Advanced Strength of Materials
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Lecture – 20

We have seen how the shear stresses are going to be acting at various points of cross section, under the action of the torsion. There are many applications, wherein you find that thin box sections are in place.

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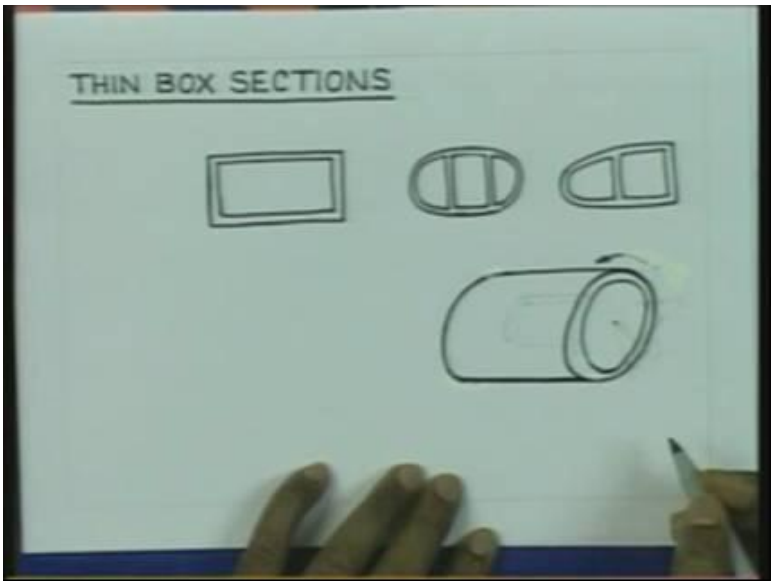

They may look like this, you can have a rectangular box sections like this with thin walls. They need not be uniform, but you could have section like this compartmented or sectioned like this compartmented. If you look into this section, it looks very similar to that of aircraft wing and this could be another section of the aircraft wing. So, these types of sections are utilized in the case of missing tools, bed and also aircraft wings, although we have, other applications too.

When, these sections are subjected to torsion. Particularly, in the case of aircraft, when the wind thrust is not uniformly distributed at the bottom of the wing. You will find that some amount of twisting comes on the section. And it is necessary to evaluate the magnitude of the stresses. Similarly, in the case of box sections, when there are twisting acting, it is possible to access the stresses, through some approximate method. So, I would like to take up those cases. On that particular type of analysis, we would like to consider.

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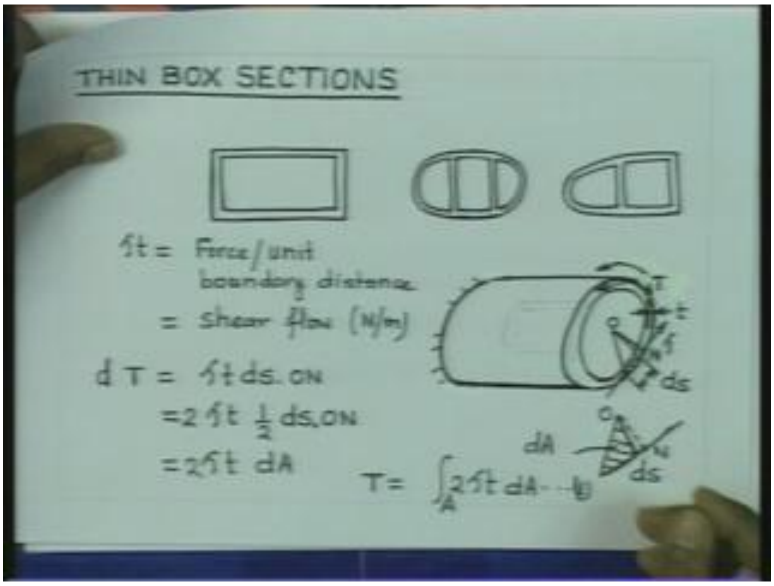

Let us look into a shaft with this sort of geometry. Wherein, the wall thickness is varying all over. Let us represent the thickness to be T and it is, of course variable. Let us also indicate that, the torque acting on the section is T. You can imagine that, this side is fixed. And we are applying a twist on the other end. Under the action of twist in the moment, from our earlier study, we can guess that the stresses are going to act, parallel to the boundary.

Since, the thickness is small; we can consider the stress to be uniform, along the thickness direction. And therefore, at any point, there will be almost a unique direction of the shear stress, which is perpendicular to the thickness. So, the stresses are going to act like this at this point. Similarly, a typical point here would have shear stresses, acting like this. Because, of the thin section, that we can assume that, the stresses are going to be uniform and it can be indicated by a single value at a point.

Let us concentrate on an element. Let us say that, we have the, we would like to concentrate on this element here, this element. This is the center. On this element, the shear stress will be acting. It is going to act, let us indicate that, this is the direction of the shear stress on this element and thickness here is T. If we consider thickness to be this length, let us say d s, d s is the length here.

So, the total force that is acting on this element is d s into T into tau. That is the force, this shear stress. If we multiply by t; that gives us force per unit boundary. So, this is nothing but, force per unit boundary distance. This force per unit boundary distance is known as shear flow. So, it is unit is Newton per meter. Shear flow, we will see more meaning, why it is called shear flow.

If I take the moment of the force, which is nothing but, t into tau into d s. Then, we have T equal to, let us say that the torque due to this length d s is d T. That should be nothing but, tau into t into d s. And the normal distance of this, let us say, so this is O, N. That is the moment and it is in the same sense as the applied torque. If we consider now, this d s, d s is this to two points and this is O, N.

You can write now, tau into t, half d s into O, N multiplied by 2. Now, these expressions indicate, half this distance multiplied by height. It is approximately area of this triangle O and this point and this, another point. So, therefore, I would like to draw this separately. So, what we have here is, this is d s, this is O, N. So, this triangle is the area here. So, you can write this thing as, d A twice tau into t into d A. So, the total torque T is equal to integration of t to t into d A, over this area.

Let us consider, this is equation number 1. This is equation number 1. Again, I would like to consider an element of the shaft, let us consider this element. Let us consider this element. That torque, that shear stress there at this point is going to be acting like this; shear stress here is going to be acting like this. Let us say that, this shear stress is equal to tau 1 here and the shear stress here is tau 2. And this thickness here is equal to t 1; thickness here is equal to t 2.

Since, the shear stresses are acting on this face. On the perpendicular face here, we must have shear stresses acting like this, which is again, tau 1. And shear stresses, that is going to act here on the space, it is going to be complemented to this stress and this is tau 2. Simultaneously, we will have the shear stress acting here in this direction. It is tau 1 and the shear stress at this point is going to be tau 2. So, on this space, you have intensity of stress acting in this direction is tau 1 and intensity of the stress acting in this direction is tau 2. So, this element; let us consider that; it is of length equal to L.

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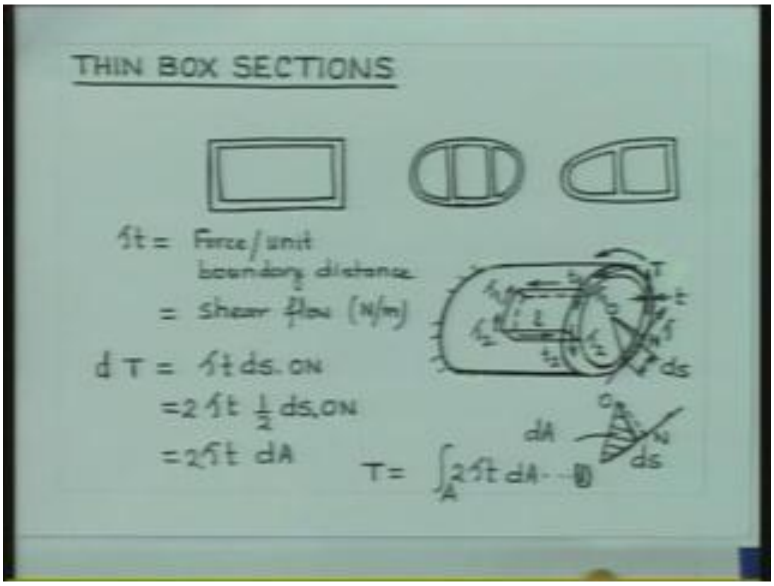

So, it will be oriented in the axial direction. And let us now try to see, the forces, net force on this element in the axial direction. So, if you would like to consider that axial direction is equal to z. This is our z direction. Then, some of the forces on this element; just take it out. And think of it, that forces are acting on this boundary, which are of intensity tau 1 there, tau 2 there.

And on this space, we have uniform stress of magnitude tau 2. On this space, it is uniform stress of magnitude tau 1 and here again, we have tau 1 here, tau 2 there. Now, if I try to consider the forces in the z direction is equal to 0. Then, forces in the axial direction is due to only this stress and that stress. This force here l into t 2 multiplied by tau 2 is acting in the positive z direction. And the force, on the other face, which is acting in the negative direction, it is l, again t 1 into tau 1.

So, these two must sum up to 0. And therefore, what it means is that, tau 1, t 1 into tau 1 is equal to t 2 into tau 2, this is a constant. So, this is nothing but, a constant. We have taken two arbitrary points with thickness t 1 and t 2. So, it is true for any thickness. That shear stress at the location multiplied by the thickness is a constant. Therefore, you can write tau into thickness into tau is shear flow, shear flow is constant.

In fact, this shear flow analogy has been taken from fluid mechanics. If you just consider a pipe, whose height, cross section is a rectangle of this height and length equal to unity. So, therefore, if you think of a portion like this. So, it is a circular sort of pipe and the cross section, everywhere is rectangular. If you think of it is, think of that fluid is flowing in the circumferential direction.

The fluid is incompressible then every section would have the same quantity of the fluid. And therefore, it is from that analogy, the shear flow is borrowed. That the, it is achieves that some quantities of fluid is flowing and that fluid quantity is remaining constant. And therefore, the shear flow is constant. That it is borrowed from fluid mechanics.

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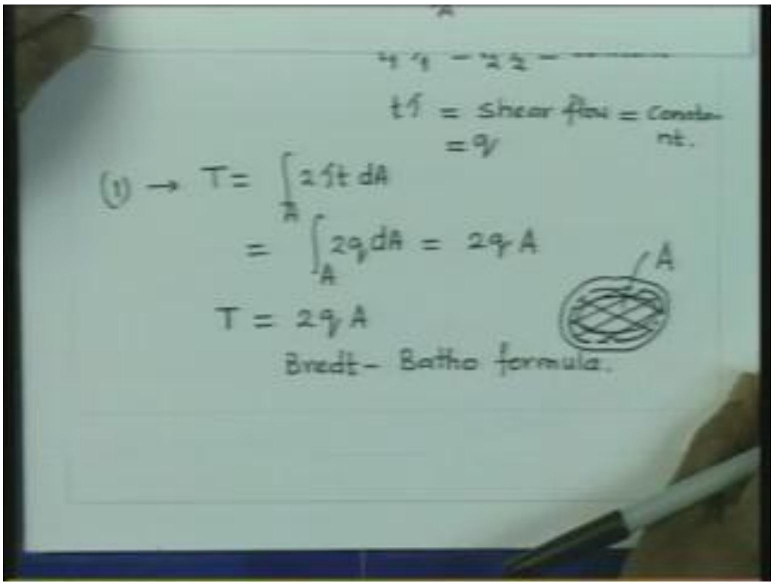

Getting back to relation 1; we have T is equal to 2 times t into tau into t into d A. And this is over the whole cross section. If we represent the shear flow to be equal to q. Let us say, shear flow is q. Then, what we have here, so this is from 1, we have A 2 q, d A and q is constant. So, therefore, this is q and integration of d A is nothing but, A. So, torque capacity is related to shear flow multiplied by this area.

What is this area? This area is nothing but, the area enclosed by the center line of the, approximately, you can take it to be the center line of the cross section. Center line of the section is nothing but, A. If we have section like this, it is usually area bounded by the center line. So, therefore, this is the area A, this is known as Bredt-Batho formula. So, we have been able to get the relationship between the torque and the shear flow q.

In many applications, it is necessary to calculate the angle of twist per unit length. Let us see, how we can calculate angle of twist per unit length. The strain energy in an element d U is nothing but, half shear stress multiplied by the shear strength, multiplied by the volume of the element. Here, you will find the shear stress is constant at a particular location on the circumference of the section. And therefore, we can now make use of the Bredt- Batho formula.

We can write the shear stress and again, we can also write strain in terms of the torque, using the formula Bredt- Batho formula, that we have derived. So, therefore, we can now write, that this is 2 by 2 into T into A into T by 2 into t into A into G, where G is the modulus of rigidity into d V. We can write this thing as let us say t into d s and in the length direction is this 1.

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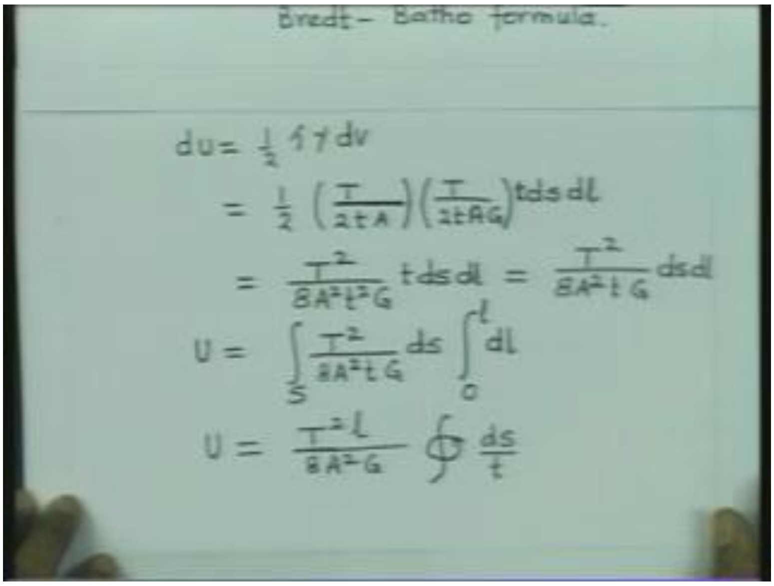

This gives us T square, 8 A square, t square, G, t, d s, d l which is nothing but, T square by 8 A square into t into G, d s, into d l. The total energy is therefore, obtained by integration of T square 8, A square, t, G, d s, over the boundary. And this l is, 0 to l into integration of this. And since we have l can be integrated easily. So, therefore, we have now, T square, t is constant and 8, A square, G and this is integration of the quantity d s by t. So, u is this 1.

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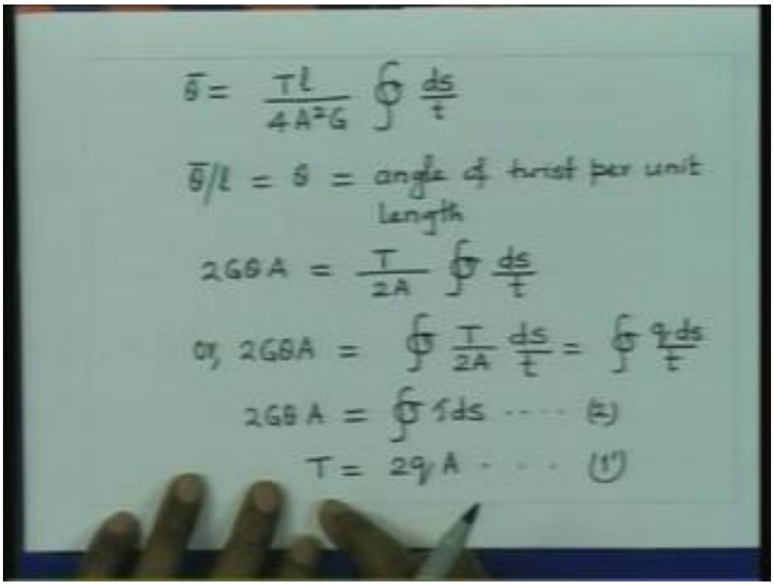

If you consider that the angle of twist of the whole bar is theta bar. So, theta bar, total angle of twist. That we can obtain from, application of Castigliano Theorem, which is nothing but, derivative of u, with respect to the applied stress, which is torque and in this case, it is d U, d T. So, you get now, theta bar is equal to T l by 4, A square, G integration of this quantity, d s by t. We like to write this thing, theta by l as equal to theta, it is equal to angle of twist per unit length.

So, we now can write twice G theta into A is equal to T by 2 A, T by 2 A integration of this quantity, over the closed control. We can write this again, twice G theta into A, integration of T by 2 A, d s by t. You have got already the relationship between the shear flow and the torque T was given by 2 times A into q, where, q is the shear flow. So, you can replace this thing by shear flow q and we get now, this is q, d s by t and since, q by t is tau.

So, we have now, tau d s, twice G theta into A is equal to the integral of tau, over the whole of the boundary. So, this relationship is to find out the angle of twist per unit length. So, we have first relationships, we have already indicated Bredt-Batho formula as the relationship, which is 1. So, we indicated that thing as 1. So, therefore, this will indicate as 2.

We can now write the two relationships, T is equal to twice q into A. That is, let us say again, 1 dash. So, this 1 dash and 2 dash are the relationship, which are sufficient to help us to solve for the shear stress at any point, and also the angle of twist, per unit length for a given torque.

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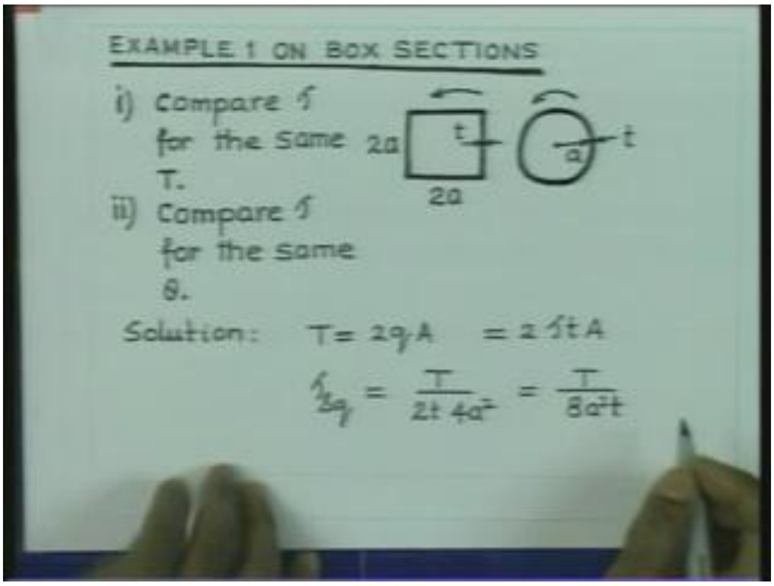

Let us solve, some simple problems, applying this derivation. Let us consider, one example of simple square box sections, where the sides are 2 a, thickness is t. And we have round section, here the radius is a, thickness is t. What you are required to is that; compare the shear stress, maximum shear stress for the same T. In the second case, you are asked to compare the shear stress for the same angle of theta.

Let us now, work out the solution. We invoke the Bredt-Batho formula T is equal to twice q into A and this is nothing but, twice tau into t into A. In the case of thin sections, you can take the cross section here in this case, it is going to be 4 A square and the stress tau will indicate by tau square. So, t s q, t s q is nothing but, T by 2 into t and area is 4 a square. So, it is going to be T by 8 a square into t.

If you consider now circular cross section, then again, you will have T is equal to thickness, A is equal to pi a square. So, t for circle, T by 2 t, it is pi a square, T by 2 pi a square t. So, if we now take the ratio, tau square, tau for the circle, T by 8 a square t, 2 pi a square t by T and this is pi by 4. What you find here is that, the shear stress, in the case of the circle is going to be same, everywhere.

And for the square, it is going to be higher at the end of the, at the middle of the sides and this is the stress, which is going to be higher. In this case, you are going to get the stress in the circle to be higher than the stress in the square. So, the stress is going to be higher in the case of the circle.

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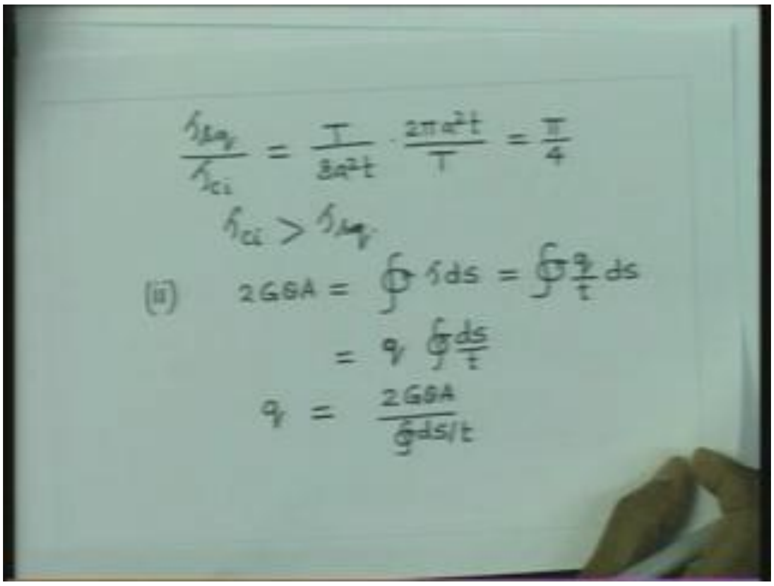

So, t c i is greater than t square. The reason is that, the length of the boundary, here in this case, it is 8 a. In the case of the circle, it is 8 a, whereas, in the case of the circle, it is twice pi a, which is less. So, therefore, the stresses are going to be more in the case of circle. Let us obtain the solution for the second part. So, this is our solution for the first part. We will now look into the solution for the second part for that, we make use of the equation number 2, which is 2 times G theta A is equal to tau d s.

If we consider now, the thickness is uniform here, we can write this thing as, q by t, d s and this q is constant. So, therefore, it is d s by t and finally, we have q is equal to 2 G theta A by d s by t. You can now substitute the value, A is given by 4 a square and in the case of circle, it is pi a square. And this circumference is 8 a, in the case of square and it is 2 pi a, in the case of the circle. So, therefore, we can now get the shear flow for the square is equal to 2 G theta, 4 a square and this is nothing but, 8 a. So, 2 times 4 a by t, so we will have G theta, a t.

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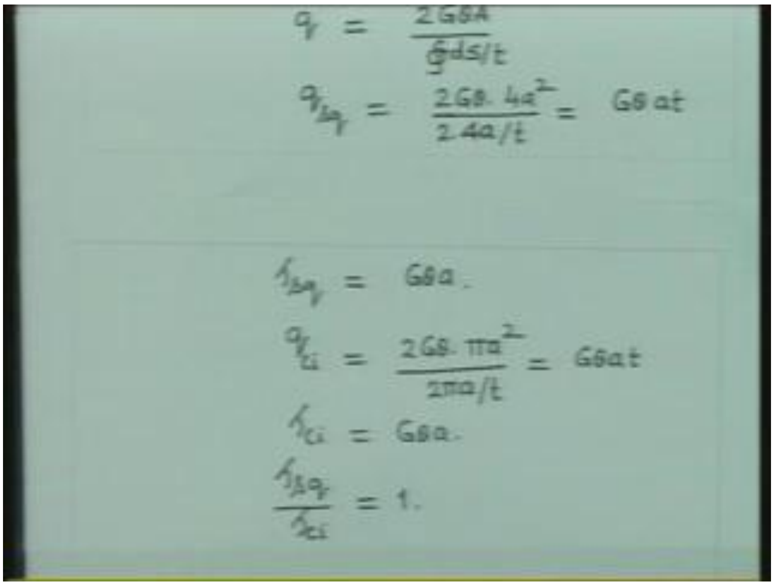

And hence, shear stress in the case of square is equal to G theta into a. Now, for the circle, q circle is 2 G theta, pi a square and we have 2 pi a by t. And once, you simplify, it is G theta a into t and hence, we can write tau circle is equal to again G theta into a. Hence, the ratio tau square by tau circle is equal to unity. So, for the same angle of twist, you find that, the stresses in the two sections are going to be the same. Whereas, in the case of same torque, we have seen that, the stress in the circle is going to be higher.

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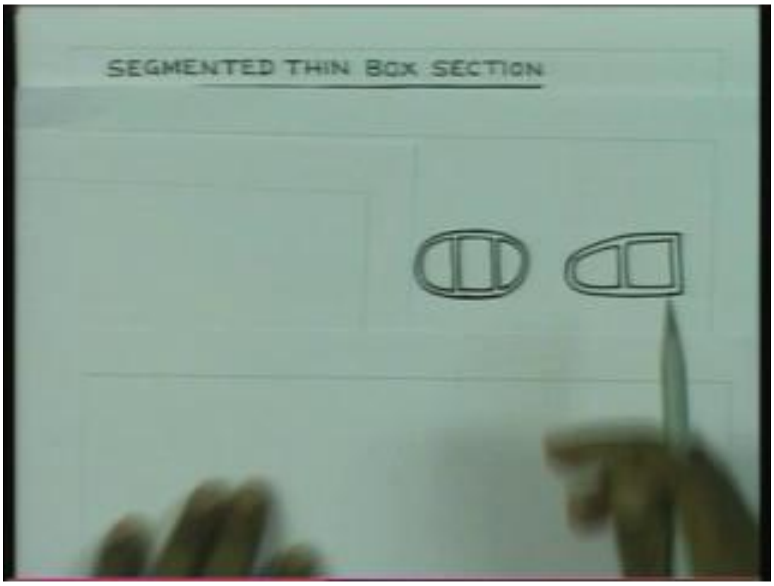

So, we have done the analysis for sections, wherein you had only 1 box. You could have situation, where number of segments are there in the box section. How to analyze these problems? So, let us look into analysis of such sections.

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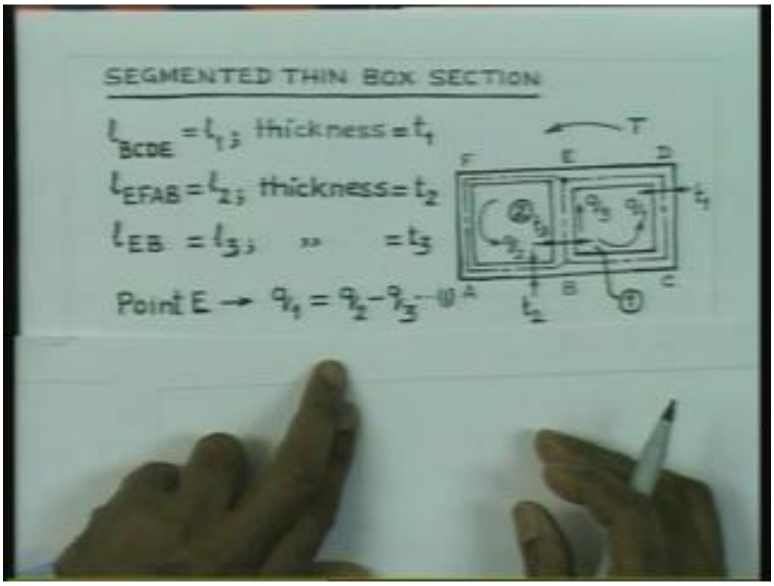

Typically, what we are trying to concentrate upon in sections like this, where we have, one box consisting of these four walls, another box consisting of these four walls. The total torque, that is acting on the section is T. Let us say that, this part or box would indicate by box 1. And this box, which is consisting of the portion, let us say E F, A B E is 2 and here, it is B C D E B is box 1.

Let us assume that the flow, which is in this circuit is q 1 and the flow in this circuit is q 2. Thereby, we will have a net flow in this arm; B E is equal to q 3. For simplicity, let us try to also take this symbols, that length in this particular case B C D E has got some thickness, which is equal to t 1. So, everywhere this thickness is equal to t 1 and length let us say that, this whole part is equal to l 1 and it is a constant thickness t 1.

Similarly, E F A B is of length l 2, and then this arm E B is of length equal to L 3 and the thickness is equal to, let us say, that is t 3 and this thickness is equal t 2. Concentrate on the point E, we would now get, from the concentration up flow; this is the junction, wherein, you think of it, that three branches of the pipe are connected. Flow here coming in is q 1, flow here coming up is q 3 and flow wing out in this direction is q 2.

So, we have flows, q 1 and q 3, coming towards this point. So, q 1 plus q 3 is the incoming flow and outgoing flow is q 2. So, from the flow analogy, we can get from the point number 1. That q 1 is equal to or q 1 plus q 3 is equal to q 2 or q 1 equal to q 2 minus q 3. That is equation number 1. We can get this equation, also from the fact of from the consideration of equilibrium at the point E. This is something I would like you to note that this flow equation is equivalent to the equilibrium equation of an element around the point E.

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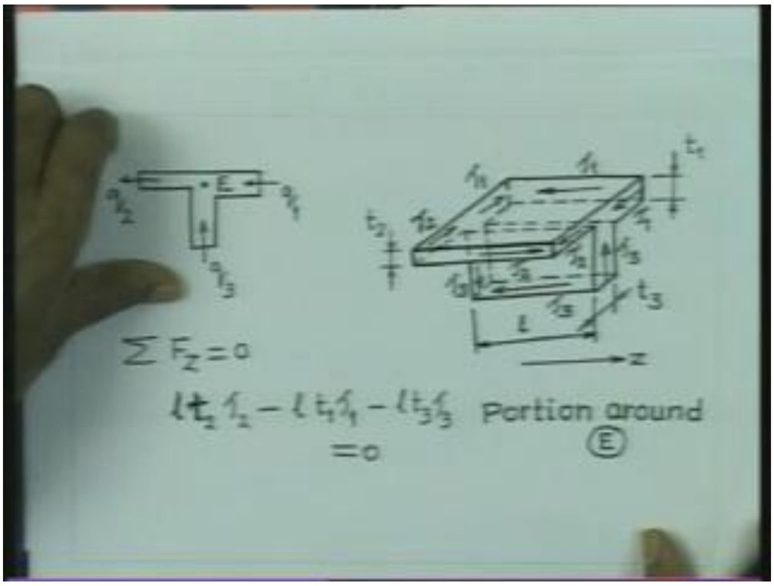

Let us look into, therefore, the point, this is the point that we are talking about. Now, our beam is on the box section shaft is going to be in the length direction. In this direction, I am just trying to take a portion of the section like this. And let us assume, that the flow, since flow is in this direction, it is of magnitude q 1, here it is q 3, here q 2. When, I take this section, the thickness here is t 1 and the shear flow is here in this direction.

So, therefore, we will have a shear stress here. Let us indicate that shear stress is equal to tau 1. Similarly, here the thickness is equal to t 2 and the flow is q 2. And hence, we will have a shear stress here. Let us say, this is of magnitude equal to tau 2. This thickness is equal to t 3 and the flow is q 3. And therefore, flow direction is this; we would have shear stress acting like this tau 3.

If this is the shear stress acting here, then on this face on this vertical face, we are going to get stress acting like that and this is also of magnitude equal to tau 1. Here, the shear stress acting towards this point is q 2. Then, on this face, which is vertical, we are also going to get stress equal to tau 2. All over the space, we have uniform stress of magnitude equal to tau 2.

By similar consideration, complementary stresses are going to be around this point, directed like this. This is of magnitude; tau 2 complementary shear stress here is of magnitude equal to tau 1. Since, this vertical stress is subjected to shear stress, tau 3. This face, bottom most face is also going to be subjected to shear stress, tau 3 equilibrium constant. And this vertical face on the left is also going to be subjected to shear stress of magnitude tau 3.

So, the stresses acting on all the faces are now shown here in this diagram. So, this is just around the point E. If you consider the x, the z axis to be oriented along the length of the shaft or the shaft with box section. Then, the sum of the forces in the z direction, must adapt to 0. So, if we now write that sum of the forces in the direction z is equal to 0, which are the stresses, which are going to contribute to the force in the horizontal direction or z direction. Let us say, that this length of the segment is equal to l.

So, the forces are going to come from this bottom face, this face at the back and this is the third face, which is at the front. Let us now write this area is l into t 2. So, l into t 2 is the area and the intensity of stress is tau 2. So, therefore, the force is l into t 2 into tau 2, and then the other force is nothing but l into t 1 into tau 1. So, it is acting in the negative z direction, t 1 into tau 1. And from the bottom face, which is nothing but, l into t 3 into tau 3, it is also acting in the negative direction. So, this is equal to 0.

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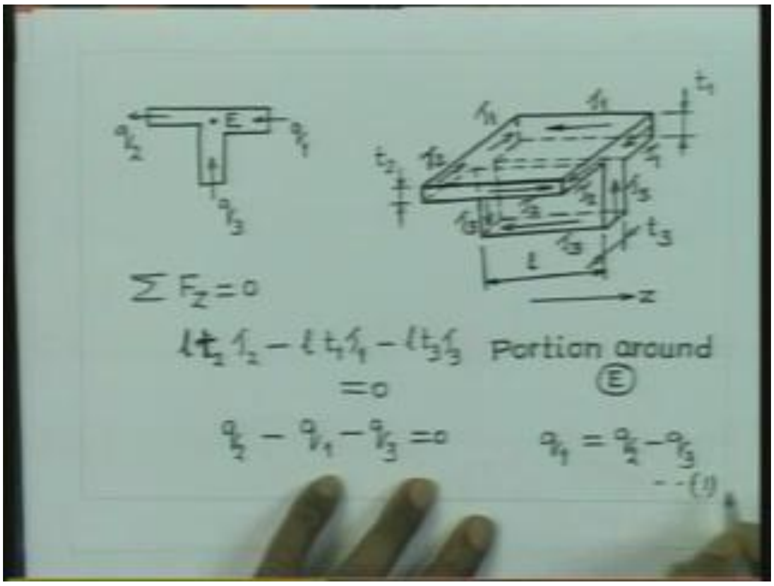

And if you now write that t 2 into tau 2 is q 2 minus this is t 1 into tau 1 is q 1, t 3 into tau 3 is nothing but q 3. And that is equal to 0, which is same thing as q 1 is equal to q 2 minus q 3. From the flow consideration, we derive the relationship also, which is here. From the flow concentration, we derive that q 1 equal to q 2 minus q 3. Here, the flow incoming is q 1 plus q 3 and outgoing is q 2 and from that, we found that q 3 or q 1 is nothing but, q 2 minus q 3. And now from the equilibrium, we find that q 1 is q 2 minus q 3. So, these two conditions are having some similarity, this is nothing but, one can interpret this thing as an equilibrium equation and one can also consider this to be a flow equation. So, we have now one relationship involving the shear flows.

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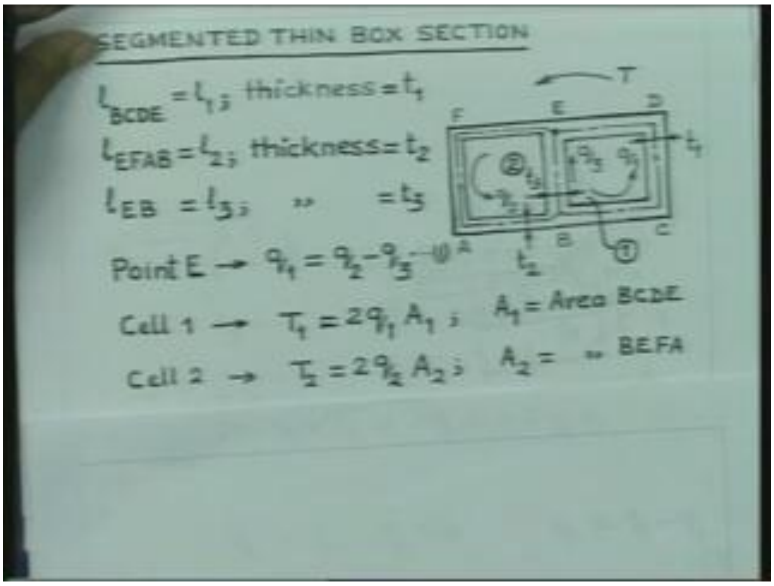

Now, in a problem of this type; that you have to handle, you will be needed to find out, the shear flows q 1, q 2, q 3 and also the angle of twist per unit length. There are four unknowns and we have to obtain all of them. So, far, we have been able to get only one relationship. Let us now consider, this cell number or let us say, first you consider this cell 1. For this cell 1, we can consider the Bredt -Batho formula.

Wherein, we can write that the torque taken by this flow q 1 is nothing but, twice q 1 into the area enclosed by the cell is a 1. So, we can write for the cell 1. Let us say that the torque T 1 taken up is nothing but, 2 times q 1 into A 1, where A 1 is the area of B C, D E B. Similarly, since we have the flow in the second cell is q 2. Again, applying the Bredt-Batho formula, let us say that the torque taken is T 2.

We can write now, that the torque is nothing but, 2 times q 2 into A 2, where A 2 is the area of again this cell 2. Since, the total torque is going to be T, we can adapt this two torques and we will get relationships involving the shear flows with the external torque T. So, if we add up the two area, two relationship, we have total torque T is equal to T 1 plus T 2 and this is equal to 2 q 1, A 1 plus two q 2, A 2.

So, we have now relationship involving flows one equation is here and the second equation is here. Again, we would like to get back to the consideration of the two cells, separately; the whole section is going to rotate at a single unit. If we consider that the angle of twist per unit length is theta. We can write that 2 G theta into A is equal to integral of the quantity tau into d s.

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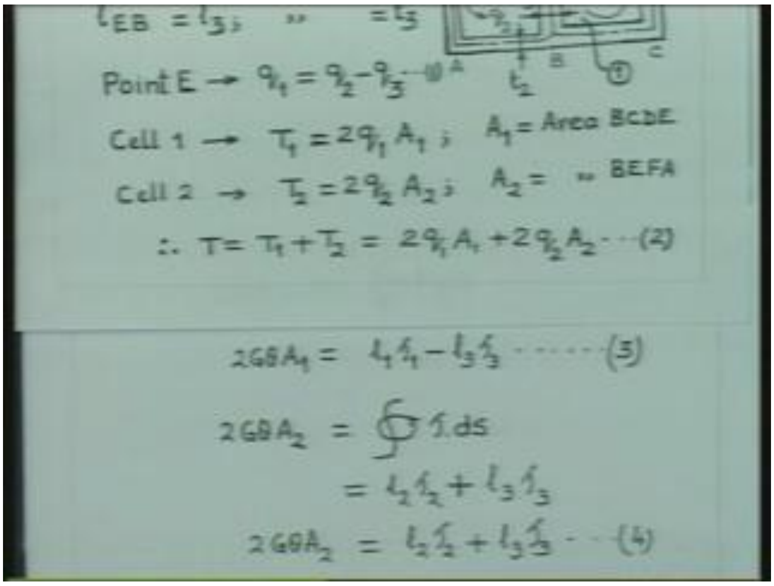

If I consider now this cell number 1. We can write now 2 G theta into area of this cell A 1. That must be equal to integral of this quantity tau into d s. So, we have to take the route B to C, C to D, D to E and again E to B A 1, of course, which is the area enclosed by the contour B C D, E B. Now, let us look into, what this integral is finally, going to lead to, we have the thickness all over the length is constant and the flow is q 1.

So, therefore, this shear stress is going to remain constant from B to C, then C to D, then D to E. So, you can now write, this tau is constant and therefore, this integration is nothing but, integration from B to E, which will give us l 1 into tau 1, because l 1 is the length B C D E. Now, we have to go to complete the contour E to B. And when, you are going in this direction, then our flow direction or shear stress direction is different.

And therefore, this portion, since thickness is t 3, shear stress is, let us say tau 3, which will remain constant. And therefore, this integration of this quantity from E to B would mean equal to l 3, multiplied by tau 3. So, twice G theta into A 1 is related to l 1, tau 1 minus l 3, tau 3. This is relationship number 3. We would like to continue the consideration and now, we will extend it to cell number 2.

If we now consider the cell number 2, starting from E to F, then A to B, then B to E. Again, we can write for this cell twice G theta into A 2 is equal to integration of this quantity, tau into d s. For this closed contour again, you have the thickness constant from E to F, F to A, A to B and the flow is q 2. So, we can indicate the shear stress to remain constant. And therefore, this will become l 2 into tau 2, l 2 into tau 2.

And from B to E, we are moving in the same direction as the shear stress and flow direction. And therefore, it will be now l 3 into tau 3. So, relationships, that we have got out of this is 2 G theta, A 2 is equal to l 2, tau 2 plus l 3, tau 3. This is number 4. So, to sum up finally, from this derivation, what we have got are the four relationships. We have this condition 1.

Then, torque related to the flows and the angle of twist related to the shear stresses here. And therefore, you have the unknowns like, three shear stresses and three shear flows and the angle of twist and you have four equations, you can solve for them. So, this is how, we can address the problem of segmented box sections. It is solving problems of this type is a routine application of this relationship, four relationships that we have derived here.

We would like to illustrate this by considering one example in the next lecture. One point, I must tell you, that this analysis is approximate, it is not to be taken as exact. The very fact is that, the stresses at the corner, we know that, the stresses at the corner is going to be 0. But, in this analysis it is assumed to be non zero. For the more, there is complexity of stress distribution around the point E, but we have approximated.

So, therefore, to that extent, the analysis is approximate. But, it gives quickly some idea about the stresses, coming under the action of the torque of the section and the angle of twist. So, what we have done in the last 2 lectures, we have taken the case of thin sections. And also these sections can be consisting of one single cell or it can have multiple cells.

In the case of single cell, the solution can be obtained by just using the Bredt-Batho formula and the relationship, which relate the angle of twist per unit length to the integration of the product of length d s along the boundary. In the case of segmented sections, we are going to get an equation from the equilibrium at the junction. And you are also going to get one relationship relating the torque to shear flows. And then you are going to get 2 or as many relationships as the number of sectional boxes. The relationship will involve angle of twist and the shear stresses.