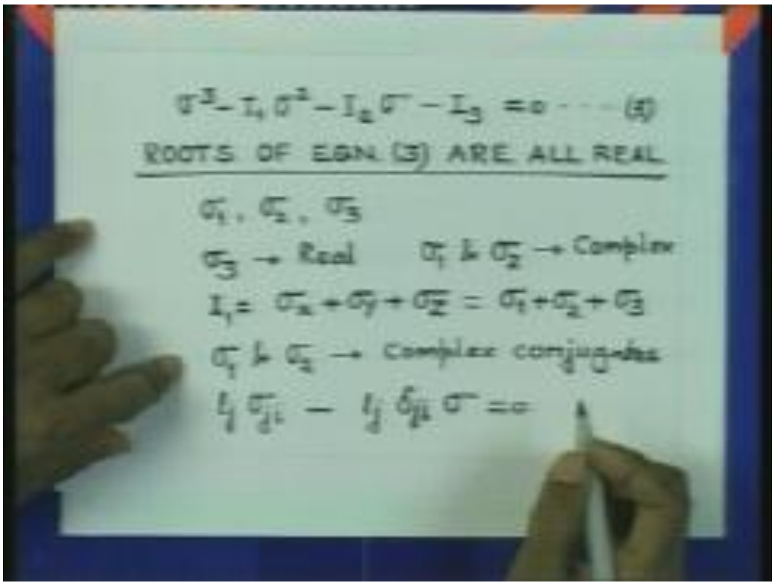

Now, we would like to show that, the principal stresses are real. They are not going to have complex quantity. Since, the equation, that we had gotten by writing the characteristic equation, it was nothing but sigma cube minus I 1 sigma square minus I 2 sigma minus I 3 equal to 0. The roots of this equation are going to be all real. So, this equation, if we indicated as equation number, let us say 3. So, roots of equation 3 are all real.

Let us assume that, the three roots are again, sigma 1 sigma 2 and sigma 3. Let us consider that, sigma 3 is real, but sigma 1 and sigma 2 is complex. Since, I 1 is equal to sigma x plus sigma y plus sigma z. This is real quantity and at the same time, we have seen that, this I 1 is also equal to sigma 1 plus sigma 2 plus sigma 3. If sum of these three quantities, have got to be real. It demands that, sigma 1 sigma 2 must be related.

In fact, one should be the complex conjugate of the other. So, it is sigma 1 and sigma 2 is complex to get the imaginary part of this sum of these two roots to cancel. We must have sigma 1 sigma 2 must be conjugate complex conjugate. Sigma 1 and sigma 2 are complex conjugates. Now, you would also recollect, that the equation, that we are going to be, that we satisfied by these are l j sigma j i minus l j delta j i into sigma equal to 0. So, this is what, is the equation to be satisfied by these roots.

Now, we find that, this component yields complex one of the component is complex and therefore, this equation now going to involve complex coefficients. And hence it will indicate that the direction cosines have got to be complex. So, sigma 1 and sigma 2 are complex conjugates, and then this equation is to be satisfied, by all the roots.

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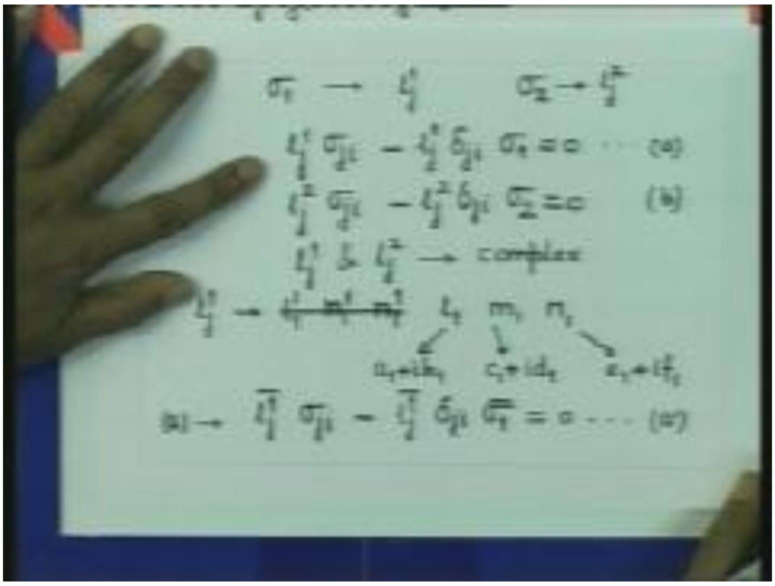

Let us write that, the direction cosines corresponding to sigma 1 is l j super script 1. Similarly, for sigma 2, it is l j super script 2. Now, this equation, you will have for each l j 1, sigma j i, l j 1 delta j i, sigma 1 equal to 0. Similarly, l j 2, sigma j i, l j 2, delta j i, sigma 2 equal to 0. Now, these 2 equations a and b, these are all real quantities and here this is complex. So, also in this equation this is complex.

So, we are going to get a set of equations involving on complex coefficients. And it is natural to expect that, this l j and l j 1 and l j 2, roots of these two equations, must also be complex. So, l j 1 and l j 2 is complex; that means, if l j 1 has components like l 1, 1, m 1, 1, n 1, 1. We can write this thing as l 1, m 1 and n 1. Then, each of these components must be complex. So, this could be that l 1 is of the form a 1 plus i b 1, m 1 is equal to c 1 plus i d 1 and this is let us say e 1 plus i f 1.

Now, if we try to take the equation 1 and take it is complex conjugate. So, if you try to take complex conjugate of a, so it will look in this form l j 1 bar sigma j i minus l j 1 bar delta j i sigma 1 bar is equal to 0. So, that is the complex conjugate form of equation number 1.

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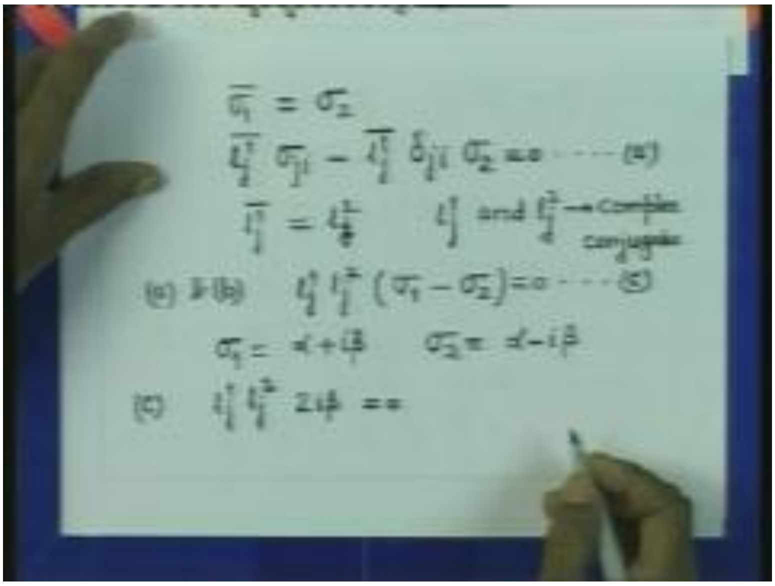

Now, this sigma 1 bar is nothing but sigma 2. That is the stress sigma 2, because sigma 2 is complex conjugate of sigma 1. So, equation a dash it now comes out to be of this form, l j 1 bar sigma j i minus l j 1 bar delta j i sigma 2 equal to 0. So, that is the form of a dash. Now, this is an equation, which is involving sigma 2 here. So, it is natural to expect that, if you compare now these two equations a b and a dash. See, it is natural to expect that these two must be related.

So, this means that, l j 1 bar must be equal to l 2 l j super script 2. So, therefore, this comparison gives us that l j 1 bar is equal to l j 2. And hence, l j 1 and l j 2 are again complex conjugates. Now, we go back to equation number a and b from a and b. If you again multiply these equation by l j 2 and these equation by l j 1 and subtract. So, if you do the subtraction, so from a and b, we are going to get l j 1, l j 2 minus sigma 1 multiplied by sigma 1 minus sigma 2 equal to 0. So, this is equation.

So, therefore, we just consider these two, consider multiplying this thing by l j 2 and this by l j 1 and subtract the resultant equation out of here from the resultant equation from there, that will you give this. Now, if we assume that, sigma 1 is equal to alpha plus I beta. Since, it is complex and since, sigma 2 is complex conjugate of sigma 1. Therefore, sigma 2 can be written as alpha minus I beta. So, from c now, what we have is l j 1, l j 2 multiplied by 2 I beta equal to 0.

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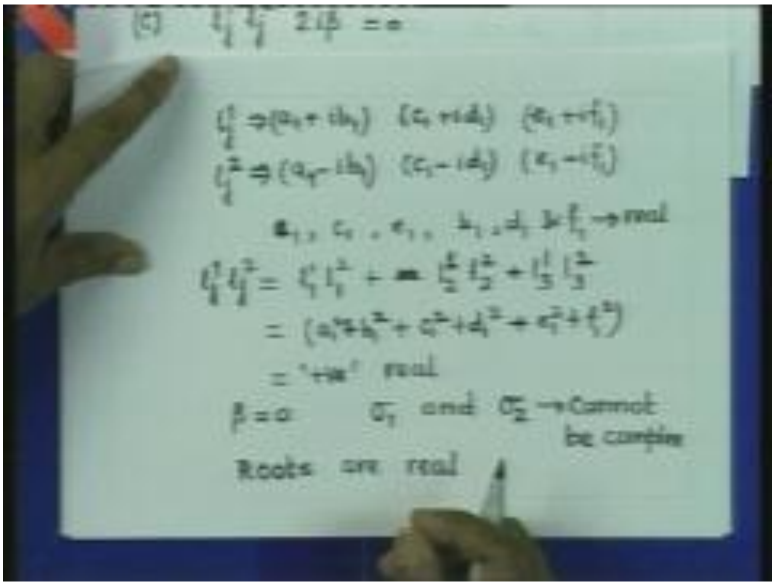

Now, since l j 1 and l j 2 are complex conjugates. Since, we can write now the coefficients of l j 1 as a 1 plus I b 1. And the components are these, this is c 1 plus i d 1 and this is u 1 plus i f 1. Similarly, l j 2 components are, since they are complex conjugates. So, therefore, it is going to be i b 1, c 1 minus i d 1 and this is u 1 minus i f 1. Now, these a 1, b 1, c 1, d 1, e 1, f 1 they are all real. So, start a 1, c 1, e 1. So, also b 1, d 1 and f 1 are all real quantities.

So, now l j 1 into l j 2 should be equal to nothing but this multiplied by these plus this multiplied these plus this multiplied by these. So, which is nothing but l 1 1, l 1 2 plus, it is l 2 l, 1 l 2, plus l 3 1, l 3 2. So, this will be, thus a 1 square plus b 1 square plus c 1 square plus d 1 square plus e 1 square plus f 1 square. Now, this expression is always some positive real value.

So, from equation c now, from equation c, that we have, it is quite clear. That beta, this product is real and this 2 I beta is product is 0. So, therefore, this is real and this simply means that, beta has got to be 0. So, this relation indicates that beta is equal to 0. And we assume that sigma 1 equal to alpha plus I beta sigma 2 equal to alpha minus I beta. It simply means that, sigma 1 and sigma 2 cannot be complex. So, the roots of the characteristic equation involving the principal stresses are real. So, roots are, thus we can show that, the other roots are also going to be real.

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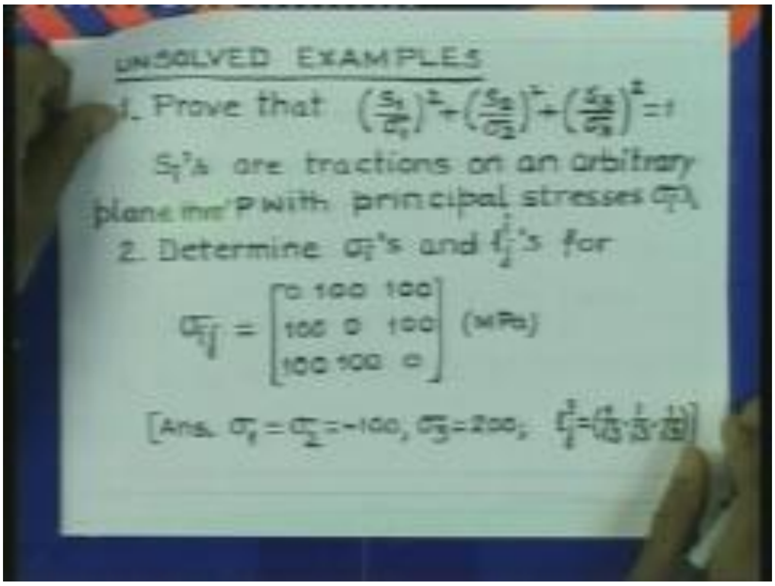

You can now consider, solving some problems yourself. You consider this example, prove that, s 1 by sigma 1 square plus s 2 by sigma 2 square plus s 3 by sigma 3 square equal to 1. Where s i's are the traction on an arbitrary plane through the point p with principal stresses sigma i's. So, this sigma 1 sigma 2 sigma 3 are the principal stresses at a point p. And s 1, s 2, s 3 is the tractions on an arbitrary plane inclined with the three principal directions; prove that, this relation is equal to unity.

Second example, you can consider determination of principal stresses and there direction cosines, where the stresses in Cartesian coordinates are given by this expression 0, 100, 100, 100, 0, 100, 100, 100, 0 MPa. And the answer for this case is going to be sigma 1 equal to sigma 2 equal to minus 100 MPa, sigma 3 equal to 200 MPa. And the direction cosines connected with the sigma 3 stresses is 1 by root 3, 1 by root 3, 1 by root 3. Similarly, determine the stresses sigma and tau on a plane, which is equally inclined with the three Cartesian axes. And here the stress tensor is given by 100, 100, 100 MPa, 100 minus 50, 100 MPa 100 100 minus 50 MPa.

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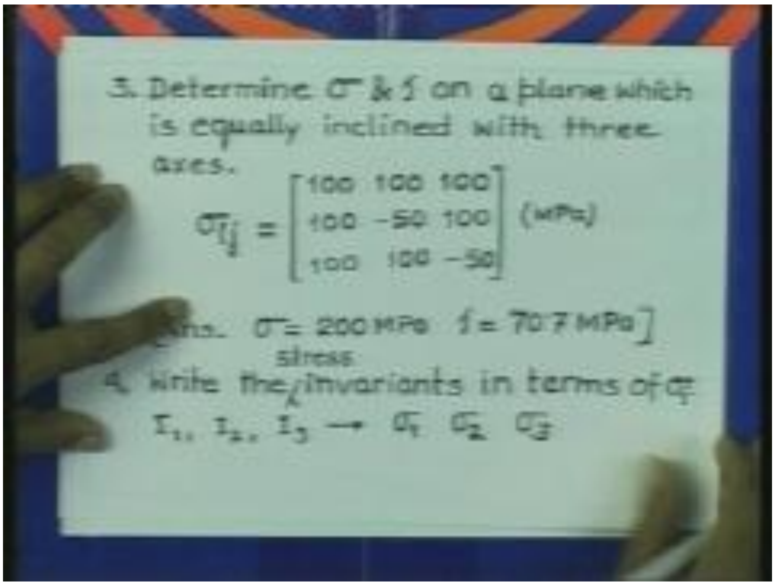

And this answer for this case, the normal stress is given by 200 MPa and the shear stress is 70.7 MPa. You can consider the fourth example, you write the stress invariants, in terms of the principal stresses sigma I. That is you write the expressions for I 1, I 2, I 3; in terms of the principal stresses sigma 1 sigma 2 and sigma 3.

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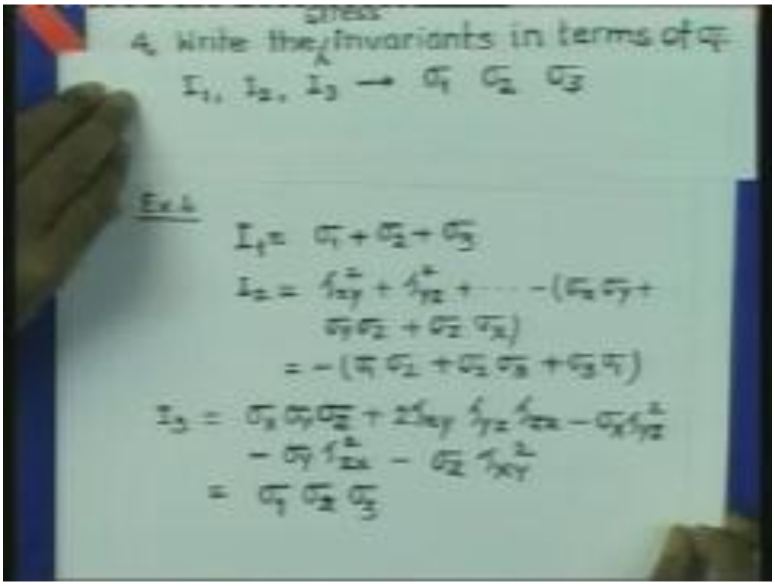

I would like to consider the solution for this example 4. So, example 4, we have I 1, it is directly related to the three principal stresses. And there sum is going to be the first stress invariant. Now, second stress invariant, I would like you to consider the relationship. We had tau x y square tau y z square and the third component minus sigma x sigma y plus sigma y sigma z plus sigma z sigma x.

Now, in this case, the shear stresses are all 0. Since, we are working in terms of the principal directions. Therefore, we are going to get directly sigma x to be sigma 1 sigma y to be sigma 2. And therefore, this is given by minus sigma 1 sigma 2 plus sigma 2 sigma 3 plus sigma 3 sigma 1. Similarly, the third stress invariant is given by sigma x sigma y sigma z plus 2 tau x y tau y z tau z x minus sigma x into tau y z square minus sigma y into tau z x square minus sigma z into tau x y square.

And since, in this case shear stresses are all absent. And the sigma x is nothing but sigma 1, this is sigma 2, this is sigma 3. Therefore, the third stress invariant is given by product of the three principal stresses, sigma 1, sigma 2 and sigma 3. So, the three principal, three stress invariants are going to be very simple expressions, I 1 is sum of the 3 principal stresses. I 2 is nothing but minus product of the two principal stresses, taken in succession and I 3 is nothing but product of the three principal stresses.