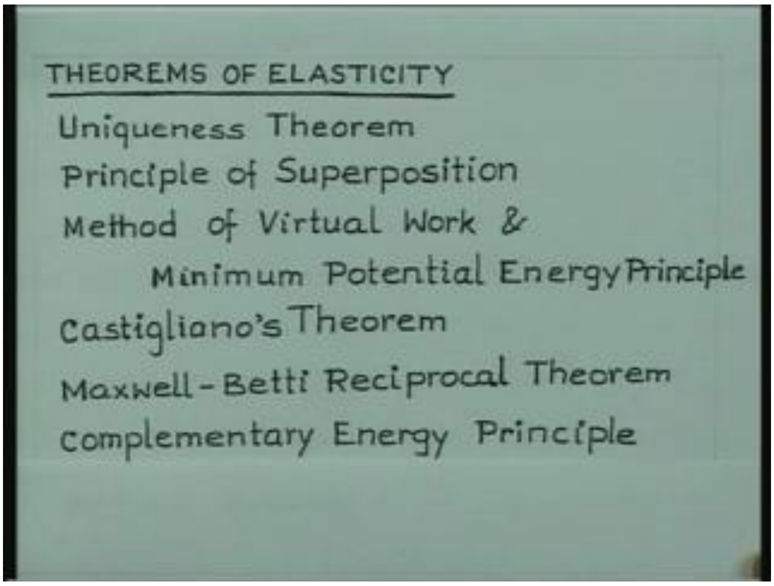
Advanced Strength of Materials
Prof S. K. Maiti
Mechanical Engineering
Indian Institute of Technology, Bombay

Lecture – 28

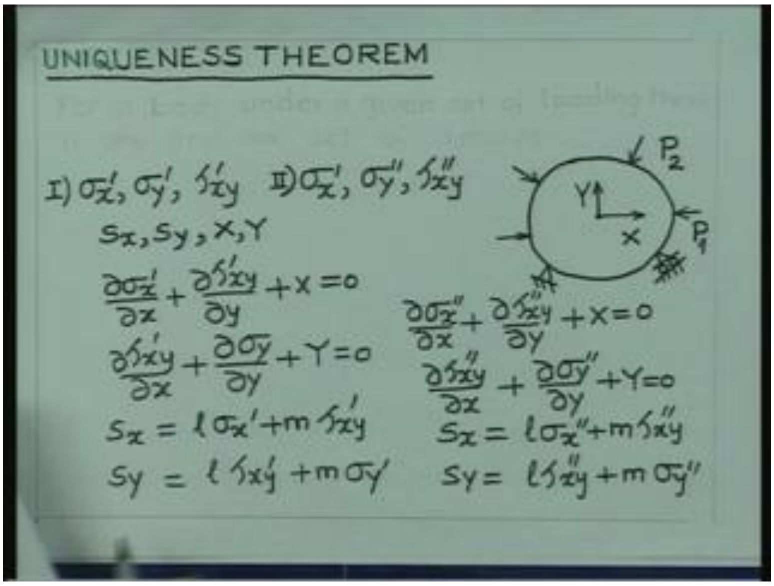
(Refer Slide Time: 00:57)



In this lecture, we would like to consider some theorems of elasticity. There are number of theorems which are very useful and we would like to consider some of them. First one is Uniqueness theorem, principle of Superposition, method of virtual work and minimum potential energy principle Castigliano's theorem Maxwell-Betti Reciprocal theorem, complementary energy principle and we will also consider some problem solving, using some of them.

So, let us consider first Uniqueness theorem. In the Uniqueness theorem, what is indicated is the following: for a body subjected to a specified system of loading, there can be 1 and only 1 set of stress distribution. That means if you have a body and it is subjected to loading like this at its boundary, it can also be subjected to body forces like X and Y in 2 dimensions.

(Refer Slide Time: 02:14)

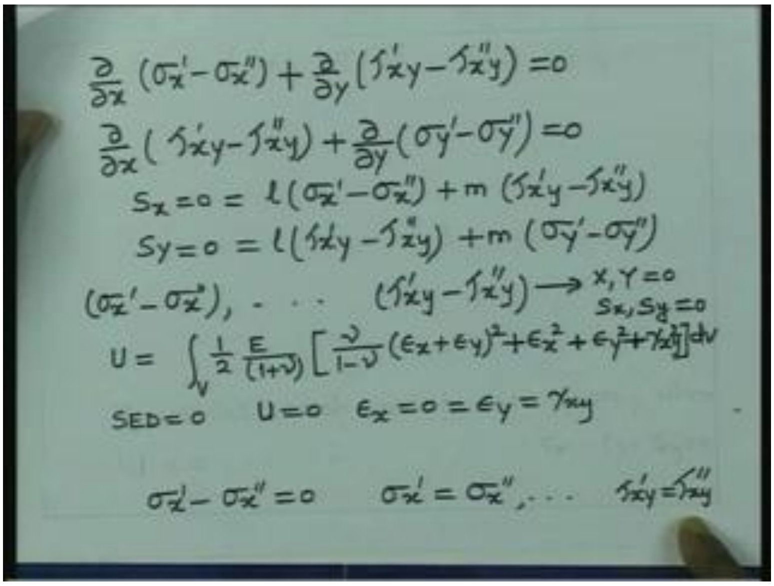


If, you indicate the loading boundary loading by P 1 P 2 and so on, for this specified set of loading P 1 P 2 etcetera and x and y, the stress distribution in the body is unique. There cannot be 2 possibilities. Let us consider that there are 2 possible solutions. And these sets are represented by sigma x dash, sigma y dash and tau xy dash, that is the first set. And let us also consider that, there is another set possible; which is indicated by sigma x dash, sigma y dash and tau xy dash. These 2 sets, they correspond to the boundary tractions. Let us say x and y and the body forces x and y.

Since, these are the solutions then we must have delta sigma. They will be satisfying the equilibrium equation. So, therefore, delta sigma x dash delta x plus delta tau xy dash delta y plus X is equal to 0. Similarly, we would have delta tau xy dash delta x plus delta sigma y delta y plus Y equal to 0. Since, the second set is also the solution, we would certainly have delta x double dash delta x plus delta tau xy double dash delta y plus X is equal to 0. And delta tau xy double dash delta x plus delta sigma y double dash delta y plus Y is equal to 0. And at the same time our boundary traction must be related if, the boundary direction cosines at a point with a length n. Then S x must be equal to l into sigma x plus m into tau xy dash. So, S x is l into sigma x dash plus m into tau xy dash. Similarly, S y is equal to l into tau xy dash plus m into sigma y dash.

Now, second set is also going to satisfy similar conditions; l into sigma x double dash plus m into tau xy double dash is S x. And S y is equal to l tau xy double dash plus m sigma y double dash. Now, let us try to subtract the first second set of equations from the first set first equilibrium equation and then we will also consider the boundary conditions.

(Refer Slide Time: 07:36)



So, therefore, if we do that, I will subtract this equation from this 1 that is going to give us; delta delta x sigma x dash minus sigma x double dash delta delta y tau xy dash minus tau xy double dash is equal to 0. Similarly, we will have delta delta x tau xy dash minus tau xy double dash plus delta delta y sigma y dash minus sigma y double dash equal to 0. So, therefore, we get a new set of solution, which is given by difference between the 2 sets which corresponds to a loading condition, where there is no body force.

Similarly, we are going to get now S x equal to 0, for this arising out of the 2 set will have S x equal to 0 which is given like this. Similarly, S y is also 0 arising out of 2 set, a new set of stresses. So, we find that this set; sigma x dash minus sigma x double dash, similarly tau xy dash minus tau x dash xy double dash. They corresponds to a set of loading where in X, Y are 0. So, also S x, S y are also 0.

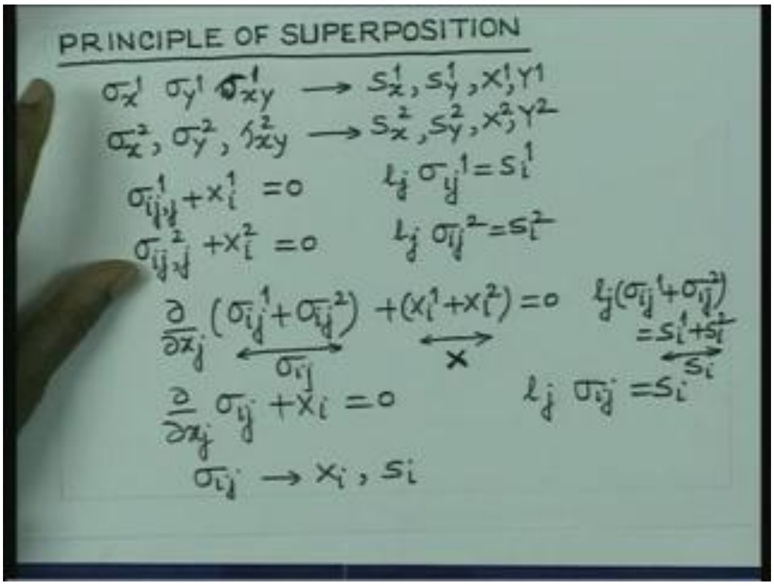
So, the set of new set of solution or rather new set involving; sigma x dash minus sigma x double dash and sigma y dash minus sigma y double dash and tau xy dash minus tau xy double dash. They correspond to a set of forces, where in the body forces are 0 and also boundary forces are 0. Now, let us try to look into the strain energy of the whole body and its form.

The strain energy can be written in a form like this. So, the expression for this strain energy, in the whole body is given by; integration of 1 by 2 e by 1 plus nu into nu by 1 minus nu into epsilon x plus epsilon y whole square plus epsilon x square plus epsilon y square plus gamma xy square. And this integration is to be done over the whole body. What we find here, that the expression is such, that irrespective of this sign of the strains, we are going to get the strain energy density. This, indicate the strain energy density positive and if, the strain energy of the whole body is to be 0, we must have each of this strain components 0. Now, the strain energy density;, so strain energy density is 0 and that will make the total strain energy 0. U will be 0 only when, strain energy density is 0. And this also indicates that, this is indicating a situation that epsilon x equal to 0, so also epsilon y, so also gamma xy.

Now, when we do not have any external loading on the component, the work done on the component in deforming is 0. Therefore, the strain energy of the body is 0. And it would indicate that if, the strain energy of the body is 0, this relationship indicates that it is possible, only when the strain at each and every point of the body is 0. And therefore, this will mean that, since strains are related to the stresses, if the strains all 3 components of strains are 0, that would mean that; sigma x dash minus sigma x double dash that is also 0. So; that means, that sigma x dash is equal to sigma x double dash.

So, by the same argument we will find that, the 2 sets will finally, give rise to a situation like this; that the normal stress in the x direction in the 2 sets are equal, so also the shear stress are equal. So, hence we have only 1 solution. So, the 2 sets that we have assumed, they are only 1 set, they are equal. And hence, we cannot have 2 solutions to a problem. So, this is the proof of the Uniqueness theorem. So, finally, what we found is that, if you are given the loading on a body specified for, the set of loading specified, you can have 1 and only 1 set of solution for stresses.

(Refer Slide Time: 14:50)



Now, we will like to consider the second theorem which relates to principle of Superposition; which is concerning Superposition principle. If, you have a body and it is subjected to 2 sets of loading, then the stresses for each set of loading can be added together, to get the stresses corresponding to the 2 sets together. I repeat again; if, you are given a set of loading, we will have a set of stresses. And if there are 2 sets of loading on the body, then you can get 2 sets of stresses. If, you apply the 2 set together, the stresses that are going to come up will be also, obtained by adding the set of stresses together.

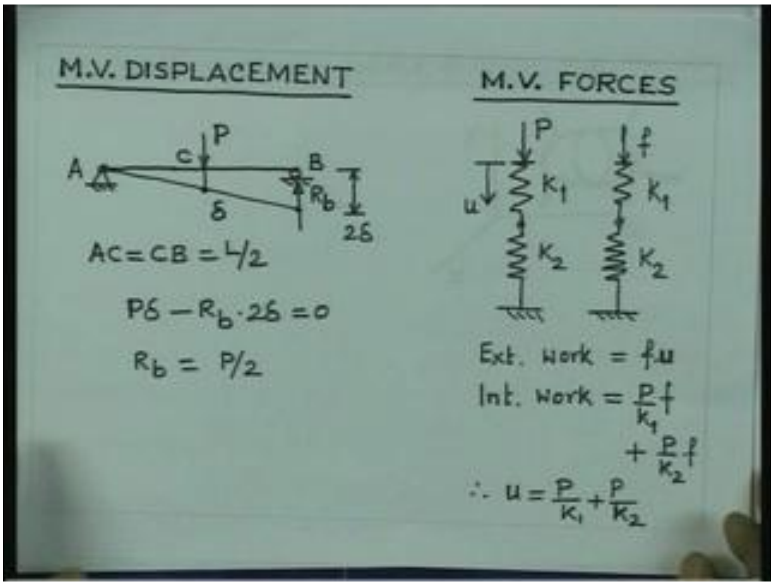
So, we would like to prove this and this is really possible because, the system is linear. So, let us consider that, we have the set of stresses: sigma x 1 sigma y 1 and sigma x and tau xy 1, so this is tau xy 1. And this corresponds to the forces S x 1 S y 1 and body forces X 1 Y 1. Similarly, let us consider that sigma x 2 sigma y 2 and tau xy 2 corresponds to, set of forces which are S x 2 S y 2 and X 2 Y 2.

Now, these set of stresses are going to satisfy the equilibrium equations. We can write now in tension notation: sigma i j, j plus X i 1 is equal to 0 for the first set. So, and so we will have Cauchy formula l j sigma i j 1 is equal to S i 1. So, this is the boundary traction. Similarly, we have sigma i j, j superscript 2 is going to be; X i superscript 2 equal to 0. Simultaneously, we will have the Cauchy formula: l j sigma i j 2 is equal to S i 2. So, these are the 2 sets.

Now if, we apply the 2 sets of loading together, then the stresses can be directly added and we get the total stresses. So, this is now can be shown like this, that if we add the 2 sets of this 2 set of equation, then we are going to get delta delta x j sigma i j 1 plus sigma i j 2 plus X i 1 plus X i 2 is equal to 0. Similarly, if we add the traction we get l j sigma i j 1 plus sigma i j 2 is equal to S i 1 plus S i 2 and finally, we if we write the total stresses sigma i j and this is the total traction X and here this is the total traction. This is the total boundary force, total body forces and this is the total traction.

Finally, we find that this equation is nothing, but delta x j sigma i j plus X i equal to 0. And we have l j sigma i j S i, and therefore your stresses sigma i j will corresponds to the sum total of the 2 sets of loading. So, therefore, the principle of Superposition is valid because, the governing equations are linear equations.

(Refer Slide Time: 20:21)



Now, like to consider method of virtual work and minimum potential energy principle. Before, I go into considering the method of virtual work and minimum potential energy principle of elasticity, I would like you to reconsider some of things, which we have done in your mechanics course: The method of virtual displacement and method of virtual forces. These are principles which are very useful and which are applicable to both rigid body system and deformable body system. And they are useful in finding out some time unknown forces and sometime unknown displacements.

So, in the case of method of virtual displacement, you can consider this principle or method of virtual displacement for the calculation of unknown forces. So, if you remember that, you know mechanics that you must have considered a situation like this; there is a rigid beam, let’s say A B. And it is subjected to some loading P at C. If, you are interested in finding out the reaction at B; 1 can consider the method of virtual displacement, 1 can apply the displacement to the load by an, by some amount. And if, you consider the displacement of the whole beam in conformity,, so if this points displaces by delta. Then, this B point is also going to displace by some amount, keeping the geometry straight. Therefore, the displacement here is going to be 2 times delta at the other end.

Therefore, we have the reaction force which is R b here that is going to move at a distance of delta 2 delta. And if, we assume that AC is equal to CB is equal to L by 2. Then, according to the method of virtual displacement; the work done the sum total of the work done by the external and reactive forces is equal to 0 or the work done by the external forces is equal to work done by the reactive or internal forces.

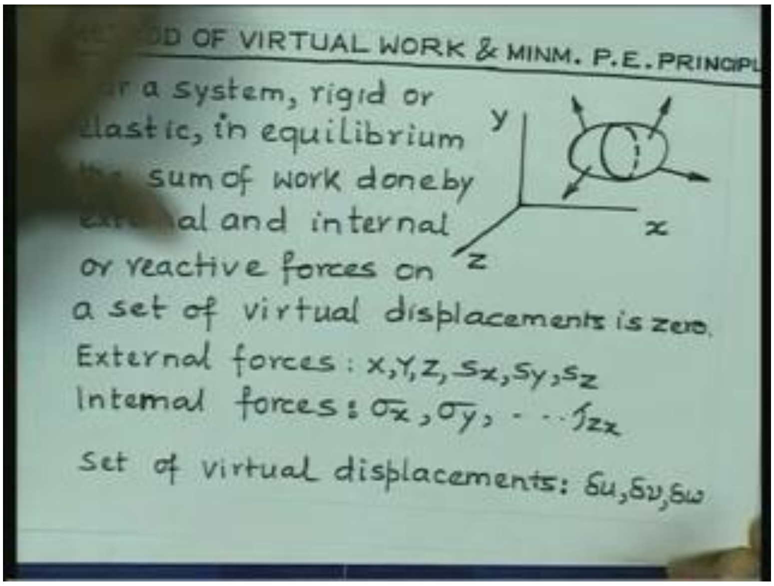
So, in this case we have the external force P. P doing a work, P will do the work while it is moving from C to this point by delta. The work done by the load is P delta and at the same time this reaction R b is moving by a distance of 2 delta. And therefore, since the movement in the opposite direction, the work done is negative. Therefore, R b into 2 delta equal to 0 and this is the sum total of the virtual work and this is 0 and that gives you directly R b equal to P by 2. So, this is the method of virtual displacement. Herein, we have been able to calculate the reaction by applying the virtual displacement.

Now, we can also have the method of virtual forces where, we apply the virtual forces and try to equate the internal work with the external work and calculate the unknown displacement. While like to illustrate this principle by considering this simple example. Let us consider that we have 2 springs connected in series like this. Let they let us consider their stiffness K 1 and K 2. And at this point, we are applying a force equal to P. Now, we are interested in calculating that deflection at this point u. To consider the method of virtual forces, we now consider that the same system here. Let us now apply a virtual force of magnitude equal to f and therefore, this force is acting on each of the springs. And the work done by the virtual force on the displacement real displacement is u. So, that is the external virtual work.

So, external work is equal to f into u, as if the virtual force has moved by a distance of u and this work will have to be equated with the internal work. And that internal work is nothing, but internal virtual forces being worked on the real displacement of the elements. So, this force which is externally applied, it is acting on each of these springs. So, therefore, the deflection of the spring number 1 is nothing, but P by K 1 and therefore, the virtual work done by the virtual force is nothing, but A fact in the spring on the displacement P by K 1. So, therefore, the fast spring would have virtual work is equal to P by K 1 into f.

Similarly, the second spring would have displacement equal to P by K 2 and therefore, the virtual work is P by K 2 into f. So, therefore, the sum of the 2 virtual works will give us the total internal work and if we equate this 2 we get the, this real displacement of this point, of application of the load which is given by; P by K 1 plus P by K 2. So, therefore, that is the total displacement. So, these 2 principles are going to be useful, I mean when we are considering the proof of the some of the principles here.

(Refer Slide Time: 27:05)

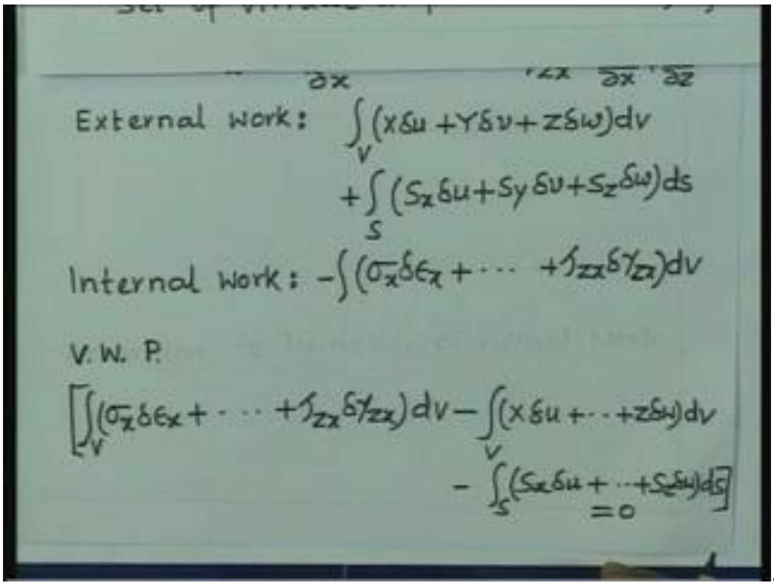


Now, according to method of, according to the minimum potential energy principle; for a system, rigid or elastic, in equilibrium, let us consider the method of virtual work, we would like to define it. For a system, rigid or elastic, in equilibrium, the sum of work done by the external and internal or reactive forces on a set of virtual displacement is 0. So, the method of virtual work is stated like this: for a system, rigid or elastic, in equilibrium, the sum of work done by the external and internal or reactive forces on a set of virtual displacement is 0.

So, let us consider, let me first write this principle; for a system, rigid or elastic, in equilibrium, the sum of work done by the external and internal or reactive forces on a set of virtual displacements is 0. So, let us consider a configuration like this. The body is in 3 dimensions and let us assumes that, the external forces are X Y Z and the boundary tractions are S y S x S y and S z. And here in the forces, gives rise to stresses which are nothing, but internal forces. So, we call them internal forces, those are reactive forces. So, that is sigma x sigma y and we will have the 6 components upto tau zx.

Let us apply a set of virtual displacements; that means, in this case the virtual displacement is given over the whole of the body. Let us say that the set of virtual displacement, let us represent them by delta u delta v delta w.

(Refer Slide Time: 31:26)

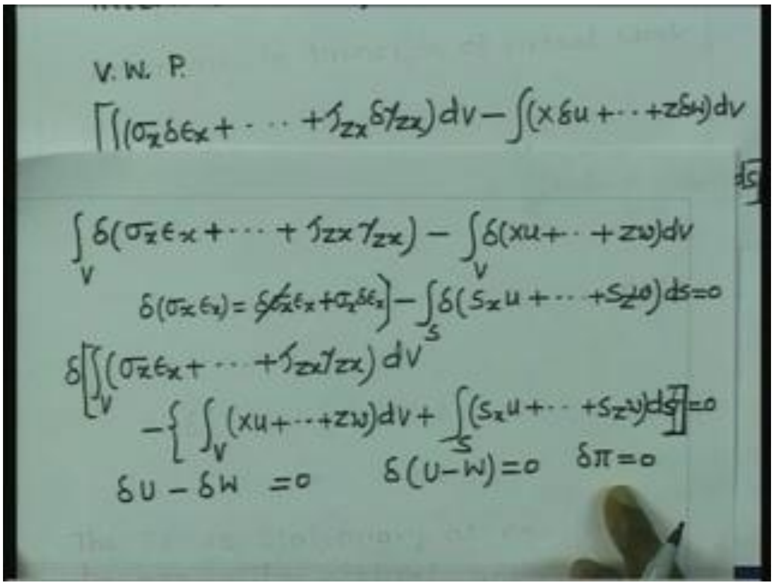


Now, these virtual displacements are going to give rise to virtual strains. We have delta epsilon x which is nothing, but delta delta u delta x. Similarly, you can write all the components up to gamma zx which is delta delta w delta x plus delta delta u delta z. So, these are the strains. Now, external work done, external work will be now given by; we have X into delta u plus Y into delta v plus Z into delta w and this is integrated over the whole volume, this is the volumetric work. And then we have the surface forces, S x also doing work 1 the displacements delta u delta v and delta w and it has got to be integrated over the whole surface.

The internal work, is equal to minus sigma x because, they are reactive forces, the work done will be negative. So, sigma x into delta epsilon x and all the 6 product terms will come. And the last term would be tau zx delta gamma zx and this is going to be integrated over the whole body. Now, according to virtual work principle, according to virtual work principle therefore, we will have the sum total of the external and internal work equal to 0.

Therefore, now we can write that v integration over v sigma x delta epsilon x tau zx delta gamma zx dV minus V X delta u Z delta w dV minus S S x delta u up to S z delta w dS and this is equal to 0. So, therefore, the sum of this expression will have, sum of this equal to 0. So, this is the final outcome of the application of the virtual work principle.

(Refer Slide Time: 35:20)



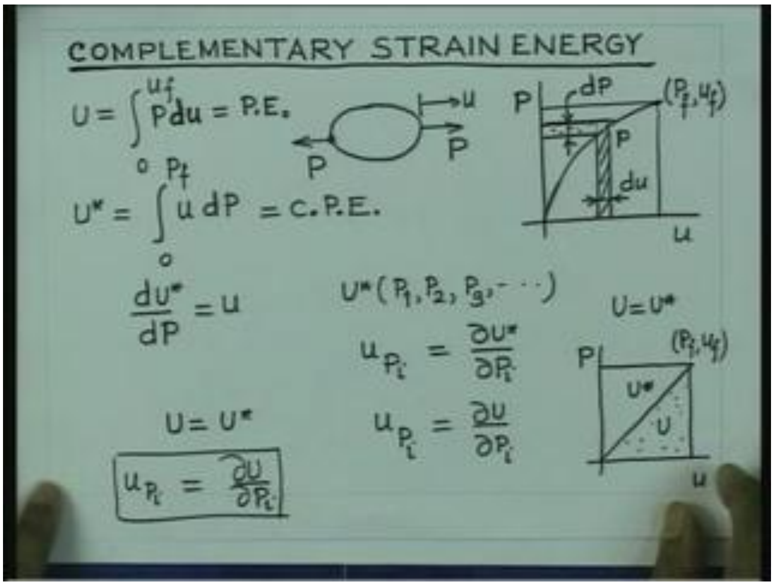
So, we have obtained this relationship and we can now rewrite the expression like this. We had integral v sigma x delta epsilon x plus up to tau zx delta gamma zx and this is integrated over the volume minus X delta u plus Y delta v plus Z delta w integrated over volume. And, then this surface forces multiplied by the corresponding displacement integrated over the surface. Now, this expression this can be written in a form integral delta sigma x epsilon x upto tau zx gamma zx minus v delta Xu zw dV minus S delta S x u S z w dS equal to 0.

So, therefore, we can take this delta out. This is because of the fact, this is a variation symbol and if, we have variation of delta x multiplied by epsilon x, this is nothing but delta sigma x epsilon x. It is like differentiation of 2 product of 2 functions. We can write first differentiation of the first in the second constant, and then we can differentiate the second keeping the first constant. So, that is the way we can do.

Now, in this case since the stresses are constant sigma x sigma y, this variation is 0 therefore this product is 0 and hence, this quantity is equal to sigma x into delta epsilon, x that is what is been made use of in writing the expression from this to this step. Finally, this delta operator can be taken outside the integral sign. And we can now write, that this is nothing, but delta integral v sigma x epsilon x delta zx gamma zx dV minus V Xu dV plus S x u plus S z w dS that is equal to 0.

Now, this expression here is nothing, but the strain energy. So, therefore, it is this is nothing, but delta of u and this is nothing, but the external work done. So, therefore, this is nothing, but delta of W and that is equal to 0. And we can write this thing as delta of U minus w equal to 0 and this is nothing, but strain energy minus the work done by external forces is nothing, but potential energy. So, therefore, delta pi equal to 0 which indicates that it is nothing, but if the body is in equilibrium, its potential energy is stationary. It is either maximum or minimum. So, in general or since the body has got to be in equilibrium, the potential energy is stationary at equilibrium position. Therefore, this follows from virtual work principle. Virtual work principle leads to the minimum potential energy principle of elasticity. So, what it finally, gives rise to that if a system, if an elastic system, subjected to set of external force within equilibrium, then the potential energy of a system is stationary.

(Refer Slide Time: 40:49)



And this principle is a is obtained from the consideration of method of virtual displacements. Now, we would like to consider some other theorems. Before, we can do that we must introduce, what is known as complementary strain energy. You consider a component which is loaded like this. Now, if we consider that, this point is having a displacement equal to U when the other point is fixed. We can plot the load versus deflection diagram.

If, we plot it for elastic body which need not be always linear, you can get a variation like this. So, in this direction we plot load, here it is displacement. Now, if we have loaded the body, up to a state let's say here, where the final load is P f and the final deflection is U f. Then, in that case while you have loaded the body gradually, we have done some work and this work done, is given by the area under the load deflection diagram and therefore, that is U.

So, U can be obtained by integrating the quantity P into du. So, therefore, if you consider the integration of these quantity, where this is du and this height is P. So, if we integrate this quantity that gives us the strain energy of deformation. And it is called potential energy of deformation. On the other hand, if we consider the other area which is bounded by the y axis and the diagram, which can be obtained by considering. We can consider integration of this area; let us say this is dP. So, if we integrate this quantity that is, 0 to P f U dP, then we get a quantity which is indicated as U star. It is called complementary potential energy.

So, this is complementary, this total area is the rectangular area and we have this potential energy plus complementary energy, is equal to the area of this rectangle. If, the system is linearly elastic, then in that case, you will have the load deflection diagram going to be like this. So, if you have loaded it up to, let's say level P f U f, then we will have this strain energy given by this area which is U and at the same time, we are going to get the complementary potential energy given by the area above, which is U star. And in this case U equal to U star.

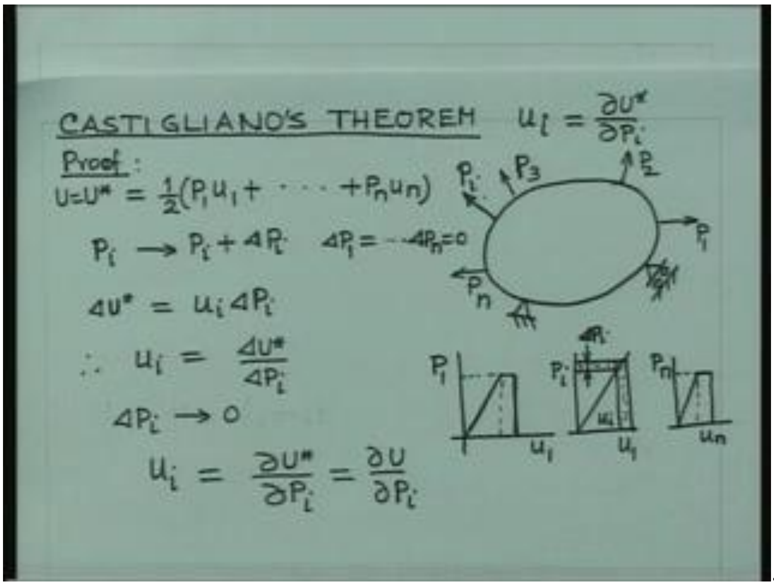
So, therefore, the complementary potential energy and the strain energy is equal in a linear system, whereas, in the case of non-linear system, the complementary energy and the potential energy of deformation are different. Now, looking into this relationship that we have written there, U star equal to integral U dP, we can now write, if we differentiate both sides what we find is that dU star dP is equal to u.

So, if you differentiate the complementary potential energy with respect to the load, then you are going to get the deflection along the line of action of the load, for deflection of the point of application of the load or deflection of the point of application of the load. So, this is nothing but Castigliano's theorem which states that; the deflection of the point of application of the load is given by the derivative of the first derivative of the complementary energy with respect to the load.

So, in case you have a system where in, there are number of loads acting. Then, in that case the complementary potential energy is going to be a function of, member of all the loads. So, therefore, here it is going to be function of all the loads applied. Then in that case, 1 can write the deflection along the line of action of a force P i given by partial derivative of the complementary energy with respect to the load P i. So, herein you have to take the derivative of complementary energy with respect to the load, for which you are interested in calculating the deflection.

So, in the case of linear elastic system, since U equal U star, we find that this u along the line of action of the load P i is going to be also given by delta U delta P i. So, the deflection along the line of action of a force is given by delta U delta P i. So, for a multi for a system with multi number of forcing acting, the deflection is given by the partial derivative. We would like to consider the proof of this theorem.

(Refer Slide Time: 47:33)



So, Castigliano's theorem u, we just make simple symbol here u i equal to delta U star delta P i. Let us consider a body, which is fixed at the 2 points like this. And it is subjected to a number of external loads P 1 P 2 P 3 and so on. Let us say there are n number of forces. Now, if we apply all these loads gradually and bring to their final level, then we would have some energy input to the system and that energy input to the system; if, the system is linear elastic then in that case, that strain energy is equal to U and that is also equal to U star and this is equal to half P 1 u 1 half P 2 u 2 and it has got to be considered up to the last 2 load and the deflection.

So, therefore, the strain energy is nothing, but half P 1 u 1 plus P 2 u 2 upto P n u n. Now, let us consider that a typical load P i is increased by delta P i. So, we have the load P i and now it is increased to P plus delta P i, keeping all other load constant. So, therefore, we do not have any increment in the other, loads they are all 0. So, we just hold the loads constant, except this P i which is increased by delta P i.

So, if we do that, what is going to be the change in the complementary energy of the system? That is going to be, complementary energy of the system is going to change because, of this change in the load and it is nothing but u i into delta P i. Let us understand this. If we consider the deflection of each of the load point like P 1 u 1, we would have got the deflection like this, after it has reached the level P 1. Similarly, if we consider the load P i, after we have reached the level P i, we would have got the deflection given by u i and this is the variation of the deflection.

Similarly, we will have P n varying with displacement like this. And now, what we are trying to do, we increase the load P i by a small amount delta P i delta P i and this is going to cause some changes in the deflection here. So, it will cause some changes in the deflection. And the other loads are going to remain constant, but they will undergo some deflection. So, therefore, there will be some change in the deflection. So, therefore, there will be some additional area under the load deflection curve. Similarly, this load would also have undergone some more deflection.

So, therefore, there is going to be some work, which is going to be represented by this area. Now, since there is no increase in the area, between the load deflection diagram and this axis. Therefore, this change is 0. So, also the change in this area between the load deflection diagram and the vertical axis is 0. There is only a change in the area here and this area is nothing, but if you consider the deflection here at this point is u i. Then, this change in the area is approximately u i into delta P i.

So, therefore, change in the complementary energy comes up because, of the load delta P i alone and therefore, this is given by this. And therefore, we have u i equal to delta U star by delta P i. So, if we consider the limit, as delta P i tends to 0. We have u i equal to delta U star by delta P i. So, therefore, the deflection along the line of action of load P i, is given by the partial derivative of the complementary energy, with respect to the load and since in the linear elastic system U star equal to u. Therefore, we can write that is also equal delta U delta P i. So, this is the proof of Castigliano’s theorem that; if you consider the energy of the system, in terms of the loads, then you take the derivative of the energy strain energy in terms of the load. That will give you the deflection along the line of action of the load.