Advanced Strength of Materials
Prof. S. K. Maiti
Department of Mechanical Engineering
Indian Institute of Technology, Bombay

Lecture – 3

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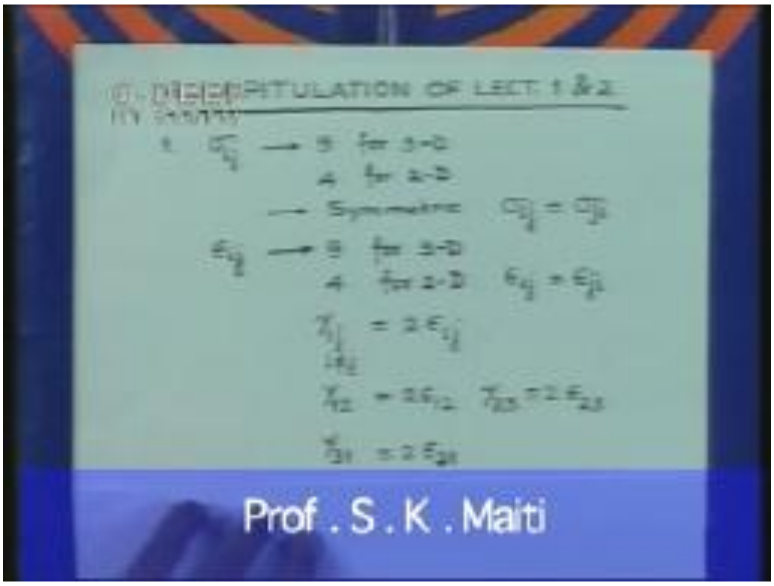

We will recapitulate fast; what we have done in the earlier 2 lectures. We have first talked about the stress at a point. It is indicated by sigma ij, it is a 9 component tensor for 3 dimensions and it has 4 components for 2 dimensions. And we have also seen that this tensor is symmetric, thereby you can indicate it in the following form: sigma ij equal to sigma ji. Similarly, we have introduced the strain tensor which is also a tensor having 9 components for 3 dimensions and 4 components for 2 dimension. It is also symmetric tensor therefore, epsilon ij equal to epsilon ji.

We have also seen that, there is a difference between engineering shear strain and tensor shear strain. Engineering shear strain is twice the tensor shear strain. Specifically we have seen gamma 1 2 is equal to 2 times epsilon 1 2, similarly same relationship same type of relationship holds good for eps gamma 2 3 and gamma 3 1.

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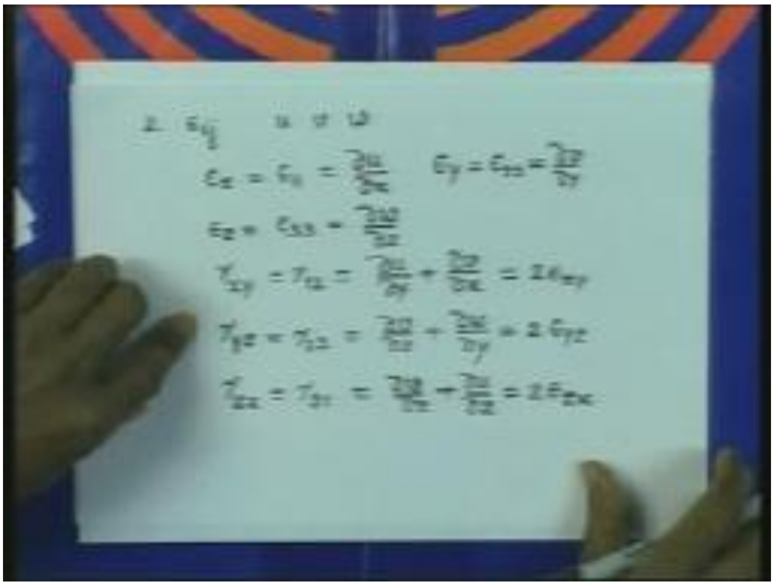

We have then define the strains at a point in terms of the 3 displacement components: u v and w. These are the Cartesian components of displacements of a point on the body and we have seen that, epsilon x is nothing but epsilon 1 1, it is given by the partial derivative delta u delta x. Similarly, epsilon y is nothing but epsilon 2 2, this is given by delta v delta y and epsilon z which is nothing but epsilon 3 3, this is given by the partial derivative of w with respect to z. Then we have seen the shear strains; shear strain if it is gamma xy which is nothing but gamma 1 2, this is given by delta u delta y plus delta v delta x.

Similarly, gamma yz is equal to gamma 2 3 which is nothing but delta v delta z plus delta w delta y. Lastly we had gamma zx is equal to gamma 3 1 which is nothing but delta w delta x plus delta u delta z. We have also noted that, this component is equal to 2 times epsilon xy, this is 2 times epsilon yz and this is also 2 times epsilon zx.

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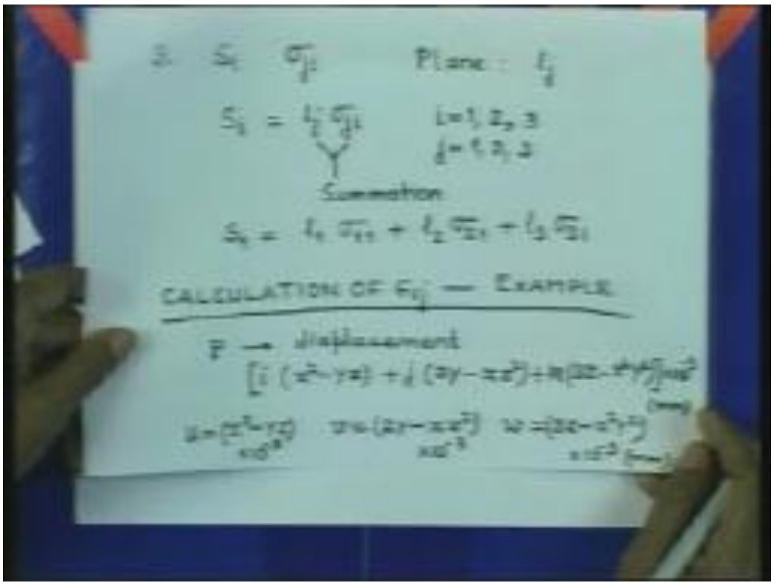

Then we took up the calculations of tractions on an arbitrary plane, so traction component on an arbitrary plane, where the stresses are given by sigma ji. And if the arbitrary plane has direction cosines lj then in that case, we have seen that Si is nothing but lj sigma ji varying Si of the components of traction and in this we have made use of the implied condition that i takes up value 1 to 3. So, also j takes up value 1 to 3. This indicates compact form of expanded expressions and herein we make use of the fact that, these 2 symbols are repeated therefore, a summation is indicated.

So, these 2 symbols are repeated and it indicates a summation. Just to the 1 case S 1 is nothing but l 1 sigma 1 1 plus l 2 sigma 2 1 plus l 3 sigma 3 1, herein i is fixed here this is fixed and l is varying from 1 2 3. So, also the stress components are varying 1 2 3. Now, we would like to take 1 example of we would like to consider the calculation of strain at a point of epsilon ij. It is given that in a body which is undergoing deformation, a point P has displacement given by the vector i into x square minus yz plus j 2 y minus xz square plus k into 3 z minus x square y square. This multiplied by 10 to the power minus 3 milli meter. So, the displacements are given in milli meter. Herein i j and k are the unit vectors in the 3 Cartesian directions.

Therefore, you should have no difficulty in identifying that the x displacement of the typical point is given by x square minus yz and v displacement is equal to 2 y minus xz square and w is given by 3 z minus x square minus y square. And all this expressions are multiplied by 10 to the power minus 3. So, these are the displacements. Now, using these displacements, we would like to now calculate the strains strain components particularly.

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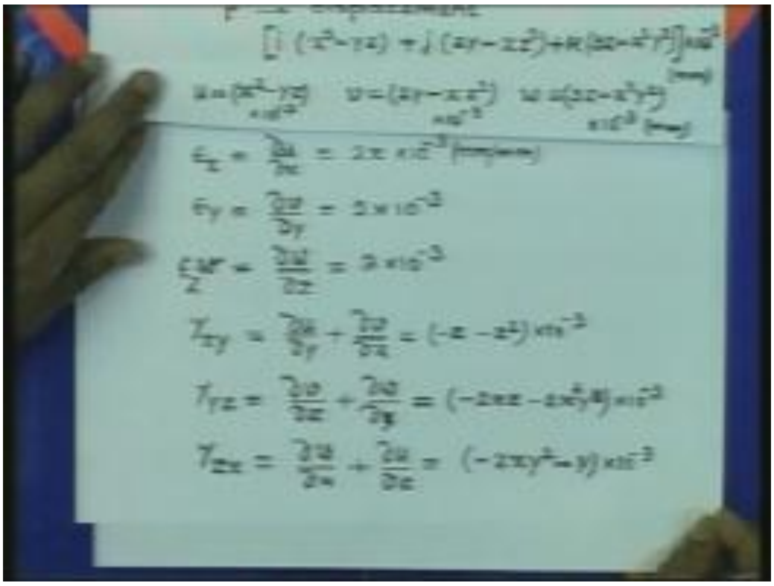

So, if I now consider epsilon x, this is given by delta u delta x. So, this will be given by twice x multiplied by 10 to the power minus 3. So, this is milli meter by milli meter. So, strains are dimensionless. So, hence forth I will not. So, the units epsilon y delta v delta y this is given by 2 10 to the power minus 3 and w is equal to w epsilon, the strain components epsilon z delta w delta z is equal to 3 multiplied by 10 to the power minus 3. Similarly, we can calculate the shear strains. Shear strains gamma xy delta u delta y plus delta v delta x, so this is given by minus z minus z square into 10 to the power minus 3 gamma yz delta v delta z plus delta w delta x.

So, this is going to be given by minus 2 xz minus 2 xy square, this should be delta w delta y, so it should be 2 x square y, so 2 x square y. And gamma zx is nothing but delta w delta x plus delta u delta z. This is given by minus 2 xy square and this is minus y into 10 to the power minus 3, here also you will have a multiplier 10 to the power minus 3. It is given specifically to calculate the strains at a point. The point P coordinates are given as 2 3 4 in milli meters unit.

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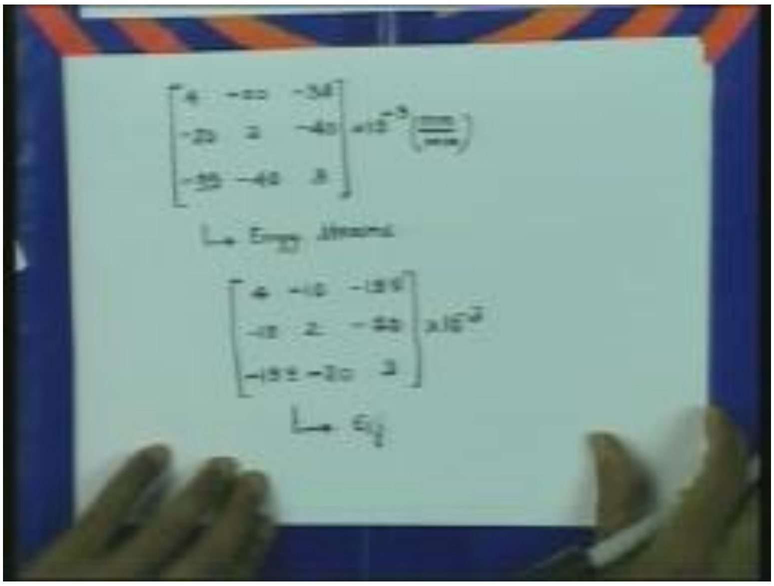

So, if you now substitute the value of this coordinates, then we get the following strains; 4 minus 20 minus 39 minus 20 2 minus 40 minus 39 minus 40 3 and all these has a common multiplier 10 to the power minus 3, this is milli meter by milli meter. So, these are the engineering strains at the point P with coordinates 2 3 4. If you want the tensor components, it is going to be 4; this will be half of what we have here, so it is 10 minus 19.5 minus 10 2 minus 20 minus 19.5 minus 20 3 multiplied by 10 to the power minus 3, this is the strain tensor at point P.

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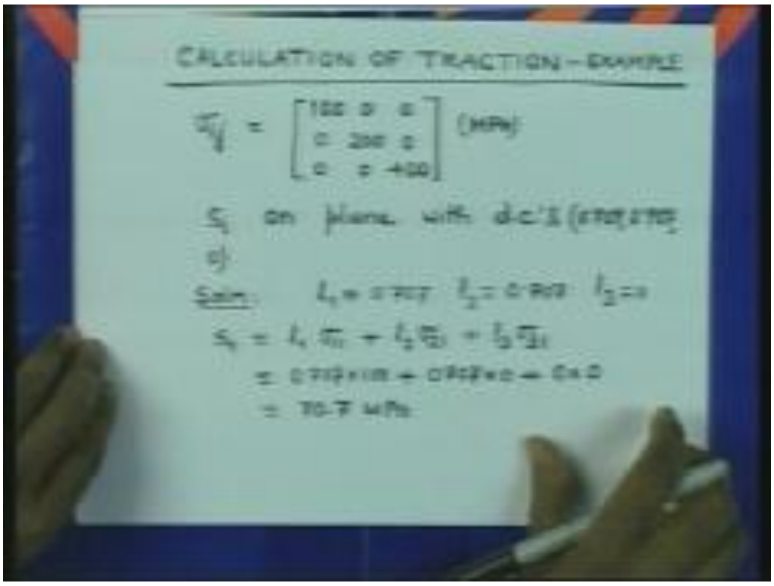

We would like to consider 1 example; calculation of tractions. Given the stress tensor at a point like this; 100 0 0 0 200 0 0 400, these are the stresses at a point. Now, it is necessary to calculate the tractions Si on plane, with direction cosines 0.707 0.707 0. This means; so a solution can be done like this; we have l 1 is nothing but 0.707, l 2 0.707 and l 3 is equal to 0. So, these are the direction cosines.

Therefore, we can now calculate the traction component 1, which is given by l 1 sigma 1 1 plus l 2 sigma 2 1 plus l 3 sigma 3 1, herein l 1 is 707 and the sigma 1 1 is 100, then l 2 is also 7.707 and sigma 2 1 is 0 and this l 3 0 and sigma 3 1 is 0. So, therefore, it gives us the value like this 70.7 MPa. So, that is the traction in the direction 1 and it has the same units as the stress.

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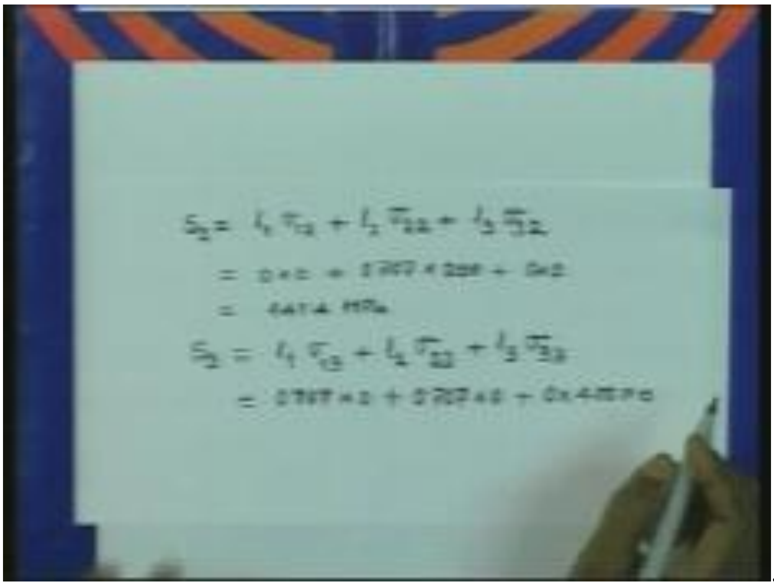.

You can calculate similarly the tractions for the other 2 directions. S 2 is given by l 1 sigma 1 2 plus l 2 sigma 2 2 plus l 3 sigma 3 2 and if you substitute the value it becomes 0 0, this is 707 multiplied by 200 0 multiplied by 0. So, this gives us 141.4 MPa. Now, S 3 is given by l 1 sigma 1 3 plus l 2 sigma 2 3 plus l 3 sigma 3 3, where l 1 0.707, sigma 1 3 0, this is 707, this is 0 and this is 0 multiplied by 400. So, therefore, this is 0. Please note that, this value should not be 0; it should be 0.707. So, this amendment you can note, but the value of S 2 does not change.

So, the 3 traction components are; finally you have S 1 is 70.7 MPa, S 2 is 141.4 MPa, S 3 is 0. So, these are the 3 traction components on the plane, whose direction cosines are 0.707 0.707 and 0.

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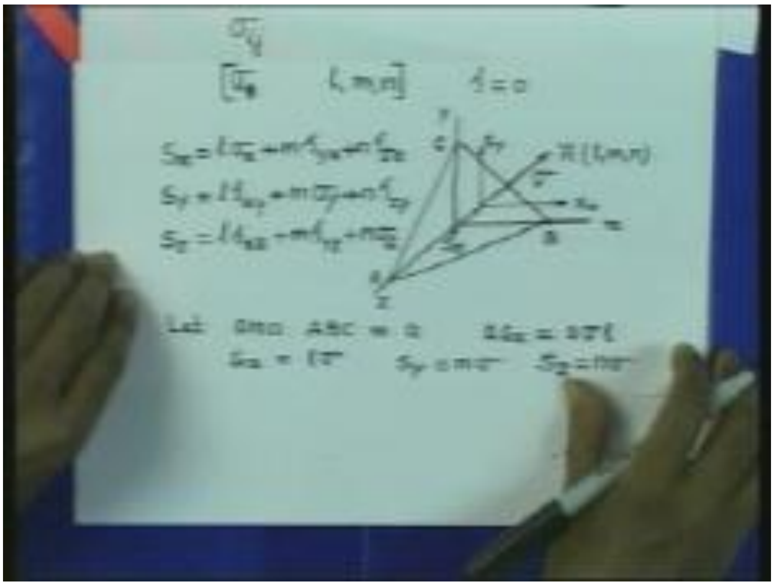

Now, we would like to consider determination of principal planes and principal stresses. So, determination of principal stresses and principal planes; do you remember what is principal stress, these are the stresses which act on plane where there is no shear stress. So, we would like to now consider determination of some, determination of such stresses in 3 dimensions. So, we have the stresses at a point given by sigma ij and we would like to find out the stresses, which are acting on planes, where there is no shear stress.

So, we are interested in determine determining the stresses like; let us say sigma 1 and its direction cosines l L m and n. So, we will indicate the principal stresses by symbol; just a indicate by sigma, so sigma is the principal stress and it is acting on a plane with direction cosines L m and n. And this plane is such that, there is no shear stress, so shear stress on this plane equal to 0. So, we are really interested in determining this set, where there is a stress equal to 0.

Now, we try to consider 1 plane: A B C. Let us say this is the plane, whose outer normal is directed like this, this is the outer normal. Let us assume that, this is the plane which has direction cosines L m and n. And therefore, on this plane we have a stress sigma acting and this plane A B C is divide of any shear stress.

Now, if I make use of the Cauchy formula then the traction components we can obtain, so we will like to guide the traction components here; these are the 3 traction components for this plane this is Sx, this 1 is Sy and this component is Sz. So, we can write now sx is equal to l sigma x plus m tau yx plus n tau zx similarly, Sy l tau xy m sigma y plus n tau zy Sz is equal to l tau xz plus m tau yz plus n sigma z these are the tractions on these plane ABC.

Let, the area of this plane ABC be represented by the symbol a, now the force which is going to act in the x direction, because of the stress stresses on the plane is going to be nothing but a into Sx. And we can also, find out the component from the stress sigma the total force on this plane ABC is nothing but a into sigma and each component in the x direction is l sigma into a.

So therefore, we can write that the total force in the x direction is nothing but ax and this should also be equal to a into sigma multiplied by a. So finally, we get Sx is equal to l into sigma similarly, you can have consideration of force in the y direction there by you will find that Sy is nothing but m into sigma and Sz is equal to n into sigma. Now, we will substitute the value of Sx Sy Sz in terms of sigma.

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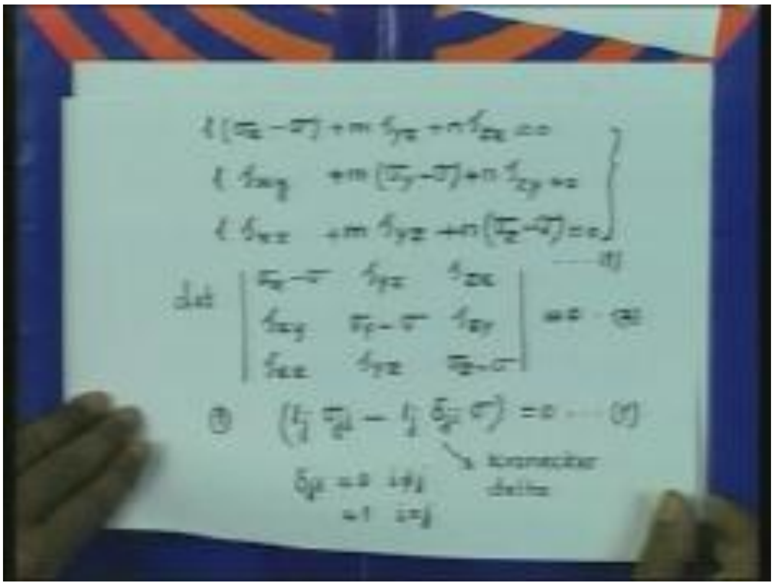

So, if I do that we finally get the following that l sigma x minus sigma plus m tau yx plus n tau zx equal to 0. So what I have done is that, I have coupled this relationships Sx and these Sx and simplified similarly, we get from the second expression m sigma y minus sigma plus n tau zy equal to 0.

Similarly, l tau xz plus m tau yz plus n sigma z minus sigma equal to 0; this shape of equations let us represent it by set 1. Now, we say homogeneous set of equations with unknown l, m and n the direction cosines of the plane and if we have to have solutions for non 0 solutions for l, m and n it is necessary to have this condition satisfied determinant of the coefficient matrix which is nothing but sigma x minus sigma tau yx tau zx tau xy sigma y minus sigma tau zy tau xz tau yz sigma z minus sigma.

So this determinant must vanish, so it should be equal to 0 so this is the characteristic equation. This equation can be written in tensor notation particularly, we can use also the tensor notation to represent equation number 1 in the following form that lj sigma ji minus lj delta ji into sigma equal to 0, where this delta ji is nothing but kronecker delta. So, this kronecker delta has the characteristics, that delta ji is equal to 0 when I is not equal to j and it is equal to unity when I equal to j. So, with those implications we can write the equation number 1 in this form, so this is really the form of equation number 1 so will indicate equation number 1 dash.

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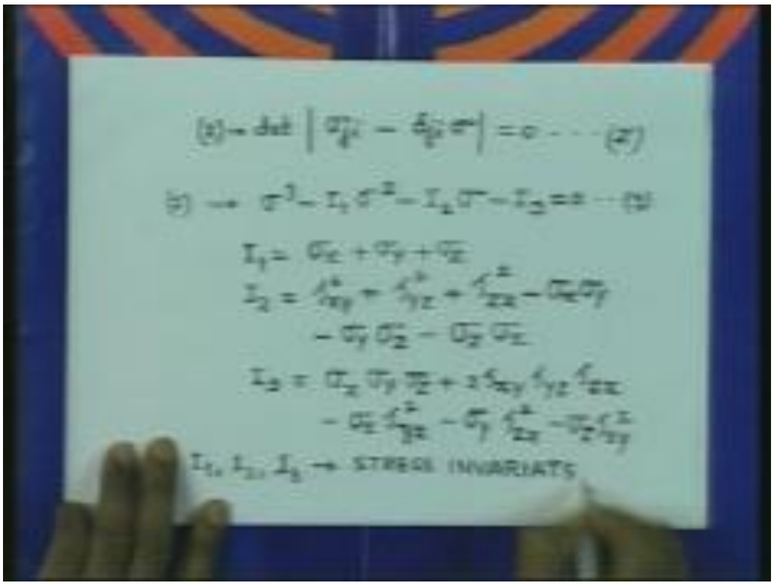

Similarly the equation number 2 is nothing but determinant sigma ji minus delta ji into sigma equal to 0, so this is in a compact form we can have. So we have seen that the equation number 2 can be written in tensor form like this. If we go back to the original form of equation 2 this can be expanded and will get an equation in sigma, which is obvious form sigma cube minus I1 sigma square minus I2 sigma minus I3 equal to 0, where these coefficients I1 I2 and I3 are given in terms of the stresses I1 is equal to sigma x plus sigma y plus sigma z and I2 is tau xy square tau yz square plus tau zx square minus sigma x minus sigma y minus sigma y minus sigma z minus sigma z into sigma x.

Similarly, I3 is given by sigma x into sigma y into sigma z plus 2 into tau xy tau yz into tau zx minus sigma x into tau yz square minus sigma y tau zx square minus sigma z into tau xy square.

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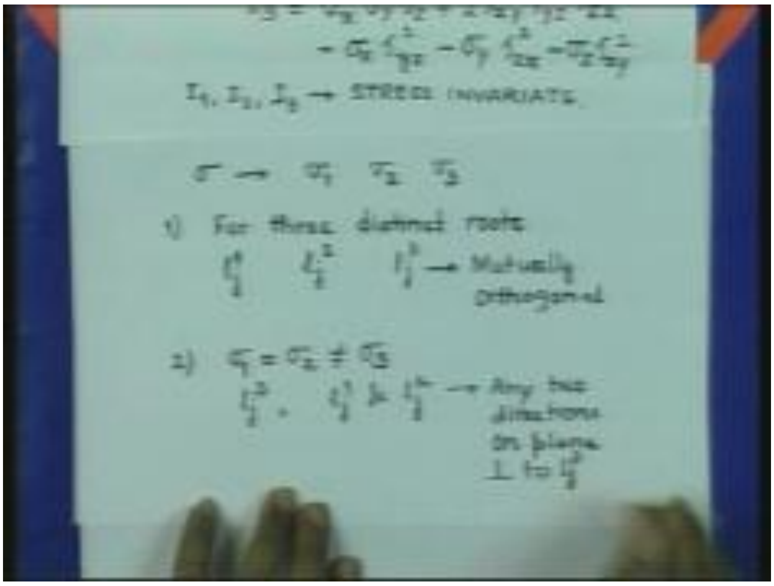

Let us, say the 3 roots are sigma 1 sigma 2 and sigma 3 if these 3 roots are distinct, then they are going to work on 3 mutually perpendicular planes. So far 3 distinct roots will have 3 directions which are going to be mutually perpendicular. So, if you consider the directions of the 3 stresses as lj1 for the sigma 1 stress lj2 for the sigma 2 stress and lj3 for the sigma 3 stress then these 3 directions are going to be mutually perpendicular.

So, these are 3 mutually orthogonal directions, if we have a situations like that sigma 1 there are 2 repeated roots sigma 1 equal to sigma 2, but then that is not equal to sigma 3. Then, you can find out this sigma 3 directions and then lj1 and 2 can be taken as any 2 directions on the plane perpendicular to lj3. So, these are the coefficients these are called stress invariants I2 I3 are known as stress invariants.

So, this equation will have 3 roots for sigma and it can be shown that these roots are real. So lj1 lj2 which corresponds to sigma 1 and sigma 2 they will be any 2 directions any 2 directions on plane perpendicular to lj3.

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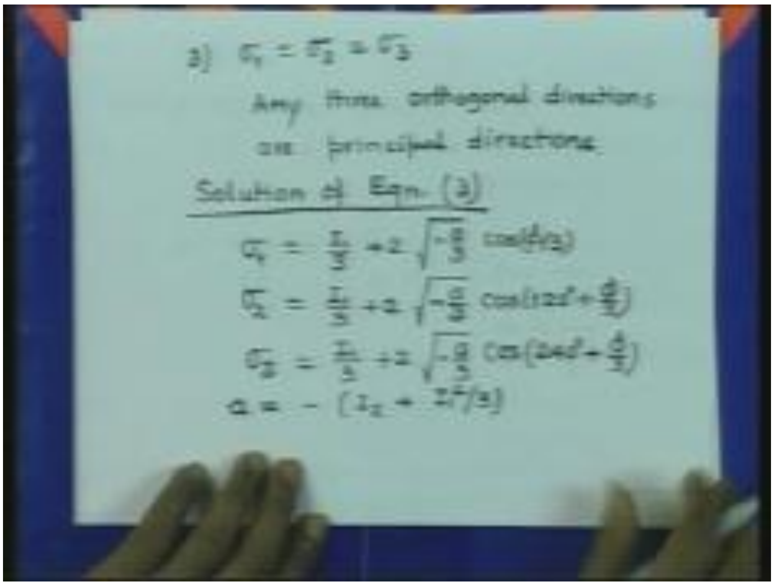

If we have 3 roots repeated, if we have sigma 1 equal to sigma 2 equal to sigma 3 then any 3 orthogonal directions or principal directions. Now, we can find out the roots of this equation 3 easily so we can solve the equation. So, solution of equation 3 you can solve this cubic equation by the following steps in fact we can write sigma 1 to be given by I by 3 plus 2 minus a by 3 cosine phi by 3 sigma 2 is I1 by 3 plus 2 times a by 3 cosine 120 degree plus and this is also phi by 3. And sigma 3 is equal to I1 by 3 plus twice minus a by 3 cosine 240 degree plus phi by 3, where we have a is given by minus I2 plus I1 square by 3.

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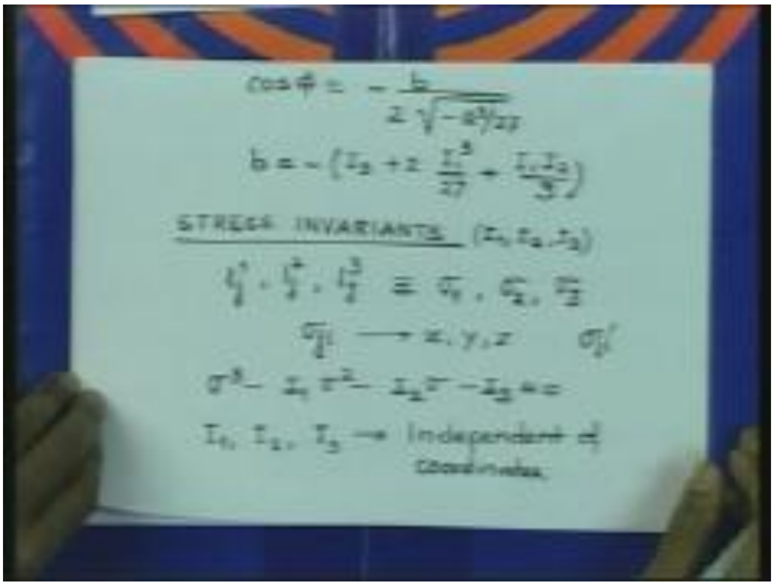

And we have cosine phi is equal to minus b by square root minus a cube by 27 and this b is given by minus I3 plus 2 I1 cube by 27 plus I1 I2 by 3. So, the solution for that equation cubic equation are given by this expressions wherein all the constants a phi and b are related to the stress invariants. Let us understand, why stress invariants? So, we have found that the quantities like I1, I2, and I3 are stress invariants.

So, you should have no difficulty in appreciating that when a body is loaded its deformation will take place and at a point there will be principal directions, there will at each point there will be 3 principal directions. And these principal directions are fixed so long as the loading on the body is fixed, the principal directions are fixed directions that means; the 3 directions lj1 lj2 lj3 which corresponds to stresses sigma 1, sigma 2 and sigma 3 they are fixed and they are dependent on the stresses stress tensor at the point.

Now, the direction of this principal stresses are fixed at a point. This will not depend on the selection of coordinates for defining the stress tensor. So, this stress tensor sigma ji is going to depend on selection of the coordinate ,x y, z, but the value of the principal stresses and there directions are fixed which implies that the cubic equation that we had retain little while ago that sigma cube I1 sigma square minus I2 sigma minus I3 equal to 0.

The roots of these equations must be unique, it will not depend on the selection of coordinate. If the 2 persons are selecting 2 different coordinates, it is going to probably have values of sigma ji for 1 person and for another person it may have value sigma ji dash, but this 2 situations must lead to a equation in sigma which should not depend on the coordinates. Therefore, this I1 I2 I3 must be independent of the coordinate. So I1, I2, I3 independent of coordinates; that is why they are called stress invariants.

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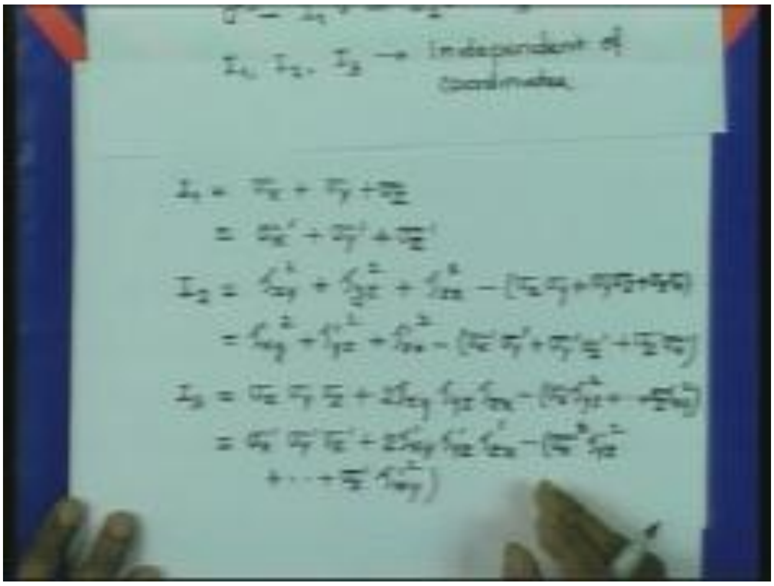

And you can therefore, finally write that the value of I1 which is given by sigma x plus sigma y plus sigma z it must be also equal to sigma x dash sigma y dash plus sigma z dash. So also, we will have I2 given by tau xy square tau yz square plus tau zx square minus sigma x sigma y minus sigma y sigma z plus sigma z sigma x this should also be equal to tau xy dash square tau yz dash square plus tau zx dash square plus sigma x dash sigma y dash plus sigma y dash sigma z dash plus sigma z dash sigma x dash.

Similarly, we can write the third stress invariant also along similar line sigma x sigma y sigma z plus y tau xy tau yz tau zx minus sigma x tau yz square plus sigma y into tau zx square plus sigma z tau xy square it is also, equal to sigma x dash sigma y dash sigma z dash plus y tau xy dash tau yz dash tau zx dash minus sigma x dash square xj sigma x dash tau yz dash square plus sigma y dash into tau yz square plus sigma z dash into tau xy dash square. So, the expression for the quantities I1, I2, I3 are going to come to the same value; that is why they are not dependent on the coordinates and it is because of this reason they are called stress invariants.