Advanced Strength of Materials
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Lecture – 34

Today we are going to talk about bending of plates. We have already studied bending of beams, bending of card beams. We will now like to see if the component is like a plate it could be in the form of rectangular plates, it could be in the form of circular plates. When they are subjected to transverse loading, how do they undergo deformation and at the same time? How stresses develop? That is what we are going to study today.

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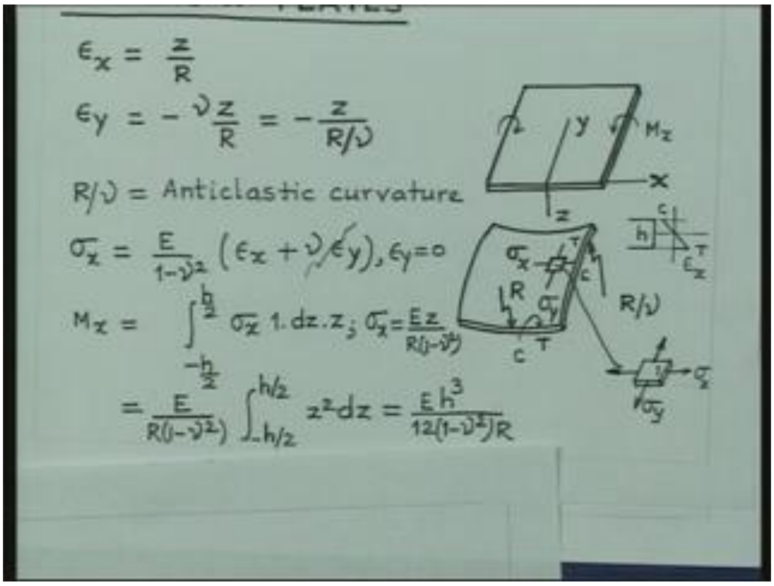

Let us consider a rectangular plate which is very thin. The height is small compared to length or the width. When, you try to apply bending load to such a plate. Let's, say we apply the bending load in the X Z plane like this and we would like to indicate this moment by M x. So, when you do that what you find is that we are trying to have bending generated like this. The surface should have curvature like that and you find that.

Because, of that curvature you are going to get some radius of curvature in this direction, while you are having this curvature you can appreciate that you will have tensile stresses at the bottom of the plate. And compressive stresses at this top fiber. So, compressive at the top fiber and tensile stresses at the bottom fiber. When the plate dimension here is substantial what you find that when, it is trying to get stretched at the bottom fiber in the direction perpendicular. In this direction there will be contraction and since, there is compressive at the top fiber in this direction there will be extension in the direction at the top.

So, that is what you see that you get contraction at the bottom fiber in this direction. So, therefore, you are going to get contraction here and the top fiber is going to now get stretched. Because, of this you find that the edges of the plate, gets a curvature like this. It does not remain straight, so you have tensile stresses there and compressive stresses there. So, it gets actually fold rather bend like that. So, there is a curvature demonstrated by the plate towards the edges and this curvature gradually, dies down as we move towards the centre of the plate.

So, we can find out this curvature very easily. If you consider that, the bending is producing a strain let us, say epsilon X. So, because of this I have a I have the bending strain z by R there by meaning that, for positive value of z we are going to have positive strain and we try to consider that, this X axis is weight at the centre of the plate. Or XY plane coincides with the centre of the plate and then we have this as a distribution of the strain.

So, what you are trying to say that this is the height. Let us, say this is the height h then we have a strain distribution which is linear. So, if you consider epsilon x to be plotted in this direction. So, this is X axis so then in that case I am going to get tensile strain here, compressive strain there. So therefore, I have this compressive strain and tensile strain at the bottom.

Now, the strain which is going to develop because, of the strain in the X direction and that is going to develop in the Y direction it will going to be nu times the strain in this direction. So, therefore, we can write this thing as epsilon Y is equal to minus nu Z by R which you can write as minus Z by R by nu. So therefore, you find that there is this is a curvature, so this is again another curvature.

This curvature R by nu we call it anticlastic curvature. So, it is known as anticlastic curvature. So, this radius of curvature that is c at the edges is going to be of value R by nu. So, what we find here is that as you consider an element on the surface of the plate you find that there will be stresses in the X direction, there will also be stresses in the Y direction. Now, this tendency of curving like this is prevented towards the centre of the plane. So, there is no such curving at the centre of the plate and you find that the stresses are going to actually come up here in the Y direction as well. So, what you find that an element of this type. So, if you draw that element you will have just tell here.

This element, this element is going to be subjected to stresses in the X direction. It is also going to be subjected to stresses in the Y direction. So, it is no longer the case of Uni-direction loading as in the case of beam bending. In the case of beam bending you have noted that the stress sigma Y zero. Now, the stresses are acting in the plane of the element and therefore, it is generally considered to 2 dimensional problem.

Then, we can now write since there are 2 strains: we can write sigma x is a plane, stress sort of situation. So, we can write now sigma x is equal to epsilon x plus nu times epsilon Y. So, that is the expression for the stress sigma x. Now, we have already mentioned that this tendency of anticlastic curvature appearing is now, there towards the centre and towards the centre you will find that the stretching of the plate or contraction of the plate in the y direction is 0

So, therefore, epsilon Y is 0 and hence, we get the stress to be equal to E by 1 minus nu square into epsilon x and since, epsilon x is equal to z by R. So therefore, sigma x is equal to E by 1 minus nu square into z by R So therefore, this term is getting knocked out. Now, if I try to consider since the stresses are varying in the Z direction I can find out, what is the moment produced by these distributed stresses on this edge per unit length in this direction.

So, we can now consider that let us consider unit length in this direction and height in h. So, what we get is that moment produced. So, we have sigma x finally, is equal to E z by R into 1 minus nu square. So, if I want to calculate the moment produced per unit length in the Y direction. Then, that moment we indicate by Mx. This is going to be therefore, given by integration of sigma x multiplied by 1 into dz into z

So, this integration of this quantity will give us the moment per unit length. So, this is a simple integration, so once we substitute the value it come out it takes the form z square dz and which is simple integration and therefore, we have Eh cube by 12 into 1 minus nu square into R. You remember in the case of beam we have Mx is equal to E by R into I.

So, we have similar expression here that Mx is equal to E by R into this quantity which is nothing but I which is equivalent to I. See you have a plate of I 1 and width is 1. So therefore, the moment of inertia should have been Eh cube by 12. So, 1 into h cube by 12, so 1 into h cube by 12. That is what is in the case of beam, but here we have an extra effect which is due to the plate action which is 1 minus nu square.

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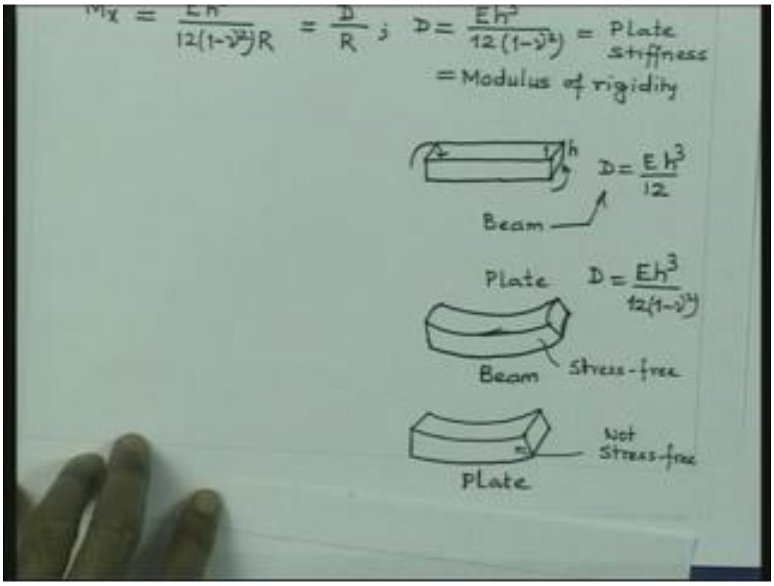

And this quantity we try to write like this: Mx is equal to Eh cube 12 into 1 minus nu square into R and we write this thing as D by R, where D is equivalent to E into I and D is equal to E h cube 12 into 1 minus nu square. And this is known as plate stiffness Modulus of rigidity. So, let’s just have something to consider here. If I had a beam like this.

This is 1 and this height is h and we are trying to apply bending moment like this. So, if we apply bending moment like this. Then, in that case you certainly would have this D. D as D for this case is nothing but E into I which is Eh cube by 12. But on the other hand, if this is a part of the plate then in that case we find that the these faces, these faces are not free of any stress and the plate gets extra rigidity and in the case of a plate.

So, this is beam and in the case of a plate, we get extra rigidity. And this D is now, equal to Eh cube by 12 into 1 minus nu square. So, there is Poisson’s ration is always less than 1 and therefore, you get extra rigidity. And that is due to the fact that, if you had the beam this beam would have bent like this. It would have taken a shape, it would have taken a shape like this. And there would have been anticlastic curvature, but in the case of, so this is again in the case of beam. But in the case of plate what we will see is that, this does not happen. It just gets bent, so it gets bent like this. This is the case of plate element and that anticlastic curvature is prevented. And that is why, there is some hysteresis. There are this face is now stress free. Whereas, in the case of plates they are not stress free and this is the reason for getting this extra rigidity.

Now, with this we can go ahead in deriving the moment curvature relationship in the case of rectangular plates. So, you have considered loading about 1 axis of the plate it could as well have loading about the both the axis. So, lets now consider bending about both the axis. So, we will consider a plate like this. These are our axis and this is as if the central plane of the plate. We are applying Mx bending moment and this Mx is per unit length.

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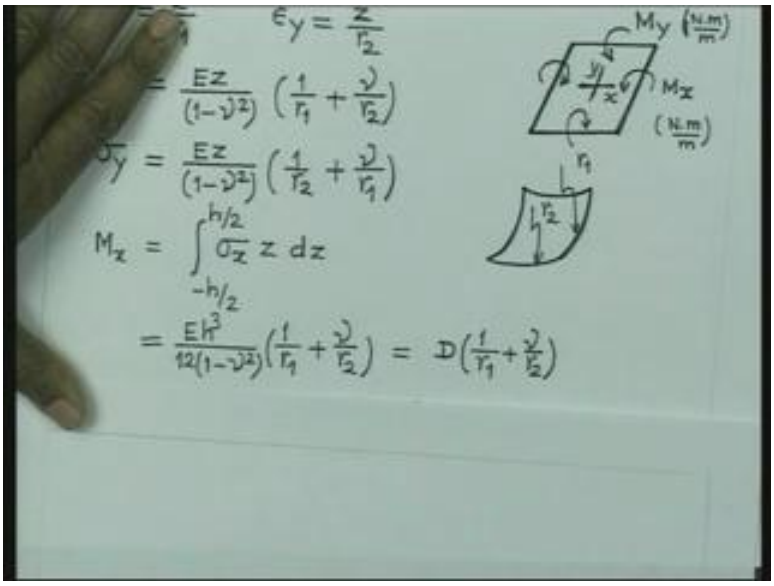

So, generally its unit will be Newton meter per meter. Similarly, M y we apply in this direction, which is again having unit Newton meter per Meter. So, we can now write if the curvature due to M1 let us, say that we have curvature equal to r1 and the curvature that we get in this direction is equal to let us say r2. Then, we can write the strain in the X direction is equal to z by r1 and similarly, epsilon Y is nothing but z by r2.

And hence, we can now write it is a plane stress problem. And therefore, we can write epsilon sigma x is equal to Ez 1 minus nu square 1 plus r1 plus nu by r2. And similarly, we can write sigma Y equal to Ez 1 minus nu square 1 by r2 plus nu by r1 So, it is nothing but sigma x is equal to E by 1 minus nu square into epsilon x plus nu times epsilon y. Similarly, sigma y is equal to E by 1 minus nu square into epsilon y plus nu times epsilon x. Now, we can calculate the moment per unit length in the X direction or in the X plane. So, we will have moment in the X plane is nothing but minus h by 2 to plus h by 2 sigma x into Z dz.

So, this will give us Eh cube by 12 into 1 minus nu square 1 r1 plus 1 by r2 and this is nothing but if you represent this quantity by as modulus of rigidity D. Then, this is nothing but 1 by r1 plus nu times 1 by r2. And by similar consideration we can write now M y is equal to D into 1 by r2 plus nu times 1 by r1.

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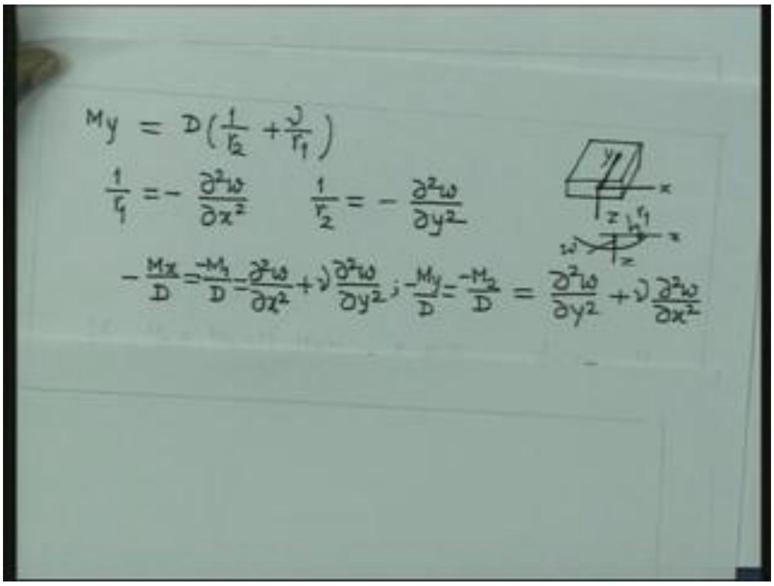

So, these are the 2 moment curvature relations. Now, because of our coordinates we have the coordinates, we have selected the coordinates which is like this. In fact, our X axis is like this, Y is directed like this and z is in this direction. So, therefore these are the 3 axes and the plate is curving in the negative z direction. So, that is what we have considered to be positive. It’s not doing like this and therefore, we would have the relationship between the curvature and the radius of curvature will be given by for small deformation 1 by r1 is equal to delta 2 w delta x square.

So, what is we are going to have is that our this is what is r1 and since, the deflection is here in this direction and this is x. So, therefore, we get double derivative of this deflection w with respect to x that gives you 1 by r1. Similarly, 1 by r2 is equal to minus delta 2 w delta y square. So finally, if we substitute all this we find that minus Mx by D. We can consider it to be let us, say minus M1 by D is equal to delta 2 w delta x square plus nu times delta 2 w delta y square. Similarly, we can write. Minus My by D is equal to minus M2 by D is equal to delta 2 w delta y square plus nu times delta 2 w delta x square.

So, these are the moment curvature relations. So, you have the relations in the case of beam bending it was M by Mx by I minus Mx by I was nothing but delta 2 delta x square, but in the case of plates you are going to get involvement of the other curvature as well.

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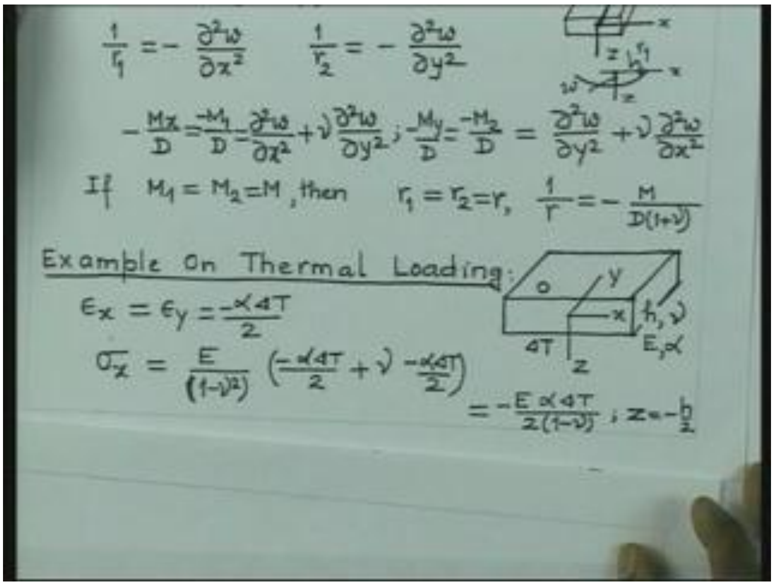

Now, for the simple case if you consider the situation like this If M1 equal to M2. And let us, say that is equal to M then we will find that r1 is equal to r2 and let us say that is equal to r then we will have 1 by r is equal to minus M by D into 1 plus nu. So, for the symmetric deformation around the centre point you are going to see this sort of relationship. Now, let us see 1 simple problem. Thermal.

Let us, consider a plate and let's see that if I have a temperature differential between the 2 faces of the plate between the top and the bottom. Let us, say the temperature differential this is maintained at 0 and this face is having a temperature of delta T. And therefore, the temperature is varying from 0 to delta T and hence, with respect to the centre this is at minus delta T by 2 and this is at plus delta T by 2.

So, let us, consider that height is equal to h and material constants are E and alpha and of course, we will also have the Poison's ratio nu. Now, if I do this then the plate will undergo strain in both the directions without any distortion. So, if you consider that this is a plate and this is your Y direction. Then, we are going to get the strain here at the top.

It is going to be let's, say in the X direction and that is going to be nothing but minus alpha delta T by 2 and similarly, in this direction also we are going to get same strength. So, epsilon x, epsilon y is equal to minus delta T by 2. So, that is at z is equal to minus h by 2. Now, from the relationship that we have got that sigma x is nothing but E by 1 minus nu square into epsilon x.

So, it is minus alpha delta T by 2 plus nu times minus alpha delta T by 2. And therefore, what we find is that this is nothing but minus E alpha delta T this 1 plus nu will cancel and therefore, we will be left with 1 minus nu. So, this is the stress that is going to be at z is equal to minus h by 2. So, look at this if we have raised the temperature here then in that case this is going to expand and this is going to have because of this expansion it is going to contract.

So, it will be actually trying to contract there and therefore, we are going to have a stress of this magnitude. So, the stress is going to be of this magnitude. Now, let us try to consider more useful examples which we come across in very often in pressure vessels. So, in the case of pressure vessels you will find that the ends of the pressure vessels are meant to be made up of plates like component. Then, another action of the internal pressure they are going to be subjected to transverse loading.

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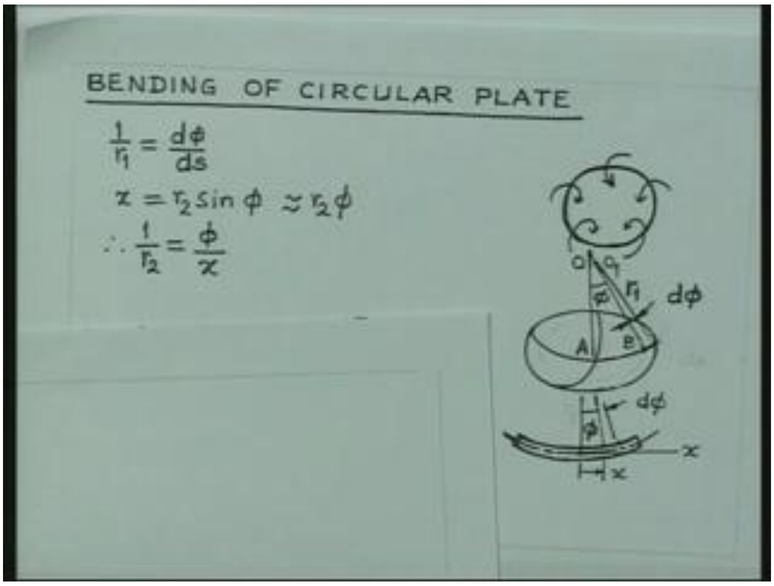

So, we would like to look into such problems, in the next portion of this particular lecture and we also continue probably in the next lecture. So, now if we consider that plate is like this and it is subjected to bending let's say of certain intensity, constant intensity. So, another action of this bending we will find that the plate is going to take shape like this. It will become curved like this.

So, this is going to curve this way and this radius will curb this way. And we should now consider a strip near the centre its going to get bent like this. Now, let us see what are the curvatures here? So, if you consider now a portion. Let's say, we consider a portion between this point and this point of the plate it was initially, horizontal it has now got rotated and this is going to actually, give rise to deformation like this in this direction. So, if we consider now the curvature which is going to come up here because, of this d phi.

Let's say we will consider a an element at a distance of angular disposition let's, say phi then this is d phi. Then, we will have some curvature coming up. This curvature where this angle is d phi and this curvature let us say, this curvature which is demonstrated here let us say, this is equal to r1. And this curvature is going to be given by rate of change of angle with arc d phi d s. Now, other curvature, so this curvature is going to be in this direction. We are also going to have 1 curvature in this direction and that is we consider that curvature to be r2 and that curvature we can calculate. You see here that this particular section 1 initially vertical. So, therefore, at a distance x we have this section it was vertical. Now, it has taken an angular orientation of phi. So, you can write now that x if the curvature is equal to let's say this portion has taken a curvature equal to r2 then this x is nothing but r2 sin phi. And since, the plate deformation is very small this angle is going to be this radius of curvature is going to be very large and this angle is very small. So, therefore, we can write this thing as well as approximately equal to r2 into phi.

So therefore, this curvature is nothing but 1 by r2 is equal to phi by x that is the second curvature. So, we have got now both the principle curvatures of an element of the plate which is located somewhere here. So, we have taken the point A and let us say that this is the point B and the centre of curvature for this portion is O. Whereas, the centre of curvature in the plane XY plane, XZ plane is nothing but this r1 the centre is somewhere here and which is made by this radial line and that radial line.

So, we have got the 2 curvatures. So, with this definition it's now possible to go in for writing the equilibrium equation for the case of circular plate element. So, let us now consider a portion of the plate and it's loading and let us, say that we have the loading which could be transverse loading on the plate or it could be some bending moment acting on the plate. Now, let us confine ourselves to a portion of the plate very close to the, engine of the close it can be anywhere located over the plate and let us, consider an element which is like this.

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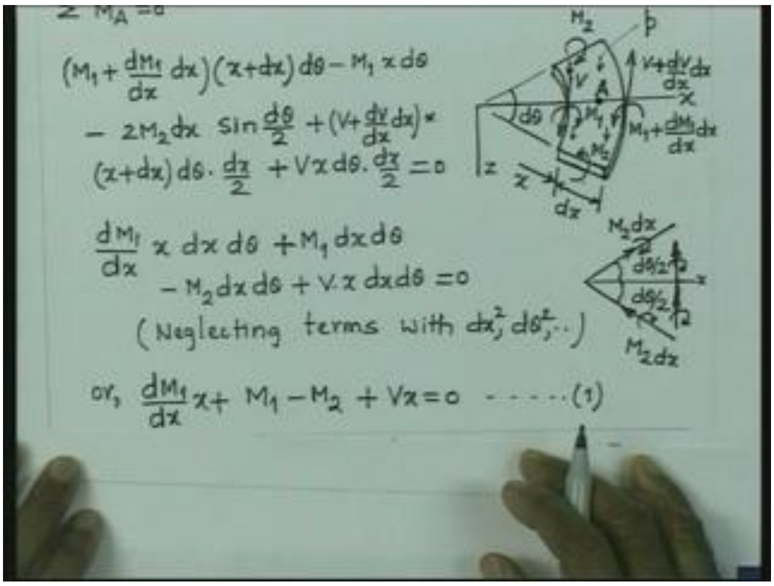

So, this is a portion of the plate that we would like to focus our attention and we will consider now our radial directions to be x and then this element is symmetrically placed above the X axis and let us say that it is making an angle of d theta at the centre So, this is our Z axis. Now, bending moment is not going to remain constant in general and therefore, let us consider that on this face theta face, which is perpendicular to radial face this is radial direction.

So, this face is perpendicular to the radial direction, so on this radial direction. Let us, consider that moment part unit length of this circumference is equal to M1 acting here and this bending moment as we move to the other end which is at a distance of let's say, delta x from the first 1. So, let us say that this is dx and this location is x. So, now here we are going to get some bending moment per unit length of the circumference. It need not be same as the other 1.

So, we can expect a change in the value of the bending moment to have a general situation. We can write now, the bending moment per unit length here M1 plus d M1 dx into dx at this stress. Similarly, we can consider now the bending moment per unit length here in the theta face which we will try to consider M2 per unit length. And for the case of axis symmetric loading if you have the bending moment per unit length here M2 we would expect the same thing to remain here, so this is M2. So, therefore, for an axis symmetric loading we are going to get the bending moment to be disposed like this or coming up like this.

Now, there can be shear forces. So, we can now consider on this face that shear force is equal to V and the shear force which is going to come up on this face is equal to V plus dv dx into dx. So, that is the shear force. These are the loading and of course, we can have our external loading which could be like this and the centre of the this vx a point let us consider A. So, now if this these are the forces acting on the plate element it must be equilibrium. And let us see, what you get by considering the equilibrium of this element under the action of the various forces.

So, we have got all the forces acting on the element. Now, can you just think about what about the shear forces coming up on these faces? There can't be any shear force coming on this faces because, the moment is not varying in the theta direction and therefore, there cannot be any shear deformation of this radial faces. The radial plane will remain always radial and therefore, there is no shear deformation.

Now, let us consider the moment of all the forces above the centre point of this element which is of dimension d theta and d x. So, we will consider now moment of all the forces which are shown here about A that is equal to 0 and now first of all let's consider, this moment let us say that that is positive and it is M plus d M1 dx into dx acting over a distance of x plus dx into d theta and of course, height if you consider to be unity that will be also possible.

So, you can now write M1 plus dM1 dx into dx x plus dx into d theta. I would like to repeat this point this is the moment intensity per unit length of circumference. This is the length of the circumference. So, therefore, we do not need this height, we can get the total moment coming from this edge by this expression and the moment which is counteracting that is nothing but M1 multiplied by x d theta. These are the 2 contributions.

Then, we have see this M2 if I show it as a vector. So, this vector if I consider this face it is like this and the other face is exactly oriented like this and this is my x direction. This angle is d theta by 2, this is also d theta by 2, this M2 we can show it as a moment vector, moment vector like this. So, this is the moment vector and therefore, this is M2 into dx total moment vector is nothing but that similarly, in this direction we have total moment is nothing but M2 into dx.

Now, we are trying to look for the component in this direction and therefore, you see it is going to give rise to component like this in this direction and which is if this is positive then the moment that will come out of this 2 will be like this, it will be directed like this, it will be like this. So, and. So, for this 1 we are going to get continuation like this. So, this is M2 dx sin d theta by 2 acting like this which is then opposite, so it is negative.

Similarly, for this also M 2 dx sin d theta by 2 negative, so therefore, both are additive. So, we will have minus 2 times M2 dx sin d theta by 2. And then we will get the moment of this about this point, so also moment of this V about this point. So, if I consider first this 1 it is nothing but V plus dv dx dx and that is the intensity and momentum is nothing but we have the momentum it of course, this the distance over which it is acting is nothing but x plus dx into d theta. So, into x plus dx into d theta and the momentum is nothing but dx by 2 and similarly, for this 1 it is V and it is producing a moment which is going to be same sense as this 1.

So, therefore, Vxd theta that is the total force due to the shear force and its momentum is dx by 2. So, that is the sum total of all the continuations. Now, if I see that cancellation this is going to cancel with this product and now if we just consider the term which are linear in dx and d theta then all the terms that will be remaining are the following x dx d theta plus M1 dx d theta minus M2 dx d theta.

We write this thing approximately equal to d theta by 2. So, that will give us this expression and then here we have Vx dx d theta equal to 0. So, this is neglecting terms with dx square d theta square. We have also had the pressure, if this is the pressure P now, that pressure will be distributed all over the plate and each centroid will pass through the point A.

Therefore, there won't be any continuation of the pressure distribution towards the moment that is why we have not incorporated in this expression. So, finally, if we do the cancellation of dx d theta which is common then we have dM1 dx x plus M1 minus M2 plus V x is equal to 0. So, this is the equation which relates the moment to the shear forces in the case of a plate. We can make the substitution for M1 in terms of the curvature terms r1 and r2.

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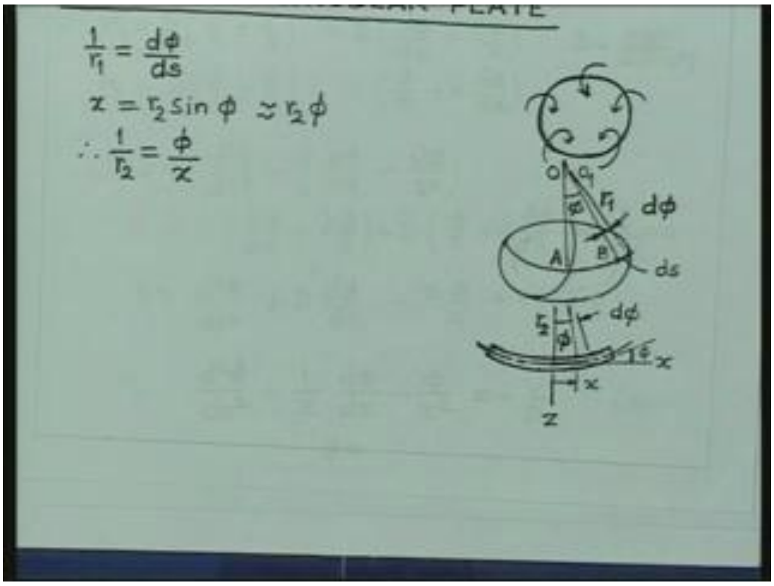

So, we can now write, before I do that I would like you to again get back to this sketch here, in this sketch we have shown that this is the curvature in the plane if you consider that this is Z direction, so this is Z direction. So, in the XZ plane are considering the curvature to be r1 and in the this is Y direction. So, therefore, in the yz plane the curvature is r2. So, this is the curvature r2 which is phi by x and this curvature r1 is nothing but if this portion is ds and this angle is d theta d phi then d phi ds. So, we have already got those.

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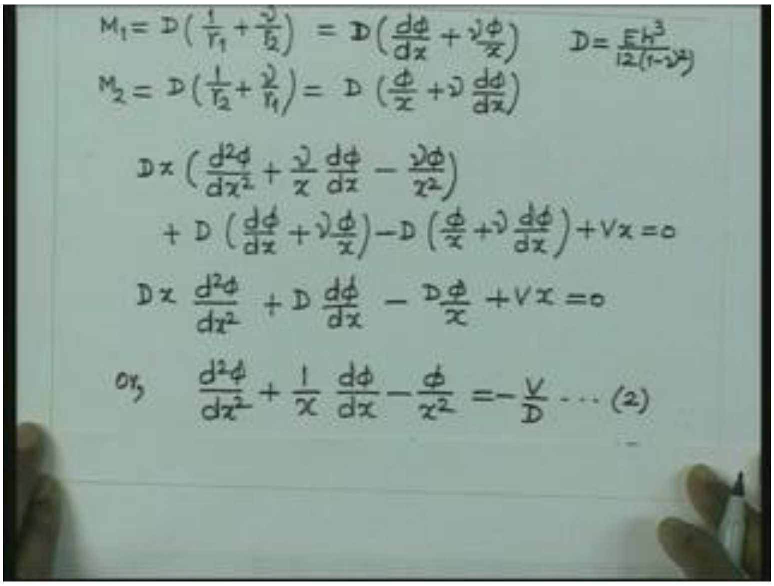

So, now let us write M1 is equal to D1 by r1 plus r2 which is nothing but since 1 by r1 is d phi dx plus r2 is 1 by r2 is phi by x. So therefore, this is nu phi by x. Nu is nothing but Poisson's ratio and therefore, this D is equal to Eh cube by 12 into 1 minus nu square. Similarly, you can write now M2, M2 is equal to D1 by r2 plus nu by r1 and this is nothing but D phi by x plus d phi dx.

So, we can now substitute this expression for M1 and M2 in the moment shear force relationship and therefore, we have now x D M1 dx it gives us D2 phi dx square plus nu times. So, you have to take the derivative of this with respect to x. So, it is nu by x d phi dx minus nu phi by x square. So, that is x D M1 dx. Then, we had M1 minus M2. So, therefore, we will have. Plus D d phi dx plus nu phi by x minus M2 which is again D phi by x plus nu times d phi dx plus Vx is equal to 0.

Now, this we find finally Dx d2 phi dx square plus D d phi dx minus D phi by x plus Vx is equal to 0. Which we can rewrite in the form d2 phi dx square plus 1 by x d phi dx minus phi by x square is equal to minus V by D. So, this is the equation of bending of plates, Circular plate particularly which has a symmetric loading. So, if you are if you now look into this phi is nothing but see here in at angle phi where considering the coordinates to be like this. This is x, this is z and this is angle phi.

So therefore, this is also angle phi and in fact, delta w delta x is going to be minus phi because, positive slope will be in this direction. So, since this is phi is considered to be positive here therefore, phi is nothing but minus dw dx. So, we have dw dx and therefore, you see that deflection of the plate is given by this equation. So, we will consider further about this particular equation and we will try to apply it to few cases and see how we can calculate the deflections and also the stresses.