Advanced Strength of Materials
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Lecture – 39

Today we are going to talk about beam on elastic foundation this deals with certain analysis which has number of applications. Let us consider some of the applications many times you will find that pipelines are made up of segments, which can be of different geometry or they can be made up of different materials.

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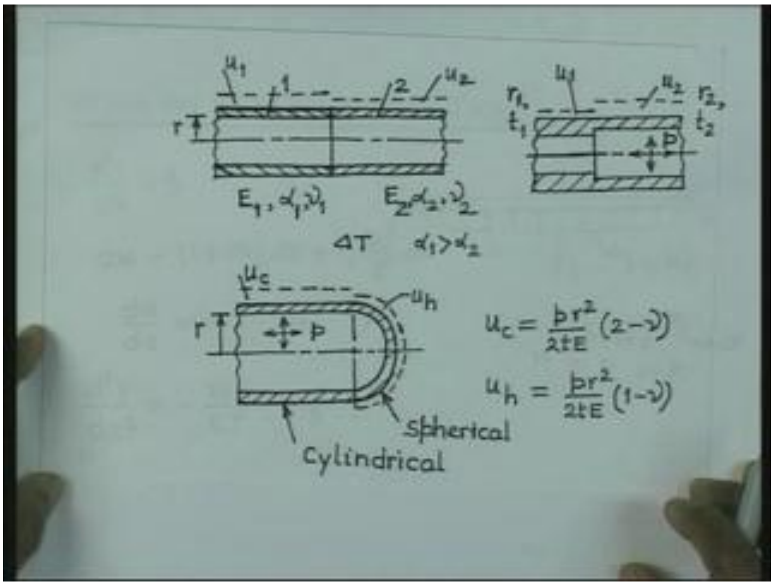

Look at this particular application here, in we have pipe of uniform cross section. The portion on the left is made up of a material, whose module of coefficient of thermal expansion ratio is different from that of the pipe which is on the right hand side. You find that in power plant applications, when the temperature is high then the vessels are made up of stainless steel and the portion of the components which are higher from high temperature region they are made up of critics steel.

So, one can have this portion made up of stainless steel, this can be made up of critic steel and these materials they have different coefficient of thermal expansion though they are modulus are some poisons ratio may not differ significantly. Similarly, you can have pipeline showing the radius of the pipeline is different from 1 segment to another segment. Although the pressure is same and also you find that the thickness of the material is different in the different segment this will give rise to some problems. Let us, look into that first of all you could think of this case.

This pipe if the temperature of the pipe is raised by delta t and think of a situation that you have this coefficient of thermal expansion alpha 1 is greater than alpha 2. So, what you expect is that, this pipe of the length is going to expand radially by some amount. Let us say u1 and the 1 on the right is going to expand by some amount, let us say u2; obviously, this expansion u1 is more than u2 and hence that this junction you find that there is mismatch in the expansion and this will give raise to some stresses in the region .So, these are local stresses look at this case here, in you have the thickness is large and therefore, you expect this type to expand radially by some amount. Let us say again it is u1 and the 1 of the right is thinner it is subjected to the same pressure.

So, it is going to expand more and this is going to be u2. So, you find that because of the internal pressure there is going to be some mismatch in the radial expansion and this will give raise to some stresses in this region. Another problem which is also very common, you find many cases cylindrical vessels have closes which can be flat, which can be ellipsoidal, which can be spherical. So, let us just look into the case, that you have the cylindrical vessel and the head is spherical they are made up of same material. And they are subjected to the same internal pressure.

Now, you know that under the action of the internal pressure in the cylinder you are going to get both hook stress and also the axial stress. Similarly, in the case of spherical vessel you are going to get both hook stress and axial stress. And if you now consider the thickness to be the same then this cylindrical portion is going to have expansion under the action of the internal pressure. Let us say, that is going to be uc is the radial expansion and similarly, the end portion or the spherical portion that is going to expand also radially and this radial expansion is going to be, let us say like this and this amount is equal to uh. It can be very easily obtained.

Since, the circumferences strain is given by u by r u by r is nothing, but sigma theta by E minus u times sigma a by E. And the hook stress is pr by t and the axial stress is pr by 2t. Therefore, from that you can get this uc for the cylindrical portion is nothing but, pr square by 2tE into 2 minus E. So, that is the radial expansion of the cylindrical part and if you consider the spherical head that is; spherical head is going to have a expansion in the radial direction which is nothing but, pr square by 2t. In this case, both the stresses are of magnitude pr by 2t and therefore, you would get the expansion is nothing but, this much.

Now, if you look into u is generally very small. If you forget, about that part the expansion of the cylindrical portion is going to be almost double. So, therefore, you get the expansion of this cylindrical portion is very large compare to the head expansion. And therefore, you find that this portion is going to have mismatch under the internal loading and that will give rise to some local stresses. So, on top of the stresses that you calculate by considering the hook stress, axial stress you are going to get some local stresses coming up here and though local stresses has to be estimated.

So, in all these application you have similarity, that you have some mismatch in the expansion of the vessel in the 2 segments. And then you are interested in trying to estimate what is the order of magnitude of the local stresses. In fact, it is possible to estimate the magnitude of this local stresses by using the concept or the analysis which is associated with beam on elastic foundation. That's shows the importance of the topic. So, let us consider how it go, about talking or analyze the stresses in the case of beam on elastic foundation.

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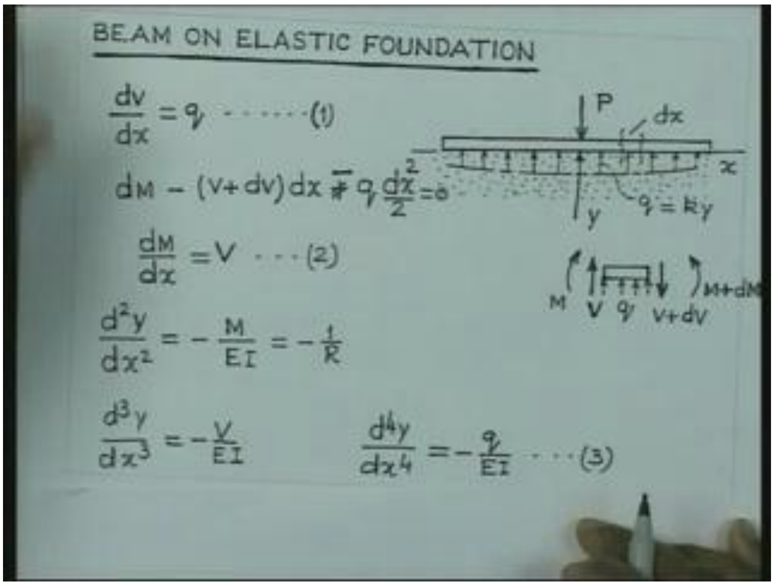

Let us consider 1 beam, which is like this and the foundation you shown here. Now, this just a beam of some dimensions it will have some height and then width and it is resting on the foundation and we subjected to some external loading P. It is not, the difference is that this foundation of the support is not rigid. And if this is not rigid, then in that case what will find is that, you are going to get this beam deflecting and it is going to deflect more under the load and its gradually going to decrease as you move from the load.

Therefore, what you find that the reaction that this supporting going to be rise to not going to be uniform. So, the reaction that is going to come up at the contact surface this is not going to be uniform. Let us consider, that reaction of this support is going to vary it is going to be varying like this. Lets you consider, that there is a variation and that variation we assume that it is proportional to the deflection that comes up in the local region. So, if you are considering the reaction here, it is of magnitude k into y where y is the deflection there.

So, we just make use of our symbol like this. This is x axis this is y axis y is positive downwards. Now, let us try to see that this is nothing but, you have this case beam like; you have a beam which is now, externally loaded like these and it is supported or it is going to have support reactions varying like this. So let us try to see, how we can get the deformation of this beam and how we can calculate the stresses in this beam. Let us concentrate, on a portion somewhere here.

So, let us just take a segment which is of length equal to dx. So, we will just concentrate on a segment which is of length equal to dx. And it will write to a scale I have shown it here, that this portion is now, subjected to load intensity q and the share force is v on the left bending moment is m there and therefore, we are going to have share force on the right hand phase V plus dV and the bending moment is m plus dV. If I now, consider the equilibrium in the vertical direction, then you find that this you find that you have dV v plus dV and this is acting apart. So, therefore, differential is dV So, let's do that. That dV is nothing, but q into dx and therefore, we have dV dx is equal to q.

So, from the equilibrium consideration in the vertical direction, what we get is that dV dx is equal to q. This is your relationship which we have also derived in your first course. The sign could be different that is all. Let us consider, this is equation number 1. Similarly, if we consider the moment equilibrium then if we consider the moment here; let us say, that anti clockwise moment is positive then this will be acting clockwise for the differential moment is dm. So, that dm is now balanced by if I consider moment about this point it is V into dx. And then q into dx into dx by 2 they are acting in the same direction.

So, therefore, we can write now that dm and then it is V plus dV into dx. So, it is in the opposite direction and then we also have, this should be negative. So, therefore, this is minus q dx square by 2. So, that is the equilibrium equation in terms of moment. And now here, in what we have is that we have V into dx dV into dx and dx x square by 2. When, you are trying to think of this dx finking to 0. Then in that case we will have no contribution. This we can forget and therefore, this is also product of 2 small quantity. We can forget therefore, what will have is dm equal to V into dx and therefore, dm dx equal to V. So, that is another relationship feature that you have already seen, in your first course.

Now, we will consider the moment curvature relationship. Moment curvature relationship in this case the beam is going to have bending like this. And therefore, center of curvature will be upward which is located in the negative y direction and therefore, the d2y dx square should be equal to minus 1 by r. Where r is the radius of curvature and therefore, now we can write d2y dx square is equal to minus m by EI. So, that is the moment curvature relationship. Now I having got it, now we can do little bit of manipulation on it. If I now try to write d3 y dx x cube nothing but, it is dm dx. So, therefore, that will become V by EI. So, I simply substitute dm dx is equal to V. similarly, if I differentiate it once more dx4 d4 ydx4 is equal to minus q by EI. So, that is the equation which is the deflection equation of the beam on elastic foundation.

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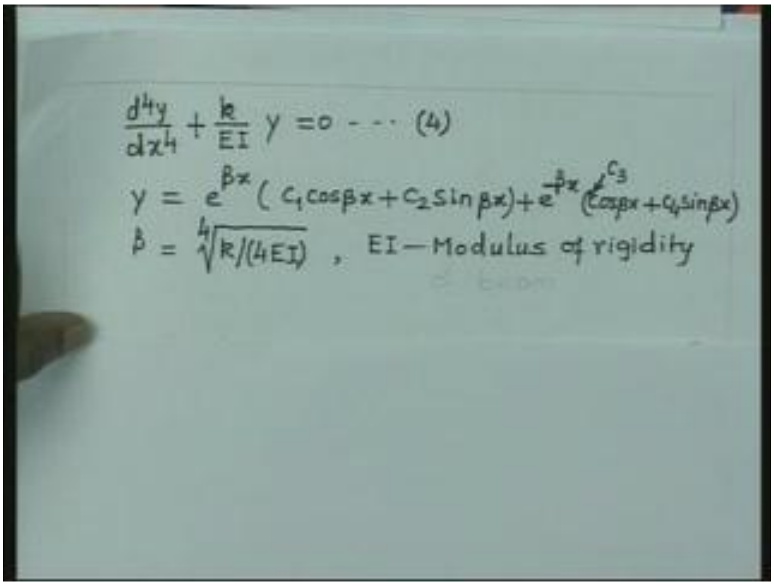

Now, since q is equal to ky you can now write, now d4y dx4 plus k by EI into y is equal to 0. So, that is the equation we can write this thing 4 that is the equation governing the deflection. Now, this is a standard equation; its solution can be retained. In this form y is equal to e to the power beta x into c1 cosine beta x plus c2 sin beta x plus e to the power minus beta x into cosine beta x plus. This would be c3 here, so we should have c3 cosine beta x and c4 sin beta x that is the solution .Where in this beta is nothing but, fourth root of the ratio k by 4 EI. So, that is the beta factor and since, it is a fourth degree equation we expect 4 arbitrary constant c1 to c4 have been bound there. This EI is modulus of rigidity, modulus of rigidity of the beam section. We has to now see, how this constants can be resolved and that is possible by taking specific example.

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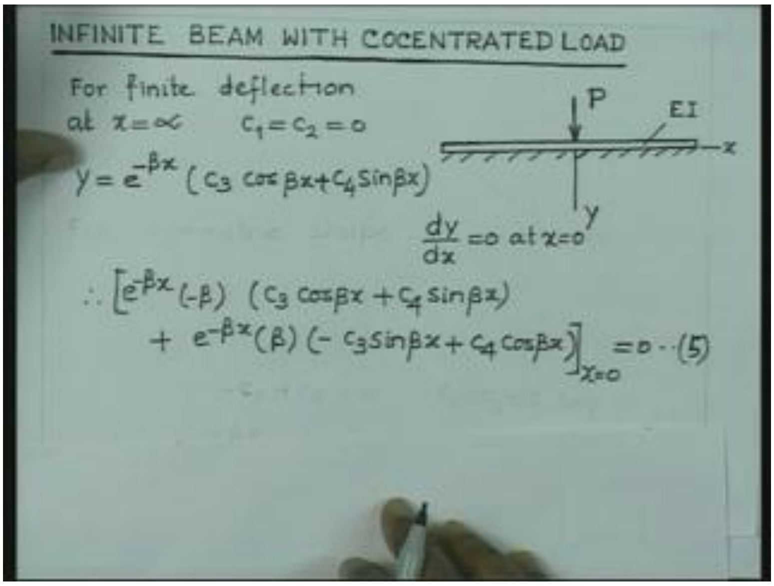

So, let us consider 1 example; where in we have a beam, we will just try to think of this case itself. That we have a beam on elastic foundation and it is subjected to concentrated load and the cross sectional rigidity is EI. Now, think of the physics of the problem you should think of the physics of this problem. The deflections as you go away from the origin, whether you move to the right or the left you expect the deflection to gradually dye out. So, therefore, with increasing x your deflection must die out. And if you see, the solution that we had obtained here, it shows y equal to e to the power beta x. c1 cosine beta x c2 sine beta x plus e to the power minus beta x into c3 cos beta x plus c4 sine beta x.

Obviously, if x increases if beta is a positive constant and therefore, this term is going to suit to infinity and therefore, we cannot have this things present for a problem of that type our solution should be certainly y equal to e to the power minus beta x into c3 cosine beta x plus c4 sine beta x. So, that is the specific case that we have and therefore, at x is equal to infinity. What we, what to have is that c1 c2 must be equal to 0. So, they must for this problem, the logic is for finite displacement or deflection at x is equal to infinity c1 c2 must be 0. And therefore, our solution for the specific case is y equal to e to the power minus beta x into c3 cosine beta x plus c4 sin beta x. Think of it, how do you resolve this constants.

We have to make use of the loading conditions and also the boundary conditions. What are the boundary conditions that we can specify; obviously, 1 condition is that this beam is going to deflect symmetrically. And therefore, we expect the slope at origin, should be 0 and at the same time, the reaction that is coming up at the bottom of the beam is going to be k into y at different location. And therefore, if we now consider the some of the forces from x equal to 0 to x equal to infinity, we must have the load double of that; obviously, it is a symmetric problem, therefore double of that must be equal to the external loading.

So, I repeat this point that you are going to get reaction to be k into y and if you now see that, that same position is going to be exactly on the other side at the same distance. So, therefore, if you sum up the reaction from x equal to 0 to infinity, And double of that, that should be equal to the external loading. So, these are the 2 things that you can make use of. So, therefore, we must have dy dx is equal to 0 at x is equal to 0 that is the symmetric shape. And this derivative we can write; therefore e to the power minus beta x into minus beta c3 cosine beta x plus c4 sin beta x plus we differentiate now, the expression inside.

So, we will have beta coming out. So, we can take that minus c3 sin beta x into beta that beta I have taken out c4 cosine beta x. So, that is the full derivative and therefore, at x is equal to 0 this is 0. So, that is equation number 5. And this gives us x equal to 0, this is unity and therefore, we are going to get minus c3 into beta and here again we are going to get c4 into beta. So, beta can be removed and therefore, we have c3 plus c4 is equal to 0 that is the expression and therefore c3 equal to c4. And let us say that, that is equal to a constant without suffix and that is equal to c lets say. So, now we have got y is equal to ce to the power minus beta x into cosine beta x plus sin beta x that is the value of deflection. Now, if you try to consider the reaction at every point and then try to sum up that will give us this expression.

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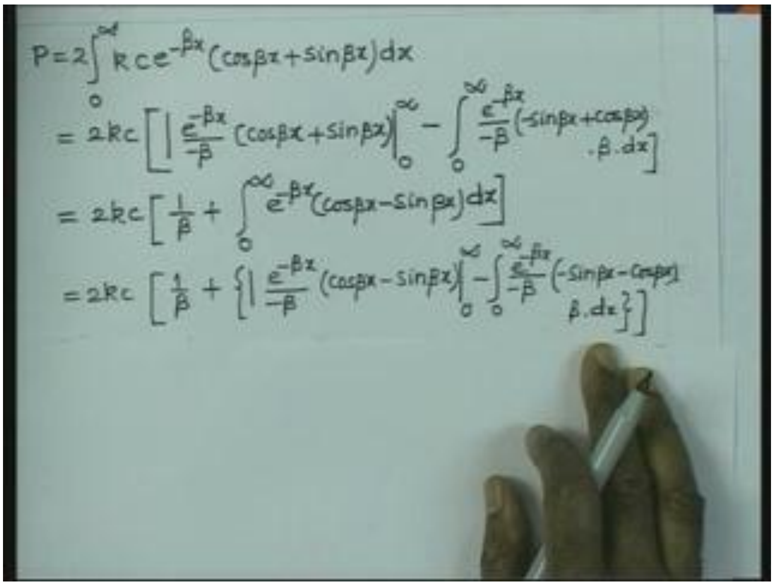

So, p equal to twice 0 to infinity k times ye to the power minus beta x cosine beta x plus sin beta x dx. So, this is a case of product of 2 function e to the power minus beta x and cosine beta x plus sin beta x. So, we can do it integration by parts. So, therefore, if we do that twice k c first you integrate this we take this as the first function. So, therefore, we will have e to the power minus beta x by minus beta cosine beta x plus sin beta x. This expression it is to be evaluated at infinity and 0 minus 0 to infinity integration of this first function differentiation of this second function.

So, sin beta x plus cosine beta x into beta into dx. So, that is the expression. And now if you try to evaluate this thing at infinity e to the power minus infinity will become 0. So, therefore, this gets knocked out and then if you try to evaluate at 0 this is unity and it is going to minus this quantity at x equal to 0. So, this will become unity and this is 1 by beta and this is going to be 1 this is 0. So, therefore, it is minus 1 by beta and that minus minus will be plus 1 by beta. So, this finally, becomes 1 by beta 0 to infinity minus e to the power beta x cosine beta x minus sin beta x into dx.

Now, we do you continue further. So, if you continue further this will give us twice k c1 by beta plus again we do the same procedure here. So, e to the power minus beta x by, beta minus beta cosine beta x minus sin beta x 0 to infinity minus 0 to infinity minus e to the power beta x minus beta minus sin beta x minus cosine beta x into beta into dx; that is the expression. Now, look at this, this 1 comes out to be this minus will be absorbed.

So, they will cancel and therefore, we are going to have 0 to infinity e to the power minus beta x by beta and this beta x going to cancel if this beta into sin beta x plus cos beta x and if you look into the initial expression. So, we have here. So, these expression it is similar e to the power minus beta x into cosine beta plus sin beta. So, therefore, this is 0 to infinity. So, therefore, we can write this thing this expression at p by 2kc.

So, this is nothing but, repetition of that and therefore, we can write now twice kc and this is going to give us again by similar concentration is going by 1 by beta only. So, therefore, we will have 1 by beta plus 1 by beta. So, let me just write that this is nothing but, e to the power beta x into cosine beta x plus sin beta x dx. So, that is the simplification. And therefore, now we substitute for this p by 2kc p by 2kc and therefore, we get the following. So, therefore, you get that it is 2kc p by 2kc and therefore, this is negative it will go to the other side and therefore, what we find finally, that let me write this 1 that this is nothing but, p by 2kc. So, this 2 k c will cancel this p can be taken to the left hand side and therefore, we have 2 p equal to 2kc into 2 by beta and hence, this constant is equal to P beta by 2k. So, that is the constant of integration for this particular problem.

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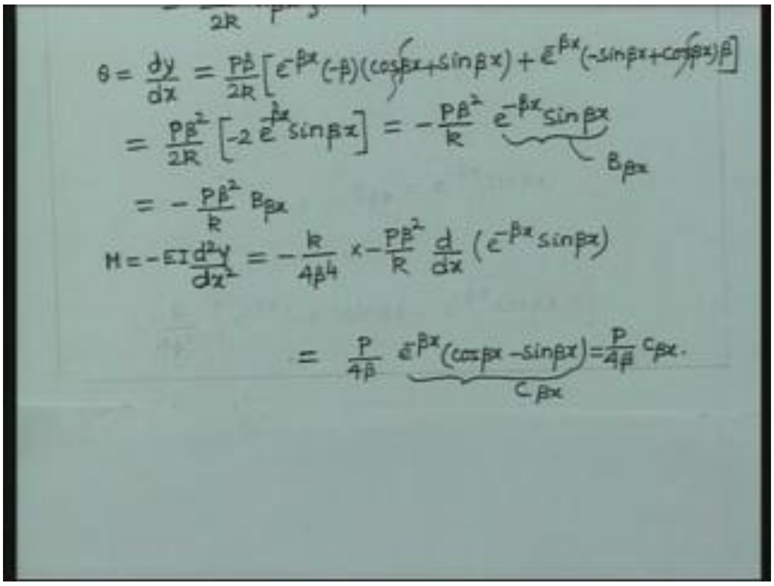

And hence we have finally, this solution y equal to p beta by 2k e to the power minus beta x cosine beta x plus sine beta x. So, that is the expression for y now, this is represented many time this is a standard function there are tables available for this function. So, it is many times represented as p beta by 2k into a beta x. So, this A beta x function is nothing but, this 1. So, that is what is A beta x. Once you have got the deflection it is possible to calculate the, you can calculate the slope bending moment and share force at every cross section; obviously, when you are interested in calculating the stresses due to bending moment you must have the value for bending moment.

So, it is what while trying to see how, the expression for bending moment share force at a particular section can be obtained. So, we will do that we write that theta slope is nothing but, dy dx. So, this dy dx simply we have to differentiate once. So, if we differentiate this e to the power minus beta x minus beta for the first function. And then cosine beta x plus sin beta x plus e to the power minus beta x into differentiation of this will give us sin beta x plus cosine beta x into beta. So, that is what is theta and this we can see that, there is some cancellation that this beta cosine cos beta x into beta is canceling with this term.

So, this 2 terms gets cancelled and therefore, we have p beta square by 2 k into minus 2e to the power beta x sin beta x. So, this is represented as P beta square by k e to the power minus beta x sine beta x. Again, this is again a standard function. In fact, we represent this thing as b beta x and therefore, with this symbol we have theta is equal to P beta square by k into b beta x. So, that is again the value of this function is available in tables. Now, bending moment we can write this bending moment as M EI d2 y dx square and therefore, it means that you have to differentiate and this you can write for EI as k by 4 beta is to be power 4. Since, k and beta EI is related.

So, we substitute that into minus p beta square, by k differentiation of this expression this expression here. So, therefore, it is of we will take directly this 1 and therefore, it is e to the power minus beta x into sin beta x this. And this finally, gives us which is quite simple. So, I will skip this step and let us write, this is equal to p by 4 beta e to the power minus beta x cosine beta x minus sin beta x. And here in again we write this function as c beta x and therefore, with that symbol it becomes p by 4 beta into c beta x. So, that is the value for the moment in terms of the sin beta x cos beta x e to the power minus beta x.

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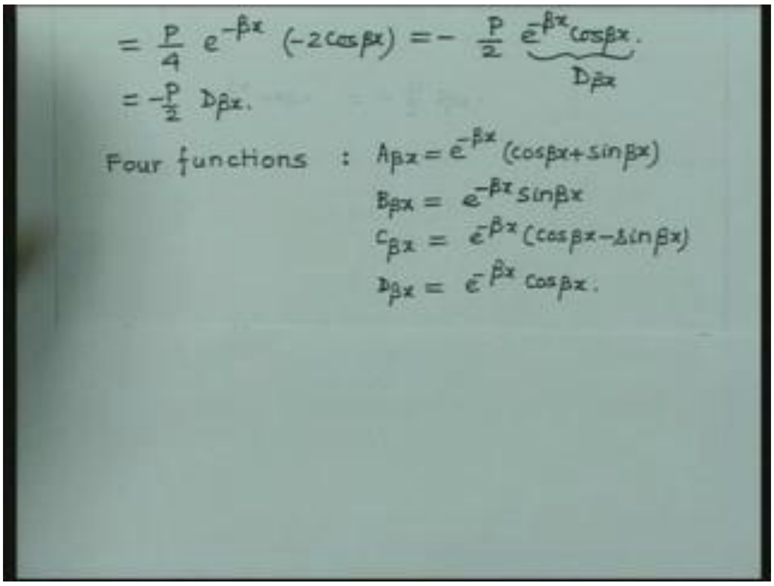

Let us try to see what is going to be the value for sheer force. Sheer force is equal to minus EI d3y dx x cube and we can again try to take the value for d2y dx square which we have already got. So, once we have to take the value of d2y dx square from this relationship. So, finally, this gives us it is nothing but P by 4 beta d dx e to the power minus beta x cosine beta x minus sin beta x. So, we have this expression and this is again root in function which you can easily differentiate, and finally what you get out of it is nothing but, P by 4 e to the power minus beta x minus twice cos beta x is equal to P by 2 e to the power minus beta x cosine beta x.

This we represent by d beta x and therefore, this expression becomes minus P by 2 d beta x. So, we have got all the quantity of interest deflection slope bending moment and share force. So, the expression finally, that we have obtained are meeting together a beta x is equal to e to the power minus beta x into cosine beta x plus sine beta x .B beta x is e to the power minus beta x into sine beta x c beta x is e to the power minus beta x into cosine beta x minus sine beta x d beta x is e to the power minus beta x into cosine beta x. mathematical tables are available where in the values of all this functions are given in terms of beta x.

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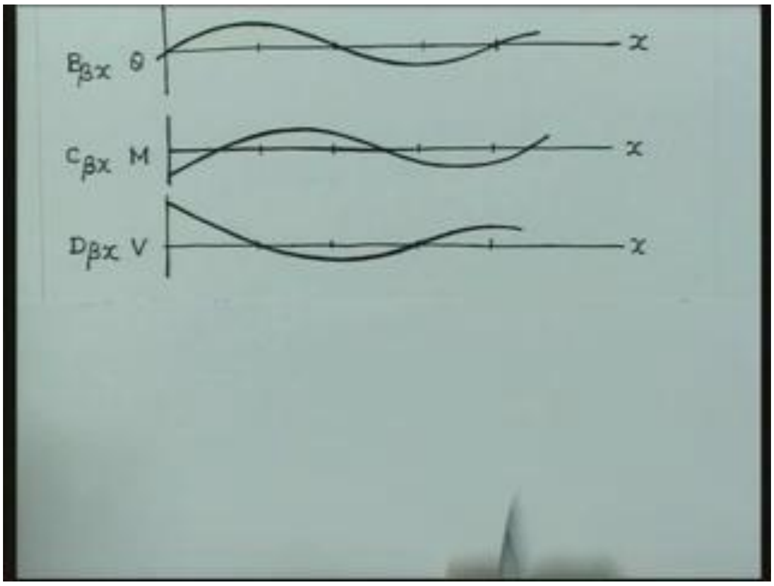

Now, this function will give us some idea if we look into them about the variation of the deflection slope bending moment and share force. So, you see that this functions are oscillatory and they have amplitude of the function gradually reduces So, therefore, if you consider the function y is directly related to A beta x. So, the plot of y, so also a beta x is like this; you expect that there will be large deflection and this point of application of the load it reduces. And then it really tries to lift up the support and then again it goes on decreasing that the deflection is actually going to positive.

So, here it is positive, negative, positive and the period of the function is 2 pi by beta and this is pi by beta. So, therefore, you find 0 to be somewhere above 3 pi by 4 beta at this is 0 and here it is 7 pi by 4 beta another 0. So, the nodes are here and here it is 3 pi by 4 beta and this 1 is 7 pi by 4 beta. The variation of the slope, this is a symmetric function a beta x are deflection is a symmetric function as we expected and this b beta x is anti symmetric function and its shape is going to be like this. And it has a node at pi by 4 another node at 2 pi by this is pi by beta and another node at 2 pi by beta. And this amplitude is gradually reducing as you move higher from the origin.

So, that is the variation of slope and variation of bending moment, bending moment is like this here it is a symmetric function and this symmetric function is going to have value like this and its having some amplitude there and again it changes. Likewise, the variation of the share force is anti symmetric and this is how it varies. So, you find that 1 function is symmetric the next derivative is anti symmetric and it goes on. So, that oscillation takes place. Now, we will try to go for consideration of a beam which is semi infinite it is not infinite it is semi infinite and try to see how we can find out the deflection bending moment variation. So, also variation of share force and the slope.

So, the beam is like this with rigidity EI and it is subjected to now, loading concentrated loading P 0 at origin and also bending moment which is positive bending moment. This share force is it is a negative share force because you consider on this phase the negative phase share force is acting upward is positive. So, this is a negative share force. Looking at the fact that the deflection should gradually diminies is the go towards infinity. And we can write the governing equation is going to remain the same and therefore, we can write the solution for this problem also in a similar fashion. As we have done in the other case.

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$$y = e^{-\beta x}(c_3 \cos\beta x + c_4 \sin\beta x)$$

$$\frac{dy}{dx} = e^{-\beta x}(-\beta)(c_3 \cos\beta x + c_4 \sin\beta x) + e^{-\beta x}(-c_3 \sin\beta x + c_4 \cos\beta x)\cdot\beta$$

$$\frac{d^2y}{dx^2} = e^{-\beta x}\beta^2(c_3 \cos\beta x + c_4 \sin\beta x) + e^{-\beta x}(-\beta^2)(-c_3 \sin\beta x + c_4 \cos\beta x) + e^{-\beta x}(-\beta^2)(c_3 \sin\beta x + c_4 \cos\beta x) + e^{-\beta x}\beta^2(-c_3 \cos\beta x - c_4 \sin\beta x)$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} = c_3\beta^2 - 2c_4\beta^2 - c_3\beta^2$$

For the semi infinite beam we can write the deflection in the form. Let us write the deflection equation in the form y e to the power minus beta x into c3 cosine beta x plus c4 sin beta x. If I now, try to write dy dx. So, we have product of 2 functions is not difficult to write the derivative. So, it will be e to the power minus beta x minus beta into c3 cosine beta x plus c4 sin beta x plus you can now keep the first function same we will differentiate now, c3 sin beta x into beta c4 cosine beta x into beta we can keep it outside. Similarly, if we now write d2y dx square this is going to give us e to the power minus beta x and this minus beta minus beta will give us beta square into c3 cosine beta x plus c4 sin beta x plus e to the power beta x. If you take the beta which is going to come from the differentiation of this we can take it together. So, therefore, this will be minus beta square into c3 sine beta x beta already we have taken out plus c 4 cosine beta x.

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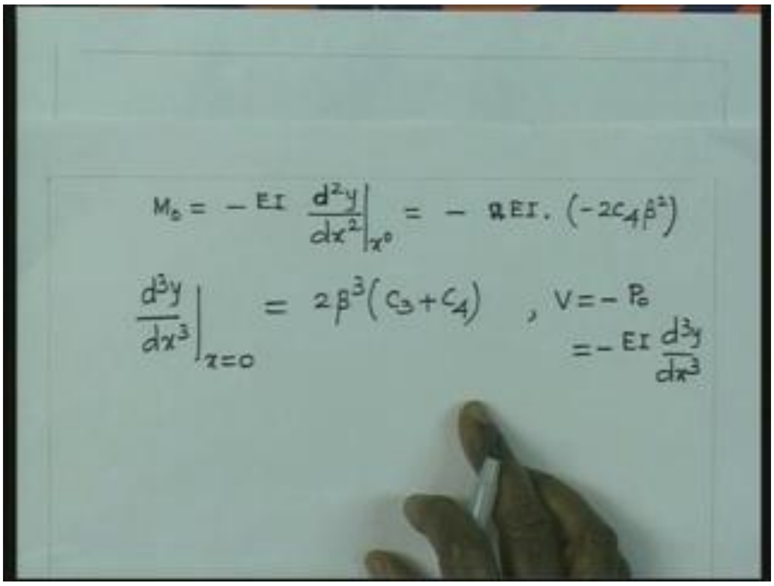

So, that is from the first expression similarly we will have from the second 1 e to the power minus beta x this will give us minus beta that minus beta will get combined. So, it will be beta square into c3 sin beta x plus c4 cosine beta x plus e to the power minus beta x and the differentiation of this will give us another beta which can be combined together it will give us beta square minus c3 cosine beta x minus c 4 sine beta x.

If you now consider that x equal to 0 the derivative at x equal to 0. Then we have this terms will become unity and then all the cosine terms will give us contribution c3 beta square and here it is c4 minus c 4 beta square and here also you are going to get c4 beta square. So, therefore, it is twice c4 beta square and here it is minus c3 beta square. So, c 3 cancels out and therefore, we can now have only minus 2 c4 beta square and we have the value M 0 is equal to minus EI d2 y dx square at x equal to 0. And therefore, now we can write that that is equal to minus EI into minus twice c4 beta square.

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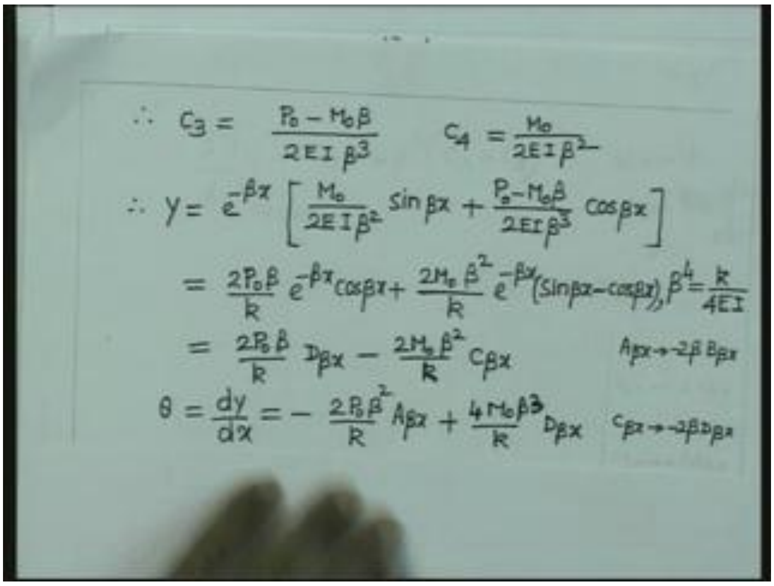

So, we will get c4 in terms of M0, similarly if you calculate d3y dx x cube at x equal to 0 this will give us 2 times beta cube into c3 plus c4, and since we have v at x equal to 0 is equal to minus P 0 and that is equal to minus EI d3 y dx x cube. Then we can relate this and this will give us finally, an equation involving c3 and c4 and we can solve for c3 now, since c4 is known.

So, finally, what we have is that c3 is equal to P0 minus M0 beta twice EI beta cube and c4 is equal to M0 twice EI beta square. And we can now, write the deflection is nothing but, y equal to e to the power minus beta x into M0 twice EI beta square sin beta x plus P0 minus M0 beta twice EI beta cube. So, we are substituted value of c3 and c4 this into cosine beta x. So, that is the expression for the deflection and we can recombine this and write the value in terms of k because beta 4 is nothing, but k by 4 EI.

So, if you make use of this then we can write this y as twice P0 beta by ke to the power minus beta x cosine beta x plus twice M0 beta square by ke to the power minus beta x into sin beta x minus cosine beta x. So, that is the value for y and this can be written in terms of the standard functions that we have introduced twice P0 beta by k d beta x into d beta x minus twice M0 beta square by k into c beta x. We can get the expression for theta is equal to dy dx that is equal to twice P0 beta square by k. If you differentiate this it will give us minus 2 beta b beta x. So, therefore, this can be written as minus 2P0 beta square by k that is a beta x once you differentiates it. It gives you twice beta into b beta x that is what I am adopted here.

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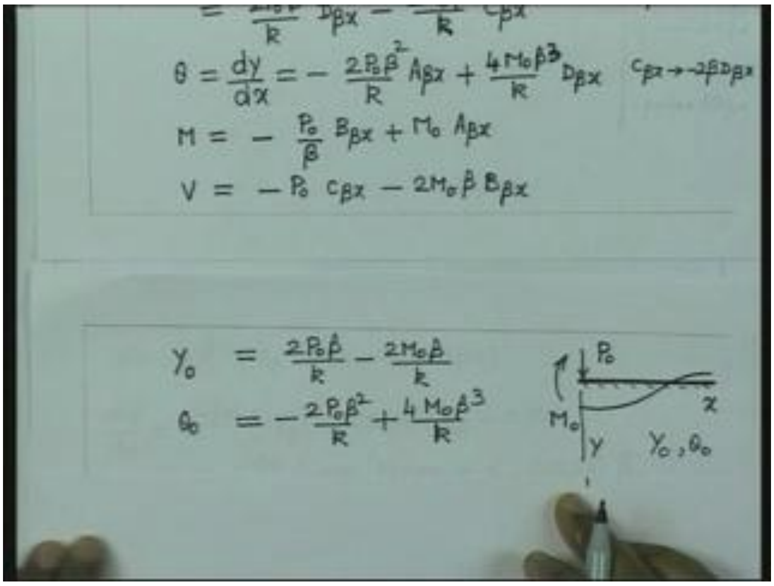

This also give us differentiation of this, will give us c beta x differentiation will give us minus twice beta d beta x and therefore, this finally gives us twice M0 beta cube by 4 times M0 beta cube into d beta x.Similarly, we can write now, M. So, I can take now double derivative and then write finally, expression for M is going to be P0 by beta this a beta on differentiation will become b beta x plus M0 d on differentiation will become a beta x. Similarly, the share force is nothing but P0 c beta x minus twice M0 beta into B beta x.

So, these are the expressions for deflection slope, bending moment and share force. If you evaluate the value of deflection at x equal to 0 and slope at x equal to 0. Then we if we represent the deflection at x equal to 0 as y0 and slope as theta 0. Then from the expressions that we have derived we get y0 is equal to it will be 2 times P0 beta by k minus 2 times M0 beta by k and theta 0 is equal to minus twice P0 beta square by k plus 4 times M0 beta cube by k. These are the deflections and slope at the origin, we will see how these vessels can be made use of in estimating the forces that developed at the discontinuity of pipes or may be junction of cylinder and the head of the cylinder.