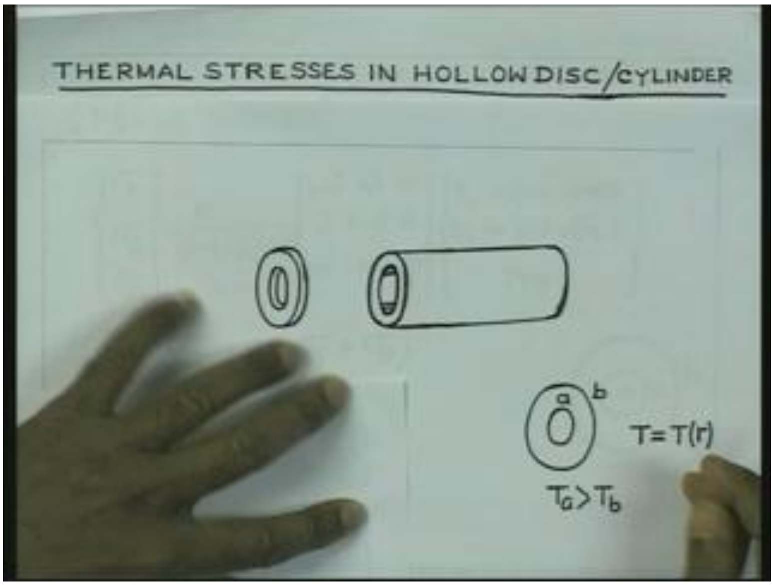
Advanced Strength of Materials
Prof. S. K. Maiti
Department of Mechanical Engineering
Indian Institute of Technology, Bombay

Lecture - 31

In our earlier lecture, we have seen how we can calculate the thermal stresses in a rectangular plate; with symmetry and asymmetric distribution of temperature in the heat direction. We will find that there are many applications, where in you are going to have cylinder or cylindrical vessels, which will consist of some fluid at elevated temperature or at some 0 temperature. As a result of which, there will be variation of temperature in the radial direction. You have already come across the problems of heat exchangers, where the temperature distribution in the radial direction is well known to logarithmic distribution. We would like to look into, how the temperature distribution in the radial direction will give rise to thermal stresses.

(Refer Slide Time: 01:43)



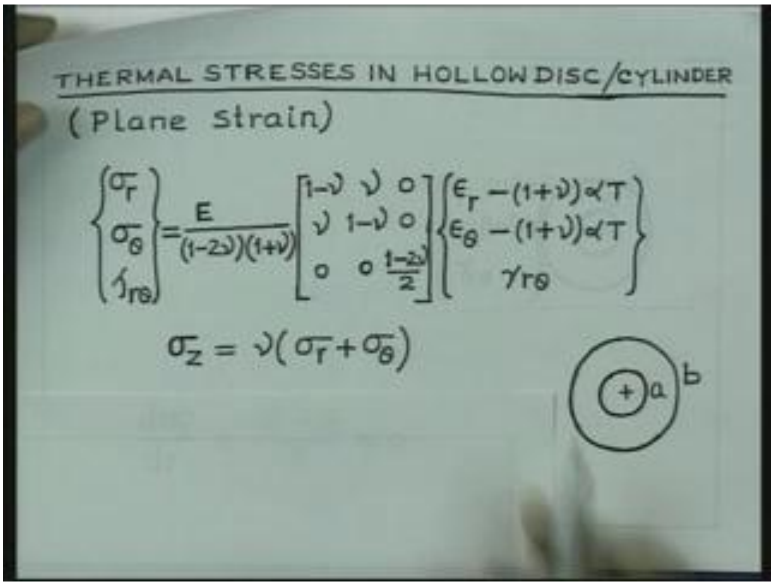
We would like to consider therefore, problem to be thermal stresses in hollow disc or cylinder. You could have a situation of this type or a situation like this. That is a pressure vessel or a cylindrical vessel, which is going to have temperature variation in the radial direction like this. And we would like to calculate the stresses. If we calculate the stresses for a situation like this, where there is internal radius and external radius.

If you are interested in finding out the stresses for internal radius equal to 0, that will give you the solution for thick disc or thick cylinder. So, therefore or rather solid cylinder and solid disc. So, therefore, we would like to consider the case of hollow disc and hollow cylinder. And we have seen earlier, that when you have in the case of hollow disc like this. Then you can have a situation conforming to plane stress, but when you are going to have a cylindrical vessel like this.

You are going to have a condition, which will conform more to the plane strain condition. We will be more concerned about this case. I would like you to think about, that if you have a temperature variation in the radial direction like this. Let us say that, there is high temperature at the inner wall and low temperature at the outer wall. So, what I mean is this, that we have T a greater than T b. I would like you to think for a while, what will be the stress there.

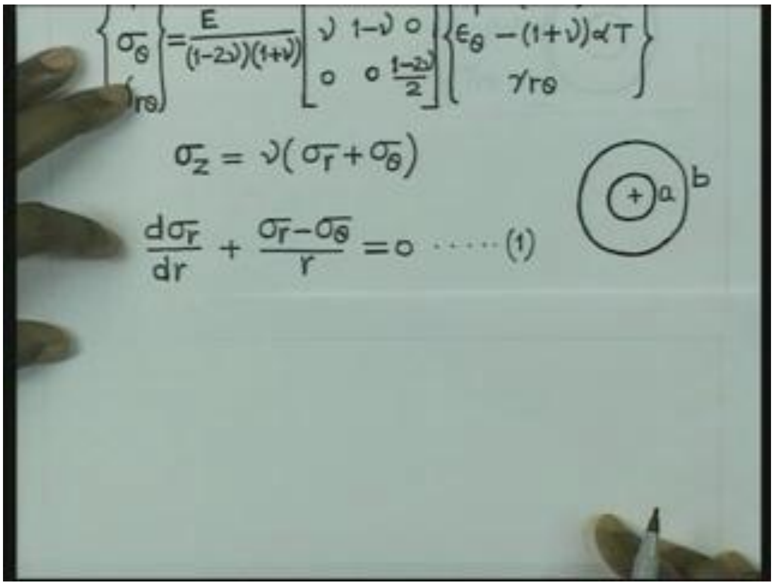
What do expect to happen to the stress at the inner wall and at the same time stress at the outer wall. So, you have high temperature at the inner wall. So, therefore, this all the circles, which are closer to a. They would try to expand more than the circles, which are at the outer edges. And therefore, you are going to have a situation, which will give rise to compressive stresses here and tensile stresses here. So, we would like to look into the, whether we really get that stress distribution as we expect. As I have said that, we will like to consider the case of long cylinder. They will have a stress ((Refer Time: 04:22)), which conforms to more with the plane strain condition.

(Refer Slide Time: 04:28)



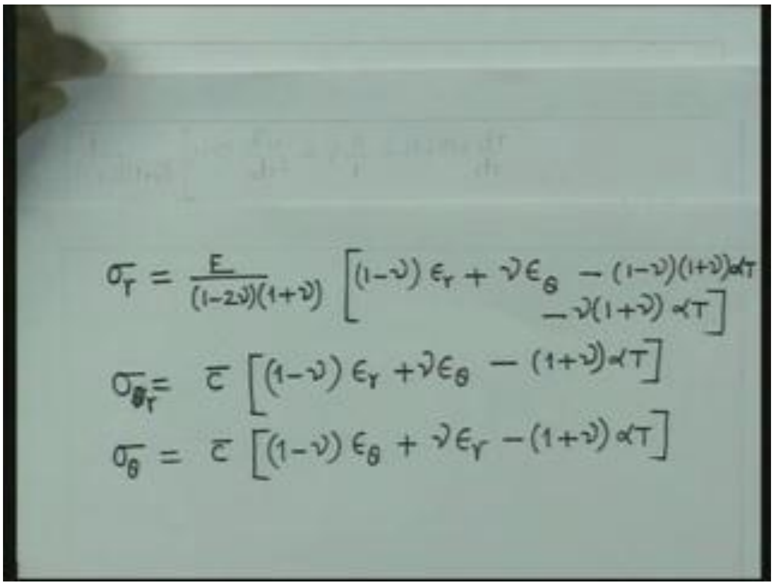
And for the plane strain condition, we can write the stress strain relationship, stress strain temperature relationship as given here. You have stress sigma r sigma theta and shear stress for r theta, which is given by E by 1 minus 2 nu into 1 plus nu into the matrix, which is their, which is ((Refer Time: 04:51)) matrix. And, then we have the strains Epsilon r minus 1 plus nu into alpha T. So also Epsilon theta minus 1 plus nu into alpha T and gamma r theta. So, here E is the modulus of elasticity and nu is Poisson's ratio. And at the same time, we are going to have this phase in the axial direction, which is related to sigma r and sigma theta. So, this sigma z is equal to nu times sigma r plus sigma theta, for the cylindrical problems or even for the problem of 6 cylinder.

(Refer Slide Time: 05:44)



We derived the relationship, the equilibrium equation. The equilibrium equation for this problem also will be similar. And therefore, the governing equation for this stress distribution is given by d sigma r d r plus sigma r minus sigma theta divided by r equal to 0. So, that is the governing equation of distribution of radial stress and tangential stress. Let us consider this equation to be number 1.

(Refer Slide Time: 06:39)

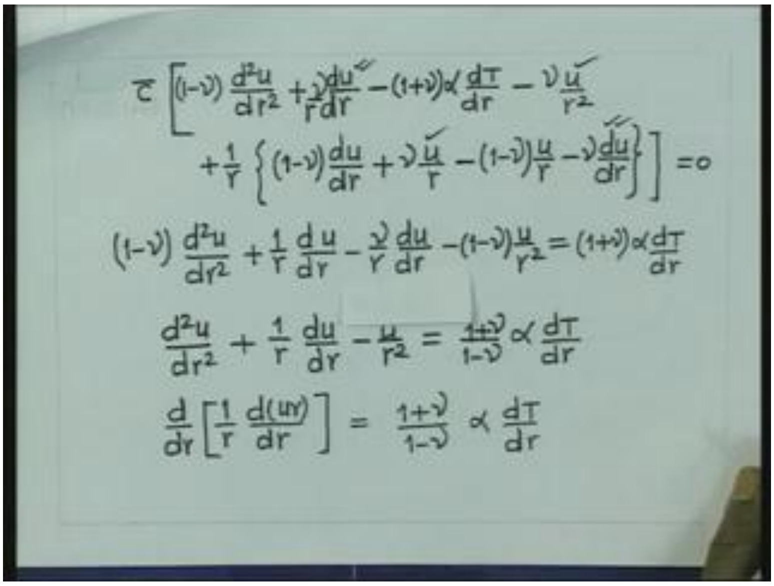


Now, from the stress strain relationship that we had shown you earlier. We can write the stresses and the stresses are going to be given by we will have this sigma r is equal to this constant E by 1 minus 2 nu into 1 plus nu. Then, we are going to have 1 minus nu Epsilon r plus nu times Epsilon theta ((Refer Time: 07:07)). Then we are going to have 1 minus nu into 1 plus nu alpha theta minus nu into 1 plus nu alpha T. This is alpha T.

So, this gives us, if we represent this constant as let us say, c bar then we have 1 minus nu Epsilon r plus nu times Epsilon theta. And, then if we take common here, you can see that we will have this, nu canceling out. So, we will be left with 1 plus nu into alpha T. So, that is the expression for sigma theta. And that is the expression for sigma r. Similarly, we can derive the expression for sigma theta. It is going to be c bar 1 minus nu times Epsilon theta plus nu times Epsilon r minus 1 plus nu alpha T. That is the expression for the stresses.

And now if we substitute this relations into the differential equation. Then, we are going to get. So, if we substitute now d sigma r d r. Noting also, the fact that we can also write the expression for the strains. We have Epsilon r is given by d u d r. And Epsilon theta is equal to u by r. It is similar to that we have done in the case of thick cylinder. So now if I try to substitute the value into the equation of equilibrium.

(Refer Slide Time: 09:42)



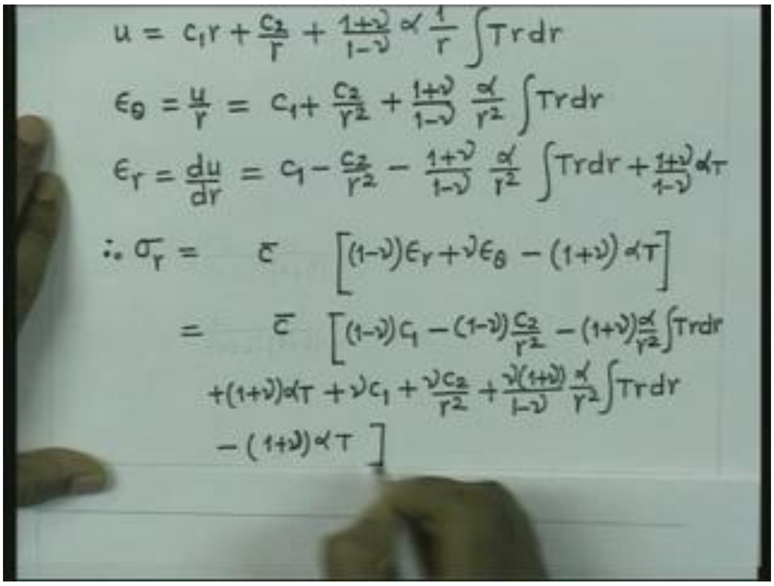
Then we will get, we will have the constant c bar, c bar into we will differentiate this 1 minus nu d 2 u d r square. Then, we are going to get nu time’s u theta. So, therefore, we will have it nu times u by r. And, then we are left we are also having the expression for this is we are also going to have this 1 plus nu. So, ((Refer Time: 10:45)) d sigma d r, if you give as the derivative of this. So, we will have derivative of this, so therefore 1 plus nu alpha d T d r.

So, we will have then minus nu u by r square. Then the other terms 1 by r into 1 minus nu du d r plus nu u by r minus 1 minus nu into u by r minus nu du d r, this is whole thing equal to 0. So, if you try to cancel some of the terms here, you will see that this term will get cancel with this one. And, then we are going to have this term canceling with this term. So, rest of the terms will be remaining and when you try to simplify them.

It becomes 1 minus nu d 2 u d r square plus 1 by r du d r minus nu by r du d r minus 1 nu u by r square equal to 1 plus nu into alpha d T d r. And these two terms can be combined to give us again 1 minus nu into du d r by r. So therefore, finally, we can write this equation in the following form. d 2 u d r square plus 1 by r d u d r minus u by r square is equal to 1 plus nu by 1 minus nu alpha d T d r.

And this is the expression on the left hand side, which you have already come across in connection with thick cylinder. And this expression can be written here like this, d d r 1 by r d u r d r, 1 plus nu by 1 minus nu alpha d T d r. So, that is the final expression for the displacement that is going to come about in the temperature field. And we can integrate this relationship very easily.

(Refer Slide Time: 13:48)



And finally, we are going to get u in the form, u equal to c 1 r plus c 2 by r plus 1 plus nu 1 minus nu alpha 1 by r integration T r d r. And since, the cylinder is prolong between the domain r equal to a, r equal to b. This is going to be, this is really have if you are interested in finding out the displacement at a point. Let us say, radial distance r from the centre then this integration should be a to r.

Now, after having got this displacement, we can get our strains u theta which is nothing but u by r is equal to c 1 plus c 2 by r square 1 plus nu by 1 minus nu alpha by r square integral T r dr. Similarly, Epsilon r, which is given by du dr. So, it is going to be equal to c 1 minus c 2 by r square minus 1 plus nu by 1 minus nu alpha by r square integration T r d r. And when you integrate, when you differentiate this expression you get another term 1 plus nu 1 minus nu alpha T.

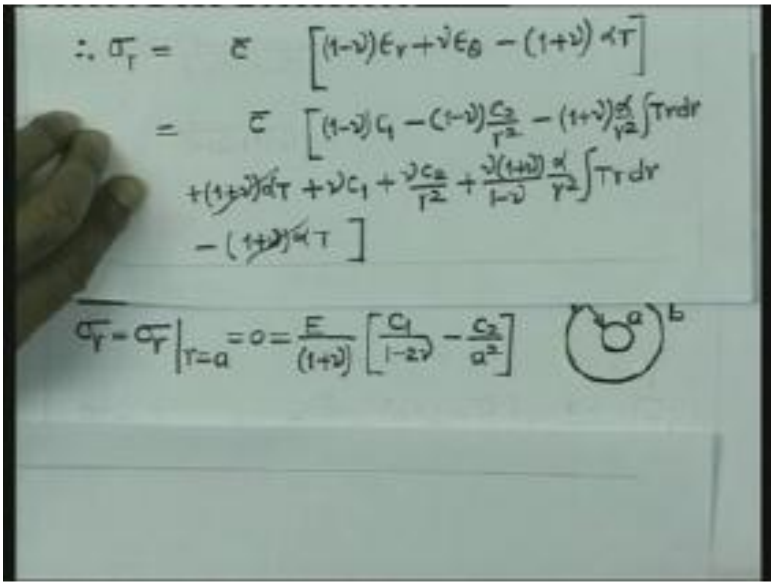
Here, what I mean is that, our Epsilon r is nothing but du dr. So, when you try to differentiate this expression, you have two terms. One is 1 by r, other one is integral T r dr. So, when we differentiate this, it gives us a term equal to alpha by r square. And when we differentiate this integral you get T r. That r cancels with r. So, therefore, you are left with alpha T.

So, these are the strains and once you substitute the strains in the expression for the stress. We get this stresses as follows particularly, sigma r is equal to we can again write, you have the constant c bar 1 minus nu Epsilon r plus nu times Epsilon theta 1 plus nu alpha T. Noting that, we have Epsilon r is equal to du d r. And u is shown there. We can now write, that this expression it is nothing but c bar into 1 minus nu c 1 minus 1 minus nu c 2 by r square minus 1 plus nu alpha by r square T r d r.

And, then we also have the differentiation of this term will give us, 1 plus nu into alpha T. Then, we have nu times Epsilon theta, which will give us Epsilon theta is u by r. So, therefore, it will give us nu times c 1 plus nu times c 2 by r square plus nu times 1 plus nu by 1 minus nu alpha by r square T r dr. Also, we are going to have one more term, which is 1 plus nu into alpha T, so minus 1 plus nu into alpha T. So, therefore, you see that this terms, gets cancelled with this one.

Finally, we are going to get the expression for sigma r. Sigma r is going to involve the constants like c 1 and c 2. As in the case of thick cylinder, we solved for this constants by looking into the boundary condition. Here also we have to look into the boundary condition and solve for the constants. So, let us try to consider, what is going to be the boundary condition in the problem of this type of cylinder.

(Refer Slide Time: 20:03)



So, if you consider now, this typical cylinder here, just a cross section of it. You have the radial stress. There is no external loading acting on the inner wall. And therefore, the radial stress on this boundary is going to be 0. Similarly, there is no external loading on the outer boundary. So, therefore, the radial stress on the outer boundary too is 0. So, therefore, we can now consider that both the surfaces are free of any radial stress. Therefore sigma r equal to 0.

Now, let us see, how do we get the value of the radial stress at the inner radius and outer radius So, we have the expression for the phase radial stress, which is we have written here. c bar into 1 minus nu into c 1 minus 1 minus nu c 2 by r square, minus 1 plus nu alpha by r square. So, if you now try to consider, that this sigma r we will try to consider the case, that sigma r at the inner radius. So, therefore, sigma r for r equal to a. And that is 0.

And therefore, now if I see here, we are going to get now. We can take the value of this c bar. So, if you take the value of this c bar inside. C bar is nothing but 1 minus 2 nu into 1 plus nu. So, therefore, we can now write, here we have certain terms here. You can consider now, it is c 1, c 1 we will have one term canceling here. So, therefore, we can write now, if we write in terms of the modulus of elasticity and Poisson's ratio. Then this is going to be nu and then we have only c 1, c 1 divided by 1 minus 2 nu.

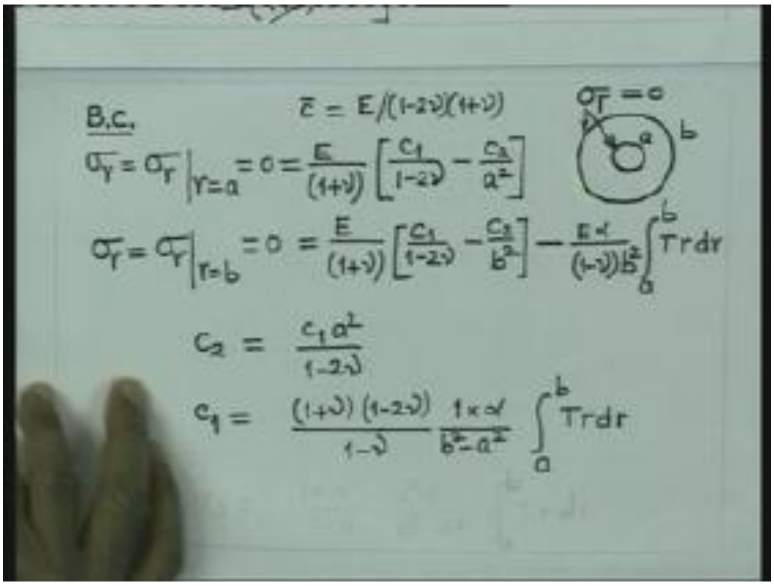
And then c 2, c 2 we have here it is nothing but 1 minus 2 nu c 2 by r square. So, therefore, if we now cancel 1 minus 2 nu, we are going to get c 2 by and r is equal to a. So, therefore, it is a square. So, that is the expression. For the first term and now coming to this part, here in our limits of the integration is a to a. So, therefore, this quantity will get knocked out. There is no contribution. Similarly here, this integral is going to be having the limits a to a. So, therefore, this is not going to be guide to any contribution.

Finally, we find that sigma r equal to 0, which is nothing but E by 1 plus nu into c 1 by 1 minus 2 nu minus c 2 by a square. I want to again to note that, we have made use of the fact that, this c 1 bar which is nothing but E by 1 minus 2 nu into 1 plus nu. So, that substitution had been made in the expression for this stress sigma r and that is how we get this terms. Similarly, if you are trying to now consider the calculation of sigma r at the point r equal to b.

What it will mean is that, we have to substitute r equal to b here. And so also r equal to be here. And the limits of integration would be a to b in both the cases. So, if we now do that. We find sigma r is equal to sigma r at r equal b, which is 0. And this would again give rise to E by 1 plus nu c 1 by 1 minus 2 nu minus c 2 by b square minus, we will have some simplification possible from the two terms. And that will finally, give us E into alpha 1 minus nu b square and this is a to b T r d r. So, this is what I said, going to be done.

Expression for the stresses at sigma r. Stresses at sigma r at r equal to a and r equal to b. And therefore, now you have got two expressions to solve for the two constants c 1 and c 2.

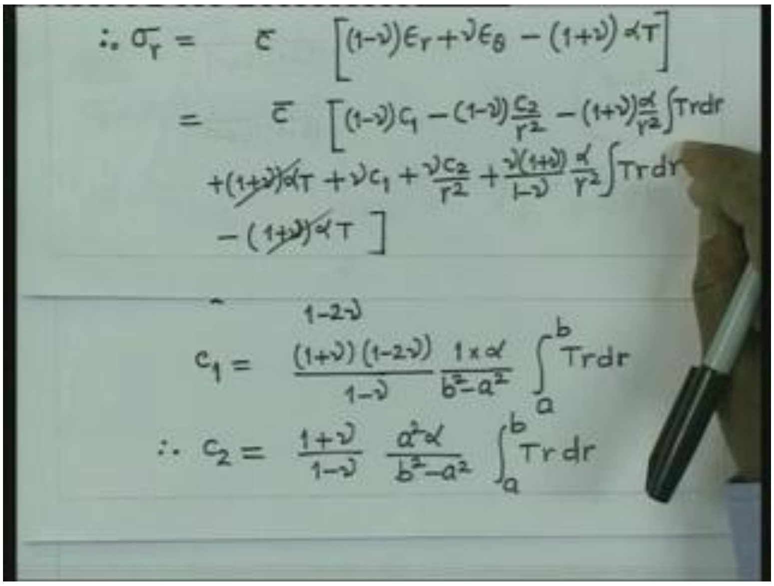
(Refer Slide Time: 25:36)



And obviously, we will find from the first relationship, that c 2 is equal to c 1 a square divided by 1 minus 2 nu. Similarly, if we now consider these c 1. So, from this relationship we will get c 1 is equal to 1 plus nu into 1 minus 2 nu divided by 1 minus nu. And this is 1 into alpha by b square minus a square a to b T r dr. That is the constants c 2. And that is the constants c 1 and therefore, c 2 is equal to 1 plus nu, 1 minus nu a square alpha divided by b square minus a square a to b T r dr.

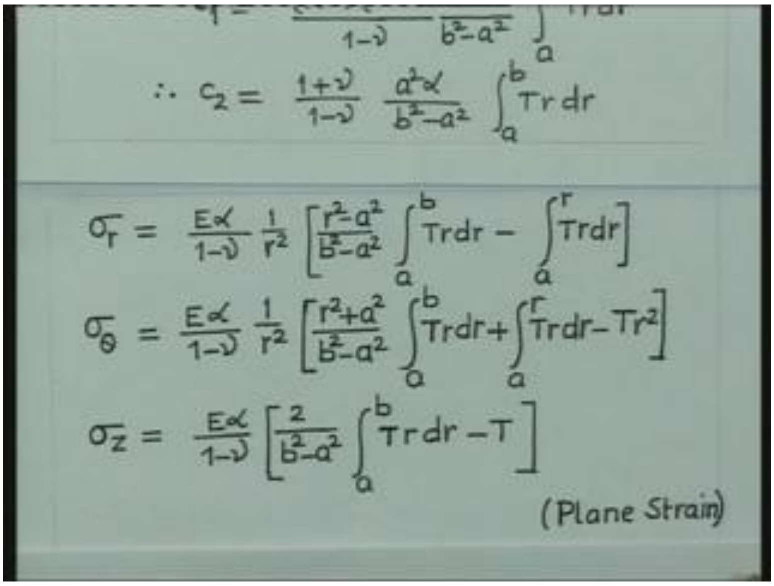
So, we have got the two constants along the lines we had taken for the 6 cylinder with internal pressure. And now, after getting this two constants we again get back to the relationship for sigma r, which we had obtained earlier.

(Refer Slide Time: 27:23)



And once, you make this substitution we will get the value of the stresses. So this are now going to be.

(Refer Slide Time: 27:59)

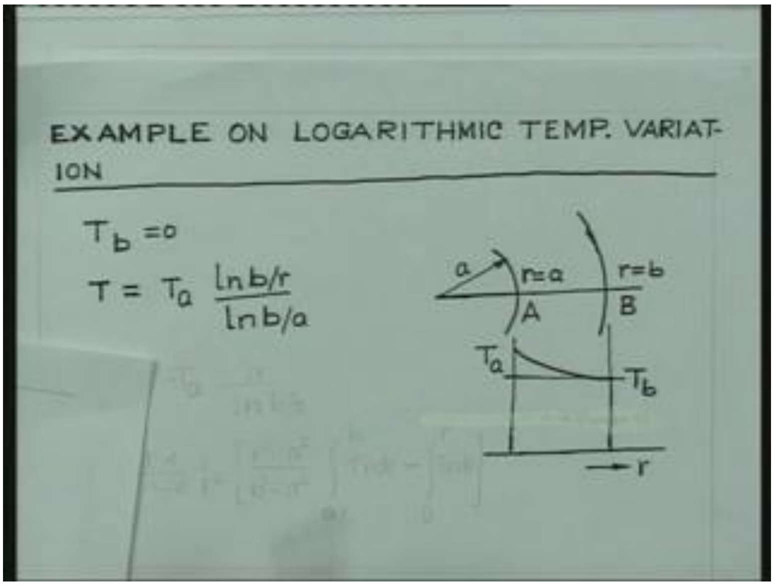


So, this sigma r is finally, going to be given by this expression. So, just I will take the final value. So, here in sigma r given by E alpha by 1 minus nu 1 by r square into r square minus a square by b square minus a square a to b T r d r minus a to r T r dr. That is the expression for sigma r. And we have already got the expression for sigma theta. And we can finally, substitute the value for the displacement. And we get the expression for the sigma theta as E into alpha by 1 minus nu into 1 by r square into r square plus a square by b square minus a square a to b T r d r plus a to r T r d r minus T r square.

So, that is the stress in the circumferential direction. And when we try to take the product nu times sigma r plus sigma theta. It gives us the value that is E alpha by 1 minus nu 2 by b square minus a square a to b T r d r minus T. So, that is the stress in the axial direction. And you must not forget that this stress distribution corresponds to plane strain. So, if you are interested in finding out the stress distribution corresponding to plane stress, you can as well go in for the steps that we have given here.

And the difference would be that these relationship between, the stress and strain is going to be different. But, all the steps are going to be remaining the same. Now, let us consider the application of this formula.

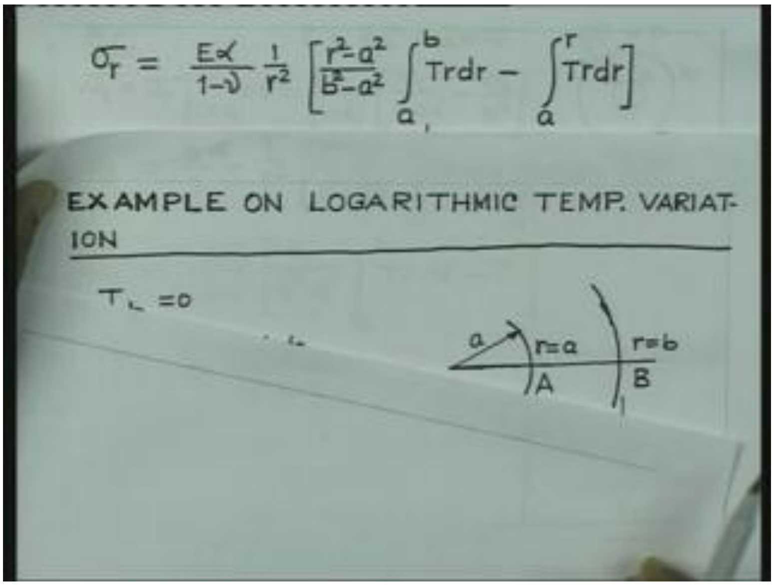
(Refer Slide Time: 30:42)



Let us consider that, we have the well known case of thermal. When heat exchangers are conducting or exchanging heat, you find that the wall temperature is going to vary from the inner wall to the outer wall. And this if we plot in this direction temperature. And in this direction the radius, then it is going to be like this. So, we have the vessel of radius r equal to a to r equal to b. And most often the temperature variation is going to be, if we take this as the data.

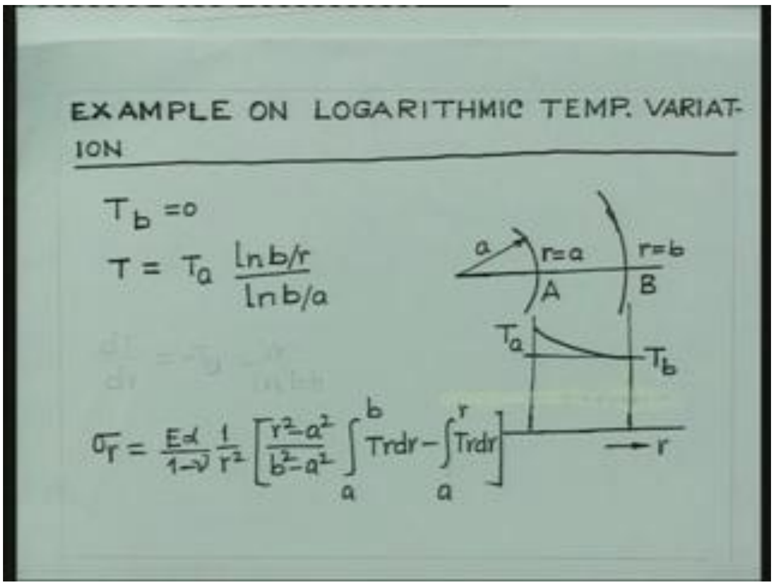
This T b temperature as the data then in that case, the temperature variation is going to be logarithmic. And it is given by T a into l n b by r divided by l n b by a. Now, we need the we have to substitute this temperature distribution in the expression for stresses.

(Refer Slide Time: 32:21)



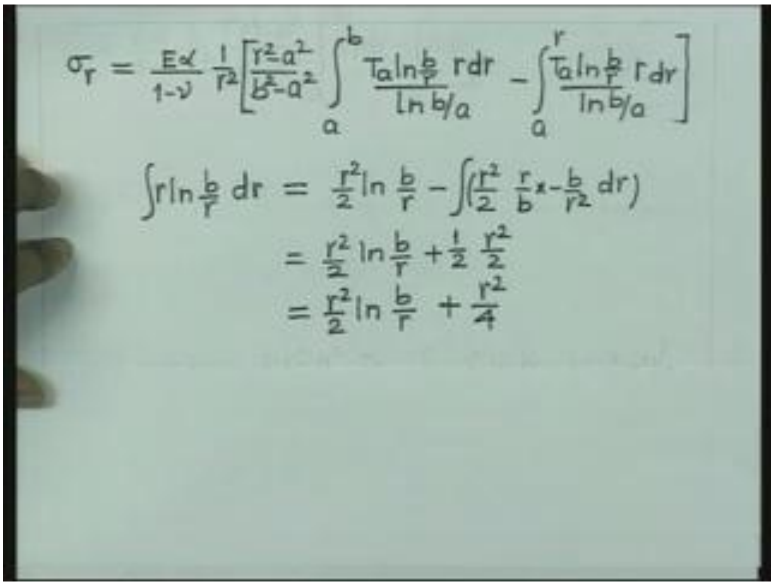
So, if we now consider the radial stress, that radial stress is going to be of this form E alpha 1 minus nu 1 by r square, r square minus a square by b square minus a square a to b T r d r minus a to r T r dr.

(Refer Slide Time: 32:46)



So, we are going to have this form and now if we substitute the value for the temperature. So, we again have the expression here. Sigma r E alpha 1 minus nu, 1 by r square, r square minus a square by b square minus a square a to b T r d r minus a to r T r dr. So, that is the sigma r stress. So, we will now solve for the temperature distribution given by Ta l n b by r divided by l n b by a.

(Refer Slide Time: 33:49)

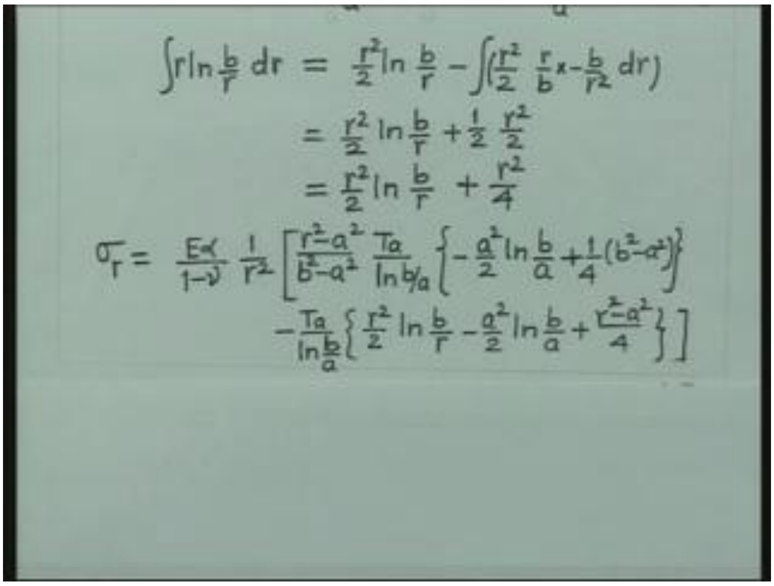


So, we can now write, sigma r E alpha 1 minus nu 1 by r square minus r square minus a square divided by b square minus a square. And we have this thing as a to b Ta l n b by r. l n b by a and we have r d r. Second term a to r Ta l n b by r divided by l n b by a r d r. We have the integration of the term l n b by r into r d r. So, therefore, let us look into the integration of this term integration of r l n b by r. So, we have product of two functions r and l n b by r.

So, if we consider this as the function, this is the second function. Then, we will consider integration of the first function r. And therefore, it will give us r square by 2 l n b by r minus integration of the first function that is r square by 2. Differentiation of this function will give us 1 by b by r. So, that will be r by b. And differentiation of b by r will give us minus b by r square d r. So, that is the expression, which we get and this has got to be integrated.

And now, if we rewrite this is r square by 2 l n b by r plus this is now, we are going to get cancellation. It is simply going to be r by 2 and therefore, we will have this thing, this minus, minus will cancel it will give us plus. So therefore, it is 1 by 2 r square by 2. And finally, this gives us r square by 2 l n b by r plus r square by 4. So, that is the expression for the integral and therefore, if we now substitute the value here. You see, that limits of integration are, a to b.

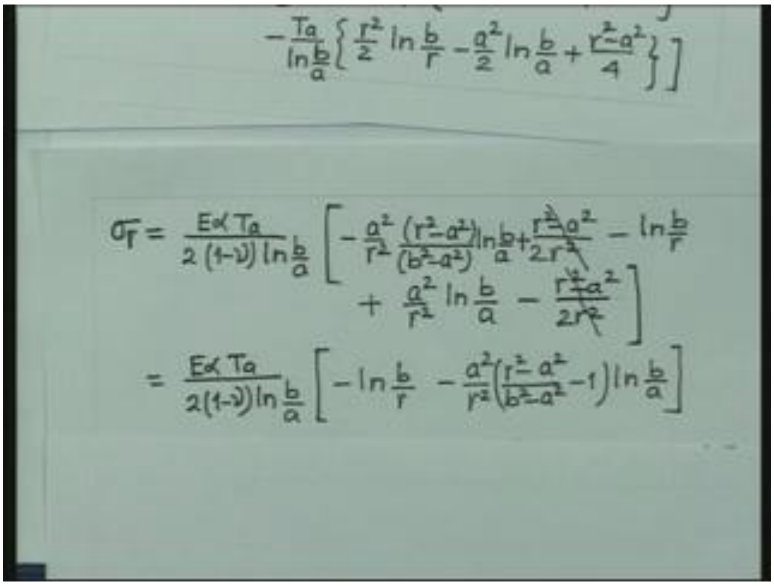
(Refer Slide Time: 37:24)



So, we will have two terms coming for each of them. And therefore, sigma r is equal to E alpha 1 minus nu 1 by r square, r square minus a square, b square minus a square. Then we have T a by l n b by a common multiplier. And now, these terms have to be between the limits a to b. So, we can now write, that will give us. So, for the limits b it will be 0. So, therefore, we will be simply having a square by 2. For limit r equal to b, it will be b square by 2 l n b by b, which is 0.

So, therefore, it is minus a square l n b by a. So, that gives us the terms for a and b. And, then this one will give us 1 by 4 b square minus a square. So, that is the term that we get considering the first term of the expression for sigma r. And the second term will give us. T a by l n b by a and this is going to be limits a to r. So, therefore, it will be r square by 2 l n b by r minus a square by 2 l n b by a plus r square minus a square by 4. That is the expression for sigma r. Finally, there is some cancellations possible.

(Refer Slide Time: 40:09)

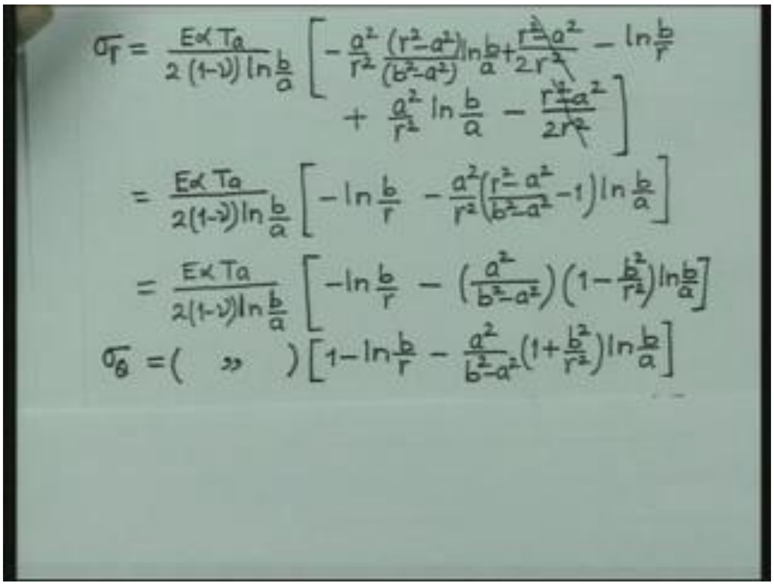


And we can write finally, sigma r to be given by E alpha T a divided by 1 minus nu l n b by a, into a square by r square, r square minus a square divided by b square minus a square l n b by a. Then, we have r square minus a square divided by 2 r square minus l n b by r. Then, a square by r square l n b by a minus r square minus a square divided by 2 square. So, these two terms are going to cancel.

Finally, we will be left with the expression they are and which is in simplified form. E alpha T a by 2 into 1 minus nu l n b by a minus l n b by r minus a square by r square, r square minus a square, b square minus a square minus 1 into l n b by a. So, this is the expression of the stress sigma r. Similarly, once you do the integration involved in the expression for sigma theta and sigma there. You can find out the expression for these components as well. So, we finally, get this components.

We can also write the expression of this sigma r in a little different form which is ((Refer Time: 43:06)).

(Refer Slide Time: 43:09)

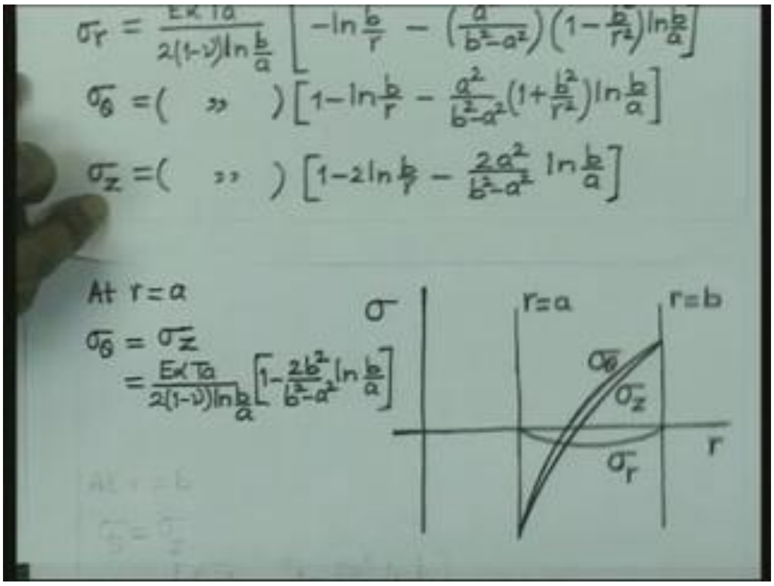


So, let us write first sigma r in slightly different form E alpha T a divided by 2 into 1 minus nu l n b by a minus l n b by r minus a square by b square minus a square into 1 minus b square by r square l n b by a. So, that ((Refer Time: 43:54)) simply we have tried to keep these two terms together. And then we have taken the other things of course, we have simplified this one. And once, we simplify this it will be a square by b square minus r square by r square into b square minus a square. So, therefore, we can write like this.

Similarly, if we write the expression for sigma theta in this format. Sigma theta is equal to the nothing but this common terms will remain the same. And, then we will have the other terms 1 minus l n b by r, a square by b square minus a square into 1 plus b square by r square l n b by a. So, that is the expression for sigma theta. And sigma z which is nothing but nu times sigma r plus sigma theta minus E alpha T. So, if I do all that.

In fact, we had already got sigma z in a form, which is here ((Refer Time: 45:19)) sigma z is equal to E alpha by 2 by b square minus a square a to b T r d r minus T.

(Refer Slide Time: 45:31)



So, if we do all this then this sigma z is going to be like this. 1 minus 2 times l n b by r minus 2 times a square b square minus a square l n b by a. So, these are the 3 components of stresses. So let me, just have everything at one place So, this is the expression for sigma r sigma theta and sigma z. Now, let us try to consider the values of these stresses, at the two extreme radial. What happens to the stresses. We find that this sigma r. Here, we will find that b, b is greater than r. So, therefore, this term is always greater than unity.

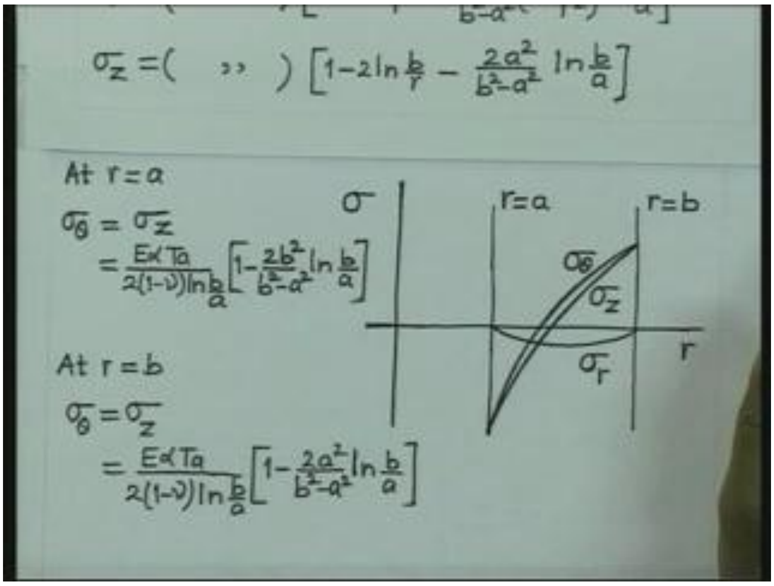
And therefore, it is negative it will become positive. And then we are going to have this term b by r is always greater than unity. And therefore, we might get some value, we are going to get something negative. Finally, what we find is that, this stress is going to have value all the time. This value is going to be negative. And the stresses sigma theta and sigma z. Let us, look into the position here, if r equal to a.

Then, we will find that this stress is 1 minus l n b by a, a square by b square minus a square into 1 plus b square by a square l n b by a. So, that will now become, it is b square plus a square by a square. So, sigma theta will get that expression. And therefore, we can write now, at r equal to a sigma theta is equal to E alpha T a divided by 2 into 1 minus nu l n b by a. That, into 1 minus so here, we are going to get. It is, this is going to be b square by a square plus 1. So, these two, so this is again b by a. So, these two terms can be combined together.

It will finally, give us 2 b square by b square minus a square l n b by a. So, that is the stress sigma theta. And if we now, calculate the value of sigma z at b r equal to a. Then also, we are going to get these two terms will get combined. And it will also give me the same value. So, it will be minus 2 l n b by a minus 2 a square by b square minus a square into l n b by a. So these, two will again give us 2 b square by b square minus a square l n b by a.

So, this sigma theta is equal to sigma z and this stress at r equal to a. You will find that, these two this particular term is going to be more than unity. And therefore, we will have a negative stress. And at the same time, if you calculate the stress at the outer radius.

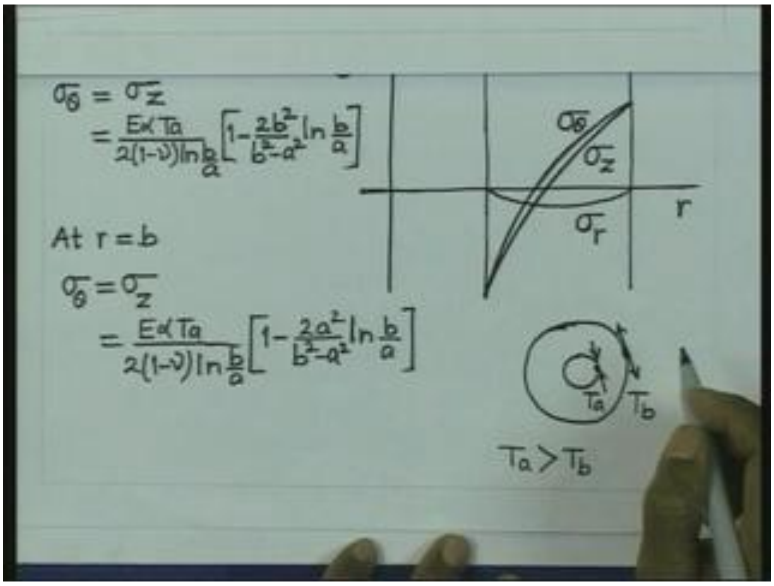
(Refer Slide Time: 49:22)



So, at outer radius it is going to be given by sigma theta is equal to again sigma z. And it is E alpha T a by 2 into 1 minus nu l n b by a, into 1 minus 2 a square by b square minus a square l n b by a, so this is the stress at the outer radius. So, this variations of the stresses, once you plot it looks like this. We have stresses plotted in this direction, radius in this direction. You will have this as the variation of sigma theta. This is the variation of sigma z.

Here in at the internal radius, here both are equal. At the external radius both are equal. And sigma r is always a compressive stress. So, this is how, we are going to find a stress variation in the cylinder with temperature at the inner wall higher than the temperature at the outer wall. So, you can now see, the stresses at the outer wall is tensile. And stresses at the inner wall is compressive.

(Refer Slide Time: 50:44)



Let us, try to see whether it is tallying with our expectations, as we started saying at the beginning. If temperature here is T a, temperature here is T b. If T a is greater than T b, then we expect the inner fibers to expand more than the outer fibers. So, their expansion will be arrested or rather constrain by these lower expansion of the outer fibers. And hence, these fibers will be subjected to compressive stresses and since these people are trying to these fibers are trying to extend.

They will try to generate a tensile stress at the outer fiber, which I am not expanding to that extent. So, if you have the inner wall at a high temperature. You expect these fibers to be subjected to compression. Because, of this molar expansion of the outer fibers. And at the same time, since these are expanding more they will try to induce tension in the outer fibers or fibers located at the outer radius. This is one of the reason, which you can understand, that why when we for high temperature water in a glass, glass it generally cracks.

So, once you are trying to put water inside the glass, you are going to have higher temperature at the inner surface. And the outer walls are still at room temperature or cooler temperature or lower temperature. And this is going to induce tensile stress at the outer radius. And this tensile stress glass being a brittle material is very dangerous and therefore, it cracks. Similarly, if you try to consider, that you have a glass particularly, if you think of a chimney and it is glass it is already at high temperature.

If you pour a drop of water, the portion where you dropped water it is getting cooler. And if it is getting cooler it is trying to contract. But, the material, which is outside it is not allowing to contract and therefore, tensile stresses will be generated at the location and hence cracking will take place. So, what I said, that in the place of a glass or a chimney.

It is already hot and if you pour a drop of water at the outer surface. That portion, which is in contact with the water, is going to be cooling down. And it is trying to contract and since the material outside is at high temperature. They cannot allow it is contraction to take place and hence tensile stresses will be generated and there will be cracking.