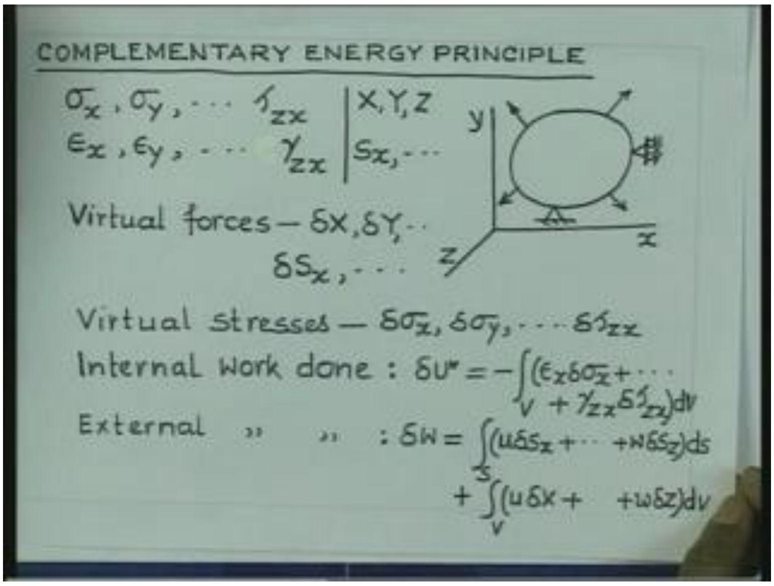
Advanced Strength of Materials
Prof. S. K. Maiti
Mechanical Engineering
Indian Institute of Technology, Bombay

Lecture – 29

So, we have been discussing about the energy theorems, last time we talked about uniqueness theorem super position principle. Then we took the minimum potential energy principle and also we have shown the proof of Castigliano's theorem. Let us now consider some more theorems. Today we will try to consider complementary energy principle this principle is similar to minimum energy potential energy principle. In the case of minimum potential energy principle, we started with the minimum virtual we started with the principle of virtual displacement. And we proved the minimum energy principle potential energy principle.

(Refer Slide Time: 01:10)



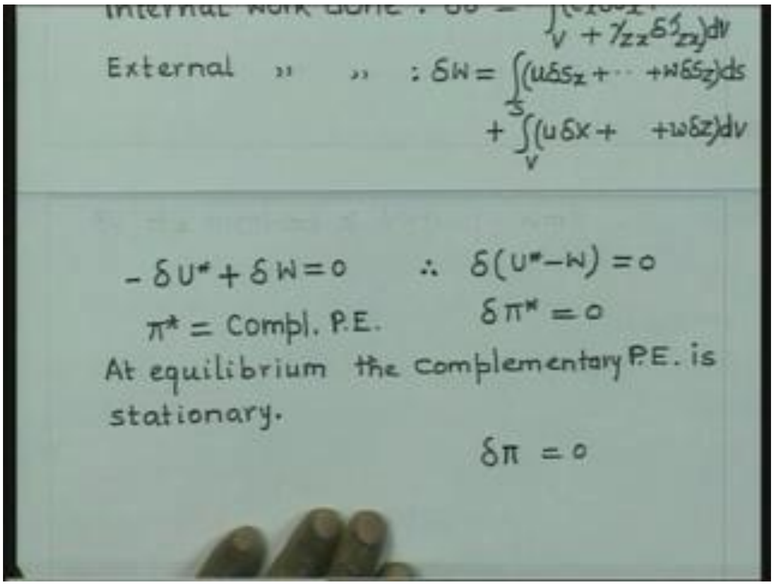
Today, we will try to start from the method of virtual forces and we prove the complementary energy principle. So, let us consider again an object like this in 3 dimensions it is subjected to forces, Body forces x y z and surface tractions Sx Sy Sz. This give rise to stresses sigma x sigma y up to tau zx similarly, strains epsilon x epsilon y up to gamma zx. Let us consider, virtual forces delta x delta y delta z which are functions of x y and z applied to the whole body and at the same time we also consider virtual tractions delta Sx delta Sy and delta Sz applied over the boundary.

Now, these virtual forces will give rise to virtual stresses. So, let us represent those virtual stresses virtual stresses like delta sigma x delta sigma y this will continue with all the stresses, 6 components of stresses. Because of these virtual stress increments we are going to get internal work done. So, this internal these are the internal stresses, which develops and it is going to work on the real strengths.

Therefore, we can now find out internal work done. Let us represent this internal work done delta u star and that is equal to integral of the product epsilon x delta sigma x plus all the 6 terms up to gamma zx delta tau zx dv. The external work let us consider it to be delta w and that is going to be integral of the product of virtual displacements; product of real displacement multiplied by virtual surface tractions.

So, u into delta Sx v into delta Sy plus w into delta Sz and this is to be integrated over the whole surface. Similarly, we will also get the contribution from the volumetric forces which are u delta x b delta y then w multiplied by delta z and this is integrated over whole volume. So, this gives us the external work done. Now, by the method of virtual work we know that the sum of the internal work and the external work would be equal to 0.

(Refer Slide Time: 05:57)

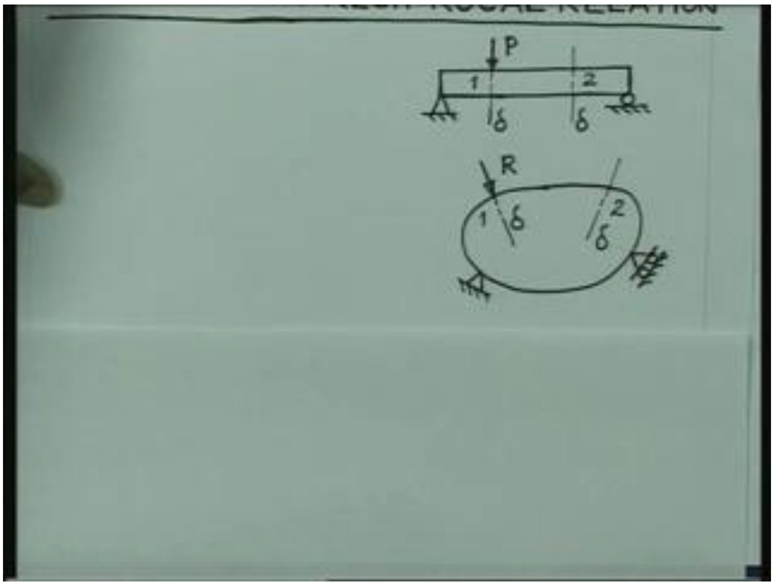


So, therefore, by the method of virtual work we would have delta u star and here in this is delta u star plus delta w equal to 0. And therefore, now we have seen that they are of opposite sign. So, we can write now delta. In fact, we find that this is negative work done and therefore, we can add here this way, and therefore delta u star minus delta w equal to 0. And this u star minus w is nothing but complementary potential energy.

So, this is delta pi star equal to 0 and this pi star is nothing but complementary potential energy. So, what we have found that if a system is in equilibrium and if they are subjected to set of virtual body forces and surface tractions. We are going to get variation of the complementary potential energy equal to 0. And therefore, we can now say that at equilibrium the complementary potential energy is stationary.

So, finally, this means at equilibrium the complementary potential energy is stationary. It is similar to what we found in the case of minimum potential energy delta pi equal to 0. And now in a case of complementary potential energy we have got delta pi star equal to 0. So, at equilibrium both potential energy and complementary potential energy are stationary.

(Refer Slide Time: 08:45)

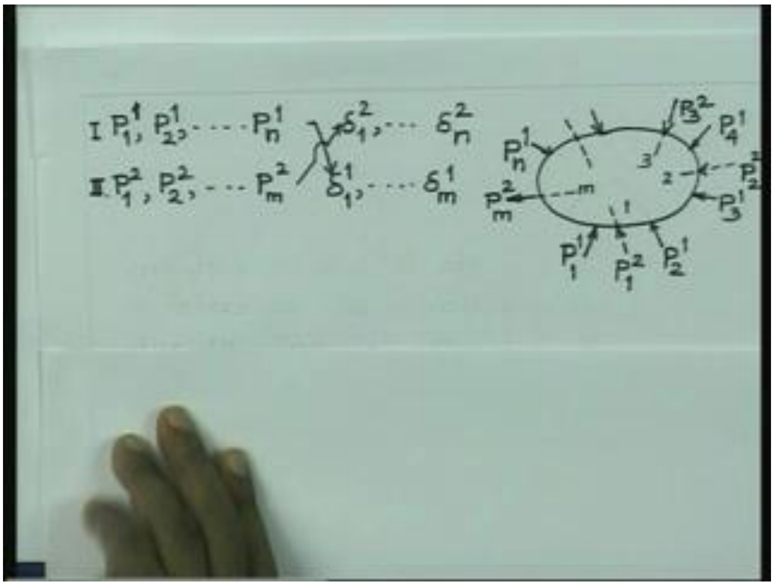


Now, we will consider another theorem which is very important theorem of elasticity it is known as Maxwell betti reciprocal theorem or Maxwell betti reciprocal relation. Let me just state what it means think of a beam like this, if the beam is subjected to some force. Let us say we have a at this location 1 a force is applied which is of magnitude equal to p. Then the beam is going to give rise to some deflection at the point 2.

If the now, if the force is now shifted from position 1 to position 2 then the deflection that will be produced at location one is going to be the same. So, what I mean that is if P is acting here and if we have a deflection here of magnitude equal to delta. When P is shifted to location 2 we have also going to see a deflection delta at this location 1. So, this is reciprocity relation or this is reciprocity.

Similarly, in the case of a 3 dimensional object also, if you think of it that you are applying some force at this location which is let us say, r and if this is going to produce a deflection here at the location let us say delta. And if we now shift and this delta is measured in this direction, now if w shift the point of application of the force r to this point. And try to monitor the deformation at this point it will be again equal to delta. So, this is reciprocity.

(Refer Slide Time: 10:52)



It has also other way of specifying this reciprocity relation now we consider a general body like this. And let us consider that, this body is subjected to forces, let us say that it is subjected to forces like this. Where 1 force we have another force acting at this point similarly, here also we have another force and we consider that we have number of such forces acting on the body. So, we consider the set of forces to be given by let us say P 1 1.

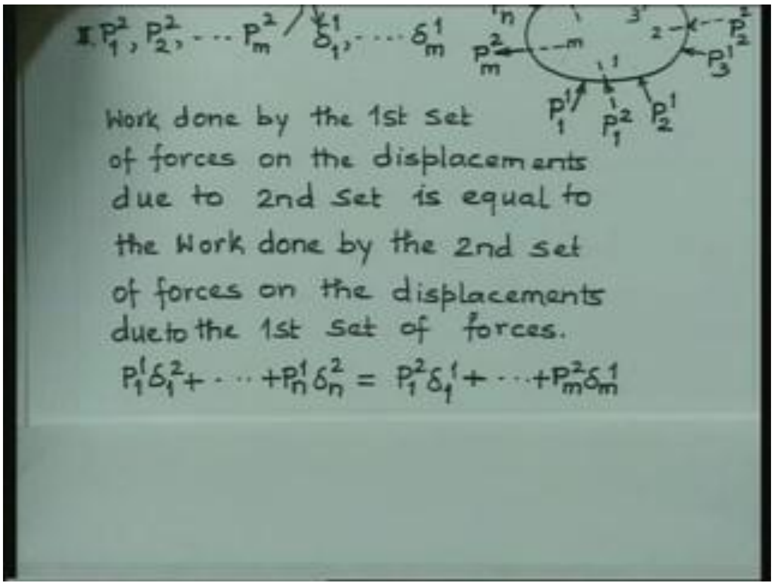
This is P 2 1 P 3 1 and similarly P 4 1 P five 1 let us say that this force which is here is let us say nth force and it is indicated by P n superscript 1. So, if we consider now this set here we have P 1 1 P 2 1 up to P n 1 this is the set 1. Now, if we monitor the deflection produced by these forces, let us say we monitor the deflection produced by these forces along these lines.

So, let us consider that this is direction 1 2 3 and let us say we consider that in m such directions that we are trying to talk about. So, we will find that deflection along these lines let us indicate them by delta 1 1 up to there are m such directions. So, this is 1 this is 2 this is 3 4 and let us say this is mth direction. So, we will have the deflection produced by the first set given by delta superscript 1 up to delta m superscript 1.

Now, if we apply a set of forces now in these directions. Let us say we will like to now, apply force in this direction of magnitude equal to let us say P 1 superscript 2 and here 1 force let us say P 2 superscript 2. And here also another force let us say this is P 3 superscript 2 and P 4 and so on. And let us consider that we are applying a force here of magnitude and P m 2. I must mention that this deflection that we have got from the first system measured in this direction, it is directed in this direction.

It is measured really in this direction. So, these forces if we consider now second set they are P 1 superscript 2, P 2 superscript 2 and P n superscript 2. So, what I am saying that now we are applying the set P 1 superscript 2 up to P m superscript 2 when the first set is absent. Now, if I try to see the deflections along the line of action of earlier force P 1 1. So, this is the force P 1 1 let us say that the deflection in this direction is delta 1 superscript 2. So, the force sets this force set produces a deflection of delta 1 superscript 2 along the line of action of P 1. Similarly, we will have up to nth force and the deflection is delta n 2.

(Refer Slide Time: 16:15)

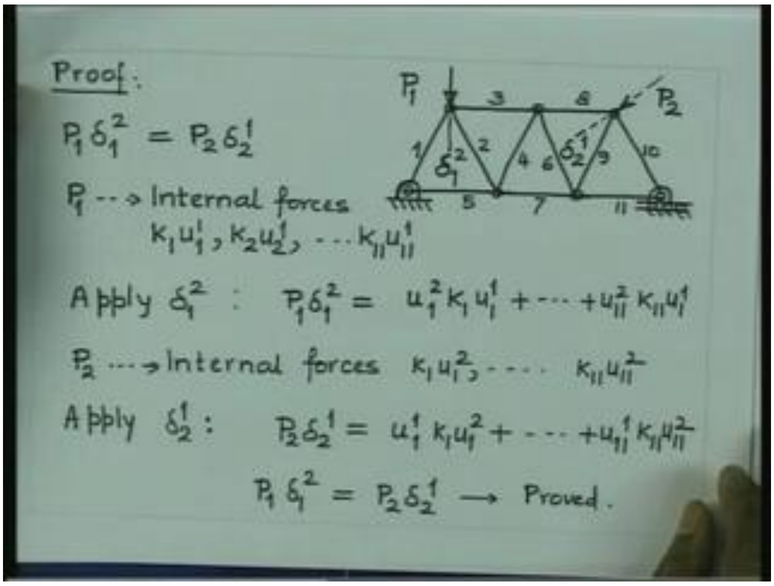


Now, according to the reciprocal theorem the work done by the first set of forces on the displacement produced by the second set is equal to the work done by the second set of forces on the displacement produced by the work done by the first set of forces on the displacement produced by the second set is equal to the work done by the second set of forces on the displacement produced by the first set.

So therefore, this is what is; the reciprocity relation. So, we will write work done by the first set of forces on the displacement, due to second set is equal to the work done by second set of forces on the displacements due to the first set of forces. So, this is the reciprocity relation and mathematically what it means is that P 1 1 delta 1 2 up to P n 1 delta n 2 is equal to P 1 2 delta 1 1 up to P m 2 delta m 1.

So, therefore, in the second set we have n number of forces and therefore, we are trying to consider m number of displacements along the line of action of those forces which are of first set. And in the first set we have n number of forces and therefore, we are considering the deflection along the same forces due to second set. Now, we will like to prove this theorem. Before we go into proving this theorem I must mention that these forces need not be only direction tension or compression, it could be as well torsion bending moment it does not matter. The nature of forces does not restrict it to only tension and compression.

(Refer Slide Time: 19:27)



So, now, let us consider proof of this theorem for considering the proof this theorem is applicable to both discrete and continuous system. So, we will consider now for simplicity a discrete system like a structure. Let us consider this planar structure consisting of elements like 1 2 3 4 5 6 7 8 9 10 and 11. So, these are the eleven members which are interconnected, each is a uniform member. Now let us consider application of a force on this structure where this is a force let us apply here in this direction P 1 and we consider the deflection along this line.

Let us say that the deflection along this line is equal to delta 2, superscript 1. Now we remove this force P 1 and we apply a force here at this point of magnitude equal to P 2. And try to monitor the deflection along the line of action of P 1; let us say that this deflection here is equal to delta 1 superscript 2. According to reciprocal theorem therefore, we must have P 1 into delta 1 superscript 2 must be equal to P 2 into delta 2 superscript 1.

So, therefore, P 1 delta 1 superscript 2 is equal to P 2 multiplied by delta 2, superscript 1. Because of the, now let us consider how to get at this relationship or let us prove this relationship. Now, if I am applying the force on this structure here this will give rise to some forces in each of these elements and if you consider that, this element is undergoing some deformation equal to u 1. And the member at an axial stiffness is equal to k 1 then in that case the force that is going to develop or the internal force that is going to develop in this element is going to be k 1 into u 1.

Now, we will just try to consider superscript 1 which try to indicate that that is due to the force P one. So, therefore, if I consider now a force P 1 is applied then the internal forces that are going to develop in this all 11 members. They will be nothing but k 1 multiplied by superscript 1 where k 1 is the axial stiffness of this member 1. Similarly, we can write for the second element k 2 u 2 superscript 1, and therefore we can write for all the 11 members like this last 1 would be k 11 u 11 superscript 1.

Now, if we consider application of a virtual displacement in this direction of magnitude equal to. We just we have the force P 1 acting and we have internal forces developed in all the 11 member. Now, we would like to consider application of a virtual displacement at this point in this direction of magnitude equal to delta 2 1. So, if you just apply this displacement then this is going to cause internal forces internal displacement for deformation in each of the members.

And the internal therefore, we will find. Now, let us I I will repeat again. I must repeat this point again I had the force P 1 acting now what I do is that I try to apply a displacement here not at this point of magnitude equal to delta 1 superscript 2. So, if we apply the displacement there then we are going to get the work done external work done is going to be given by P 1 delta 1 superscript 2. Now, this displacement which corresponds to the force P 2, it will also give rise to the deformation of each of these eleven elements which can be indicated by u 1 superscript 2 u eleven superscript 2.

Therefore, the internal work done is going to be now we have already the force acting in the element number 1 is k 1 into u 1 superscript 1. And now we have some deflection or deformation of these elements which is of magnitude equal to u 1 superscript two. And therefore, this is the internal work done. Similarly, for all the elements eleven elements we are going to get the work done to be given by this expression. So, eleventh element has the force k 1 1 u 1 superscript 1.

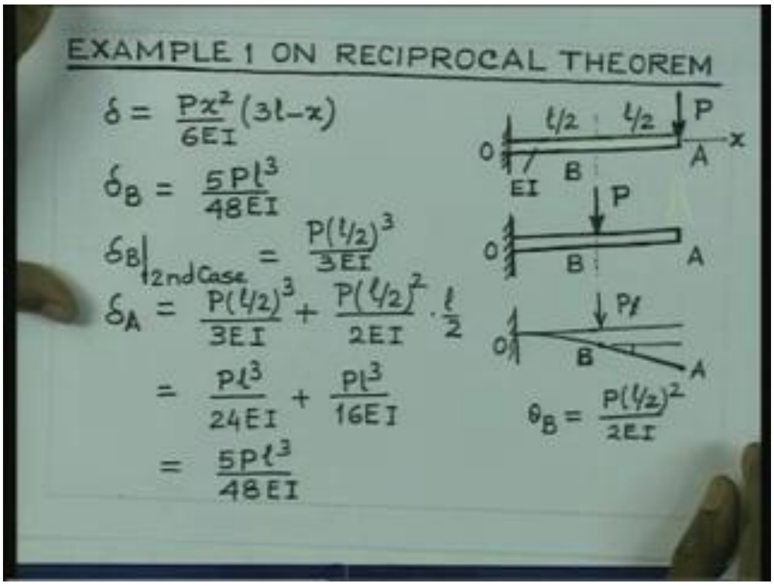
And now we have a deformation u 11 superscript 2 therefore, the work done is given by this. So, this is the first case, now let us consider that we have P 1 is not acting simply the structure is subjected to P 2. So, we apply P 2, and we now consider the internal forces that comes up due to this. So, internal forces if each of the elements are deforming to the extent u 1 superscript 2 u 2 superscript 2 and so on.

Therefore, the forces that are going to come up is are going to be given by k 1 u 1 superscript 2 and last element would have force k 11 and u 11 superscript 2. Now, we have the force P 2 acting there and we apply the virtual displacement along the line of action of P 2 which is equal to the value delta 2 superscript 1 So, the virtual work so if we apply now virtual displacement delta 2 superscript 1. Then the work done external work done is P 2 delta 2 superscript 1 and this is equal to the internal work done.

So, when we apply this displacement here the internal displacement that are going to be produced in each of elements are going to be given by u 1 superscript 1 u 2 superscript 2 and so on. Therefore the work done by the internal forces are going to be u 1 superscript 1 multiplied by k 1 u 1 superscript 2 and this will continue up to u 11 superscript 1 multiplied by k 11 u 11 superscript 2.

Now, look at these relations; so you have the expression here and here on the right hand side they are the same and therefore, we have P 1 delta 1 superscript 2 is equal to P 2 delta 2 superscript 1. So, that proves the reciprocal theorem. In fact, it can also be proved for continuous system, but that proof will be little involved. So, this is good enough for you that, reciprocal theorem needs that the work done by first set of forces on the displacement due to second set is equal to the work done by the second set of forces on displacement due to the first set of forces.

(Refer Slide Time: 29:30)



Now, we would like to consider some example, very simple example to illustrate the first example we will take it up like this. You take the example of a beam this beam is of length l it is subjected to a force P. And let us consider that the modulus of elasticity is E and the moment of inertia is I. So, therefore, the modulus of rigidity of the cross section is E into I.

Let us see if we apply the force P here and try to calculate the deflection here at the point b which is exactly at the middle we can find that. So, let us first consider that deflection should therefore, the force is acting there. Then the deflection equation for this case is going to be given by P x square 6 EI into 3 l minus x. Where in, we are considering x is to be aligned with the axis of the beam.

So, if we now consider the deflection at the point b which is exactly at the middle. So, you put x is equal to l by 2; so that gives us 5 Pl cube by 48 EI. Now, if we consider the force to be applied at this point and we try to calculate the deflection at the point a. That can be obtained by the consideration that you just think of a that, you have the beam here. And it is if the force is acting here at this point it is going to deflect like this up to this point and then it is going to deform or deflect like a rigid element from this point to this point.

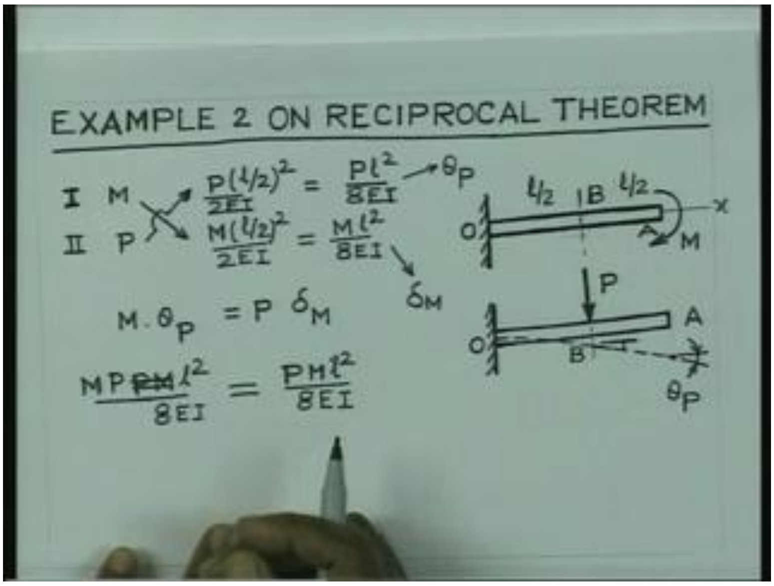
So, therefore, from b to a it is going to deflect like a rigid element. So, therefore, if you now calculate deflection at this point this is at l by 2 from the fixed end. So, if you consider that this fixed end is O. So, this deflection at this point delta b second case. So, we will like to write this is the second case is equal to we can obtain that deflection from the formula of a cantilever beam pl q by 3 EI. And therefore, this is nothing but pl by 2 cube by 3 EI. So, that is the deflection of the point b under the action of this force P by 2 here, under the action of the force P acting at this point.

Now, if I have to calculate the deflection at this point we can consider deflection at a is nothing but deflection at b plus the rotation here at point b multiplied by this distance l by 2. And this rotation at the point b in this case is given by the formula pl square by 2 l and therefore, it is pl by 2 square by 2 EI. So, that is the rotation.

And therefore, we can now calculate the deflection at the point by adding up this and for adding up the deflection due to this rotation to this 1. So, therefore, delta at a is equal to Pl by 2 cube by 3 EI plus Pl by 2 square by 2 EI multiplied by l by 2. So, we have got this delta a given by this relation and this comes out to be Pl cube by 24 EI plus Pl cube by 16 E. And the 2 together gives us 5 Pl cube by 48 EI 48 EI.

So, therefore, we see that the deflection in this case and deflection in the other case is the same, therefore that is also as we expect from the reciprocal theorem which states that. If the force P is acting here and it gives rise to displacement at the point b. And if the force is now shifted to this point the deflection that will be produced at a is the same as in the earlier case. So, deflection produced by P at this point b is equal to the deflection produced by the force P at the point a.

(Refer Slide Time: 36:21)



Let us consider another example where in we have 2 forces, but now they are different: 1 is moment another is a direct force. Let us now consider the same type of beam subjected through a bending moment m at the free end. So, let us consider that this is the set 1 and now we consider a another set where the P is the force acting at the middle of the beam. So, therefore, the second set is P if m is acting here then the deflection that is produced at this point is going to be given by. So, m is going to produce a deflection along the line of action of P is given by it is it follows the formula M x square by 2 EI.

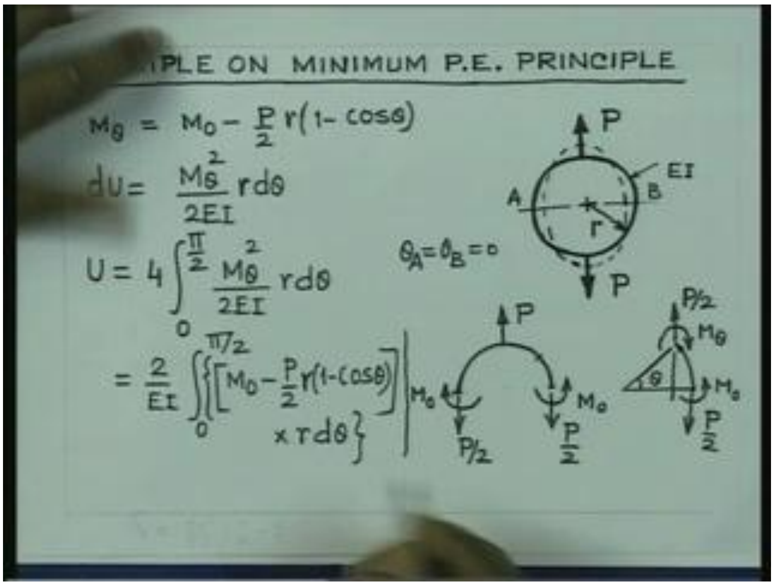
So, the deflection equation is m x square by 2 EI and therefore, at this point l by 2 it is going to be ml by 2 square by 2 EI and this gives us ml square by 8 EI. When P is the force acting here that is force is going to produce a rotation at this point or deflection at that point in the same sense as this bending moment. Look at this case here the deflection curve is going to look like this.

It is going to deflect and then it will remain like a rigid element from here to this point and therefore, the slope here is going to be this 1. So, that is produced by P. So, therefore, let us indicate by theta P. In fact, that theta P is same as the theta at this point. So, when you have the force P acting here the slope produced is nothing but Pl square by 2 EI and therefore, we can get that from the formula Pl square by 2 EI.

Since l is l by 2 therefore, we will have Pl by 2 square by 2 EI and that is again equal to pl square by 8 EI. According to reciprocal theorem, what we expect is that work done by this m on the rotation produced by P. So, that is the work done by M on rotation produce by P m into theta P and the work done by the force P on the deflection produced by M. So, we have if we indicate that P and the deflection is delta M.

Now, I think let me just also connect this way now what we have really is that this is theta P this is nothing but theta P and this is nothing but delta due to Mm. So, if we now substitute the value what we have P delta M is Ml square by 8 EI. And P into M into P which can be written P into Ml square by let me write this way this is M into Pl square by EI. So, you can see that the both the expression are same and they are equal and hence it also follows the reciprocal theorem.

(Refer Slide Time: 42:06)



Now, we will consider 1 example on the minimum potential energy principle. These energy principles are very useful in solving equations statically indeterminate problems. So, let us consider application of the minimum potential energy principle for solving 1 statically indeterminate problem. Consider the case of a ring of uniform cross section let us say that the modulus of rigidity of the cross section is EI.

And the radius of the ring is r it is subjected to tension diametrically opposite tension. So, the objective is to find out what is the deflection of the top point with respect to bottom point. And at the same time where what is the maximum bending moment acting on this ring. So, if we now take section of the ring along the horizontal diameter. And if we try to segregate that top half we will see a picture like this where the force acting P here.

And we are going to get because of symmetry we will have the reaction force here P by 2 P by 2 and at the same time we are going to get some bending moment here. Let us say that that is the bending moment m 0 acting here. Similarly, this is the bending moment m 0 acting here. Now, when the ring deforms under the action of 2 forces it is going to become elliptical may be it will take a shape like this.

So, it is going to take a shape like this and you can see that here in the slope originally it was 90 degree and the slope is after deformation also 90 degree there is no change in the slope. And therefore, the slope is unchanged. And we can write that the slope here at this point is equal to 0. So, if you call it a and b you really have a case theta a theta b equal to 0. Now, in order to solve this problem it is first necessary to calculate this reaction moment M 0.

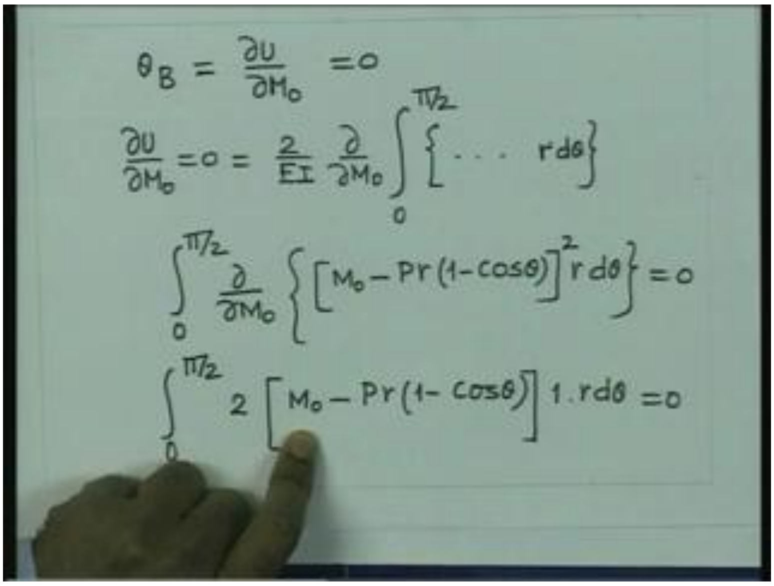
So, we can calculate that by applying the energy principle. Let us consider a sector of the ring here this much and it is making an angle of theta will be horizontal direction let us say. And if I draw the free body diagram of this and show the forces acting on the sections this is at before it is P and M 0.

And let us say that the force which is going to come up here is the force is the force let us say it is it is; obviously, going to be P by 2. So, we can write this thing as P by 2. And at the time we are going to get a moment here let us indicate this moment by M theta. If we consider the equilibrium of this element from here to here, then we have m theta is equal to it is M 0 minus this is P by 2 P by 2 into this distance which is r minus r cos theta.

Therefore r minus r cos theta so therefore, it is 1 minus cos theta. So, that is the bending moment which is acting at this section theta. Now, if we consider the strain energy of deformation of the body. The strain energy that is here that is nothing but dU dU is given by m theta square by 2 EI into the length of the element let us say r d theta. So, if we consider a small element here making an angle of d theta at the center.

So, therefore, the element energy is equal to M M theta square by 2 EI EI r d theta. So, therefore, the total strain energy in the body is going to be if we integrate this quantity from this point to this point and take the 4 times multiplication that will give us the total energy. So, because the there is symmetry here the energy here in this sector is equal to energy here so also energy in this sector and also in this sector. So, therefore, you can write this total strain energy is nothing but it is four times 0 to pi by 2 M theta square by 2 EI r d theta.

(Refer Slide Time: 48:51)

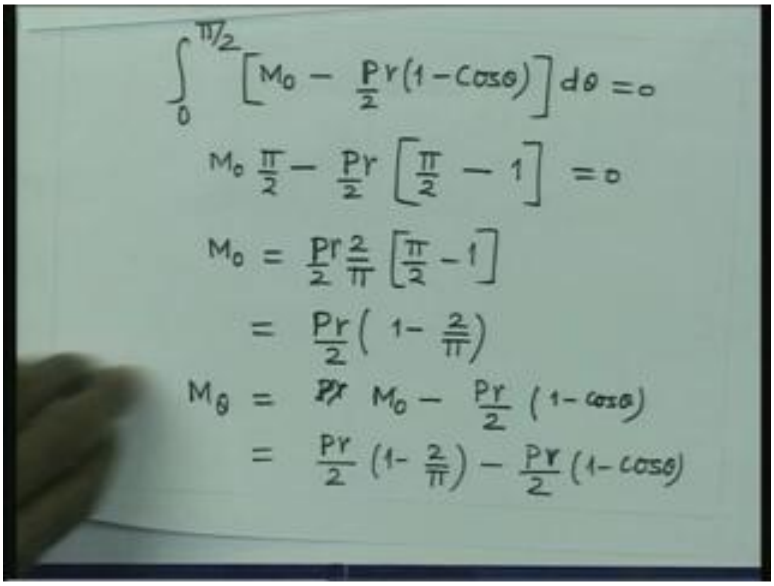


So, this is equal to 2 by EI 0 to pi by 2 we have now M 0 minus P by 2 r 1 minus cos theta into r d theta. So, that is the integration. If you are considering now the rotation at the point b or a we can write that rotation at the point b to be given by theta b is equal to delta U delta M 0 And in this case, since U is a function of both P and m 0 we can write delta u delta m 0 by castigliano's theorem. So, if we now try to consider theta b which is equal to 0 we will now get delta U delta M equal to 0 equal to 2 by EI delta delta M 0 of the integral 0 to 2 pi. We will not write the expressions.

So, I'll just write the quantity within the curly bracket like this. So, this is what is we have last we have of course, r d theta. So, we will write this that is equal to 0. So, now in this case you note that the quantity which is here there is a square here. So, the quantity which is here appearing within the integral sign is not consisting of m 0 as a variable of integration it is a parameter.

So, therefore, we can do the integration or we can do the differentiation before the integration, you can take this differentiation inside. So, if we do that, and since if this is 0 we can write now 0 to pi by 2 delta delta M 0 M 0 minus Pr 1 minus cos theta into r d theta equal to 0. So, if you do this gives us 0 to pi by 2 and this will give us 2 and M 0 pr 1 minus cos theta and integration of this quantity with respect to M 0 is 1 and we have r d theta equal to 0.

(Refer Slide Time: 51:21)

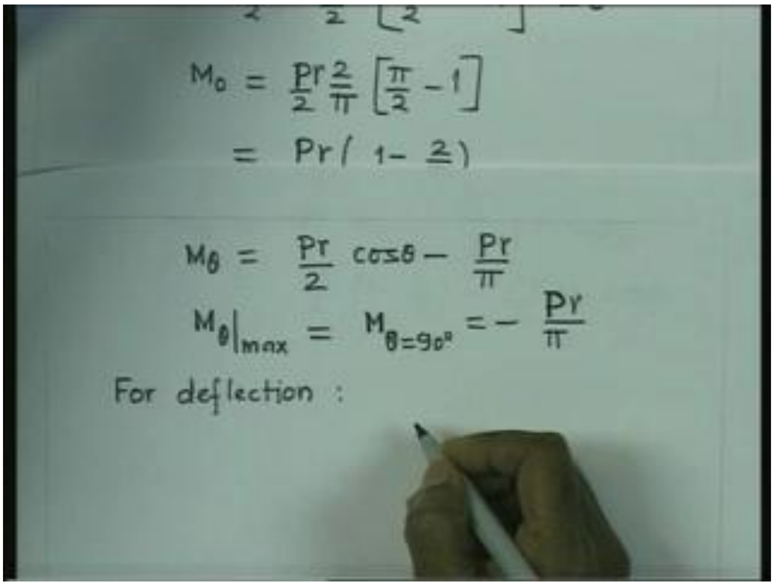


.

So, 0 to pi by 2 this is m minus pr 1 minus cos theta we can take this out 2 r also out 2 also out. So, therefore, this is equal to 0 and if you integrate this give it is going to give us M 0 pi by 2 minus Pr this will give us pi by 2. And this is sine theta and sine theta 0 to pi will give us 1. So, that is equal to 0. Finally, what we find is that this m 0 is equal to pr 2 by pi into pi by 2 minus 1 which is nothing but Pr into 1 minus 2 by pi. So, the bending moment that is going to be occurring at point b or a is going to be given by this.

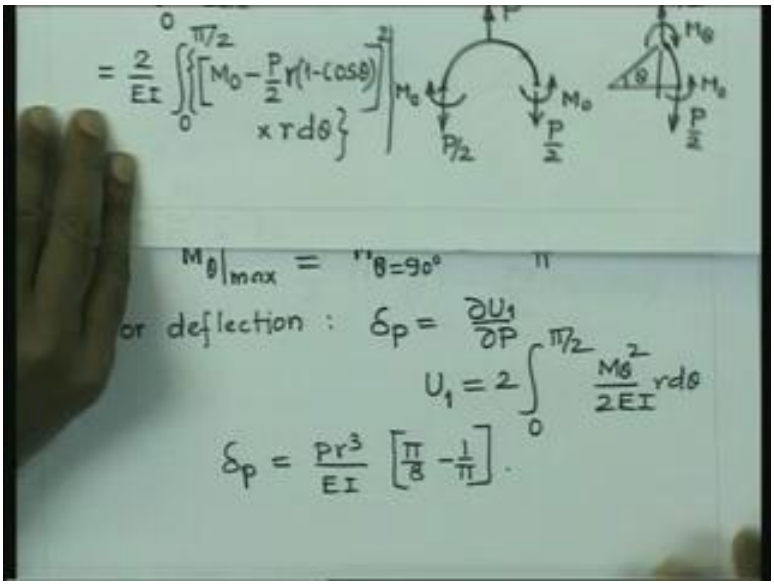
I think that we have 1 mistake here this is we have just here to take care of this we have missed this 2 here. So, if we carry forward this 2, so we have this 2. So, this will be Pr by 2 is this. So, we get the bending moment, and therefore you see its the value for m theta we try to write m theta is going to be Pr it is M 0 minus Pr by 2 1 minus cos theta and now if I write Pr by 2 1 minus 2 by pi minus Pr by 2 1 minus cos theta. So, finally, this gives us m theta is equal to Pr by 2 cos theta minus Pr by pi. So, you get the maximum bending moment m theta maximum m theta maximum when this is equal to 0.

(Refer Slide Time: 53:16)



That is possible when theta equal to pi. So, therefore, this is when you think of it theta equal to 90 degree and its value is equal to theta equal to 90 degree it is minus pr by pi.

(Refer Slide Time: 54:30)



You can calculate the deflection by considering the fact that you. So, for deflection you want to calculate the deflection of this point. So, what you do is that you consider these 2 points fixed and this is moving. And you can calculate delta P to be given by castigliano’s theorem delta U delta P. You can calculate u for this, this u let us consider this U is equal to let us say U 1 and this U 1 is nothing but the energy of the portion 0 to pi by 2 this portion.

So, therefore, it is 2 times m theta square by 2 EI r d theta. So, you can calculate this energy of the half of the ring by this formula. And then if you differentiate this with respect to P that gives you the deflection. And I would to give you this deflection you calculate yourself it is going to be nothing but delta P is equal to pr cube by EI into pi by 8 minus 1 by pi. So, you can apply the energy theorem for the solution of statically indeterminate problem and you can also calculate deflections.