Advanced Strength of Materials
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Lecture – 2

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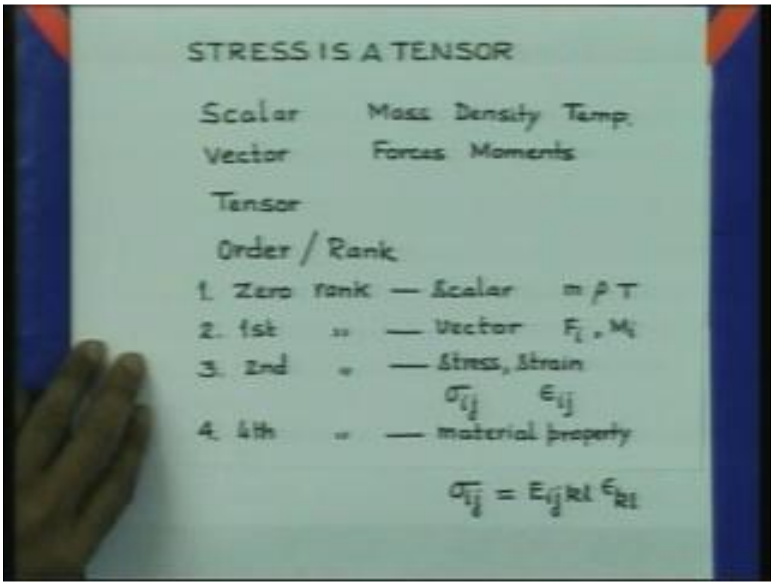

We have heard now that stress is a tensor I would like to now tell you what is a tensor? You have come across quantities like scalar you remember the examples are mass, density, temperature, they have simply magnitude. We have also come across vector examples are: forces, moments, in this case you have magnitude and also a direction. So, they have 1 more attribute in fact all quantities that engineer and physicist come across they are nothing but, tensor scalar, vector, they are also tensor.

So, tensor is a more general quantity. Example of tensor can be of different order, let us look into how we define the order or the rank of a tensor. In fact, this order or rank is defined by the number of attributes. We have 0 rank tensor and they are nothing but, scalar and examples I have already given you mass, density, temperature, and we can represent them by symbols like: mass let's say m, density by rho and temperature by T.

Then, we have first rank tensor they are nothing but, vector you can represent them by symbols like: Fi, Mi Fi in the case of 3 dimensions I can have 3 components: 1 in the x then in the y and z direction therefore, there are 3 component. So, they have magnitude plus the directions and there is 1 index to indicate the first rank tensor.

Similarly, we have second rank tensor examples are: stress, strain. In the case of stress, we have seen that, we have magnitude of stress it has a direction of action and at the same time it has also a plane of association. So therefore, there is 1 more attribute and we generally represent the stress by symbols sigma ij, so there are 2 attribute therefore, there are 2 indices.

Similarly, in the case of strain we represent by symbol epsilon and there are also 2 indices use to indicate the rank of the tensor. In fact, in phase analysis and strength of materials we will have 1 more tensor which is fourth rank tensor and this is material property tensor. In fact, the stresses and strains are related and they are related by a relationship of the type, which you can write sigma ij is equal to some property tensors eps Eij kl epsilon kl. So, there are 4 indices and it is a fourth rank tensor. So, you should be now on be careful in trying to differentiate between stress and a vector. Now, we would like to configure how we can quantify the deformation at a point or define the strains at a point.

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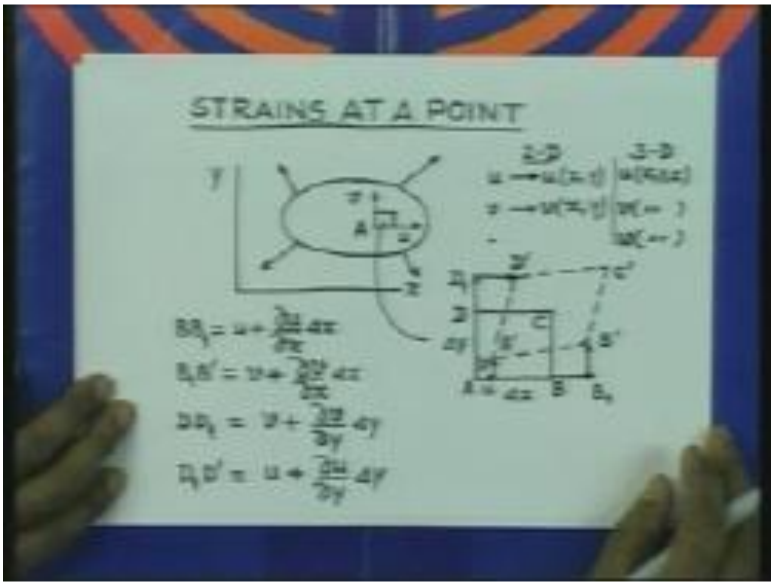

In order to define the strain at a point, I would like to go in for considering a 2 dimensional case for convenience, but it can easily extended to 3 dimensions. So let us, consider a thin disk subjected to forces which are like this and the body is in equilibrium and you can just deforming. Under the action of these forces each of the point in the body is going to have some moment in the x direction.

Let us represent by this moment by symbol u and it is also going to have some displacement under the action of the loading in the y direction which is equal to v. This value of the displacement is going to vary from point to point and therefore, this u displacement is certainly a function of position of the point. So, u x y function of x y so also v displacement is a function of the position xy.

And in case of 3 dimension so this is the picture in 2 dimension, if we consider 3 dimension then you are going to have the displacements like: u v and 1 displacement in the z direction which is generally indicated by w. So therefore, in the case of 3 dimensions you are going to have displacements u it is a function of x, y, z so also v which is going to be function of x, y, z. And then will have displacement w which is also function of x, y, z.

Now, will try to see that the summation at a point let us say point A or rather we would like to calculate the strain so the quantify the strain at the point a after the body has undergone a deformation. In order to do that, you consider a very small element like this and if we draw to a larger scale, let us draw to a larger scale this particular element so this is the element a let us say this is A B and then C and D.

Now, after deformation this particular element is going to change position and it’s going to have a shape let us, say that displaced position of this element will indicate by this. So, this is the deformed position of the element therefore, A has shifted to A dash, B to B dash and this is shifted to C dash and this is shifted to D dash. Let us, consider that the distance between AB is delta x which is small distance between AD is delta y.

Now, point A has shifted by a distance u in the x direction and it has shifted by some distance g in the y direction therefore, this is nothing but u and this distance is nothing but v. So, these are the displacement of the point A since it is a continuous body you find that this function u and v they are continuous functions there is no jump in the function as we moved from point over these area of the body.

Now, if I want to calculate the displacement at the point B I can calculate it from the displacement of the point A, from the consideration that they 2 are at a distance delta x which is very small. And you can consider expanding this u in scalar standards, and then you can write that the displacement of the point B in the x direction. Let us, say that is the displacement of the point in the x direction B to B1.

So this B to B1 you can write this thing as BB1 is equal to u of the point A plus rate of change of u with respect to x multiplied by delta x so that, is the displacement BB1. Similarly, if we calculate this distance B1 B dash it can be obtained from the displacement v of the point A. So this would be nothing but v plus rate of change of v with respect to x multiplied by delta x so that is again.

Similarly, if you consider the point D it has shifted to D1 in the y direction and then it has shifted to D dash in the x direction. So therefore, by similar configuration we can write that DD1 is nothing but v delta v delta y del y and this distance D1 D dash is equal to it is again obtainable from u delta u delta y del y, therefore these are the displacements.

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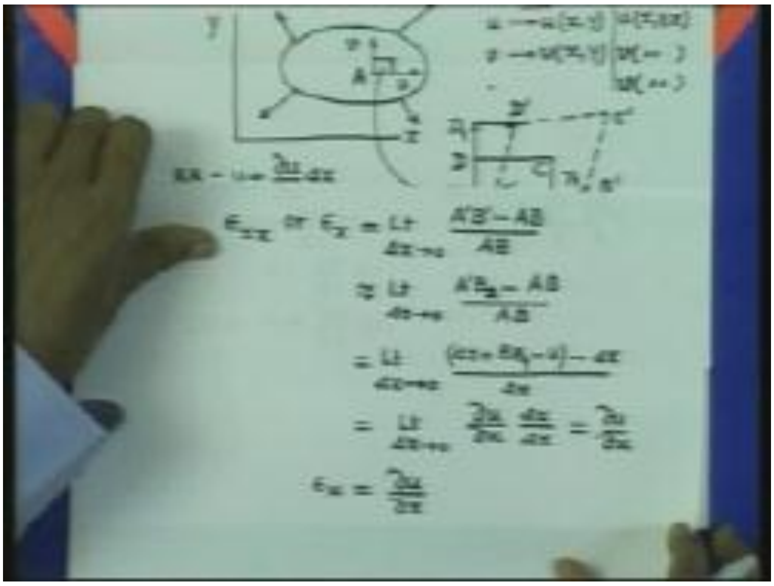

Now, we would like to calculate the strains at the point A and the strains are defined like this strains at the point A in the x direction, which is represented by epsilon xx or in sort epsilon ax which is nothing but rate of change of deformation of ab in the x direction. So therefore, it is given by and also that rate of change of deformation has delta x tends to 0 that gives us the strain attribute.

So therefore, it is defined by delta x tending to 0 it is the deformed length is A dash B dash original length is AB divided by the original length. Therefore, this is that is the rate of change of length and as delta x tends to 0 that gives us the strain at A in the x direction. In fact, this A dash B dash since this angle they are least angle is let us, say this angle is gamma 1 is very small we can approximate this distance A dash B dash by this distance A dash B.

So therefore, we can write that this limit is given by it is A dash B 2 minus ab by AB. Now, we can write this thing very easily it was given by limit delta x tending to 0 and this A dash B 2 is nothing but delta x plus BB1. So, delta x plus BB1 minus the distance u minus u that is what is this minus AB which is nothing but delta x and this is delta x.

Now, BB 1 is nothing but, u plus delta u delta x del x therefore, will cancel u and then will be also haven calculation of this delta x. So this finally, will give us limit delta x tending to 0 delta u delta x delta x by delta x and therefore, this is delta u delta x. So, the strain in the x direction epsilon x is nothing but partial derivative of u with respect to x that is the strain in the x direction.

Now, let us calculate the strain in the y direction so we will again confine our self to the deformation here and the strain in the y direction is defined as the deformation of AB in the y direction as this length delta y tend to 0. That is the strain in the y direction at point A.

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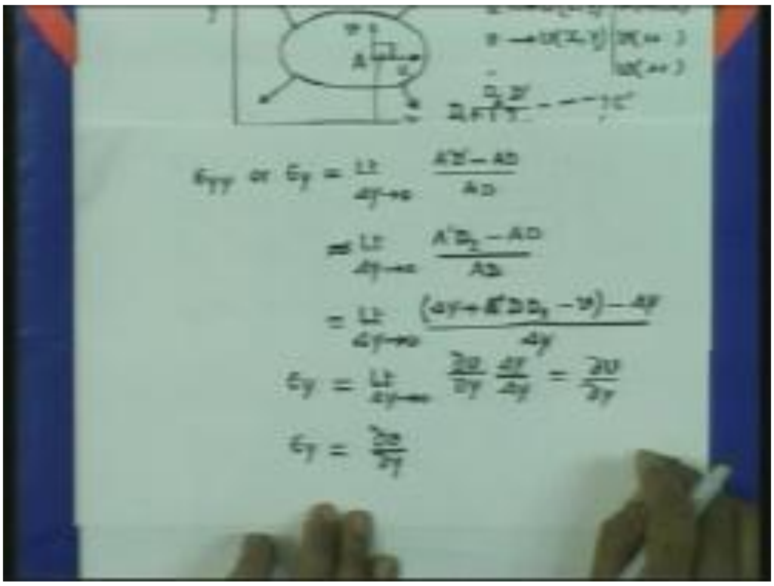

And it is given by epsilon yy or in sort epsilon y and that is equal to limit delta y tending to 0 this is A dash D dash minus AD divided by AD. Now, you can write this thing again as we can approximate A dash B dash again by consider some like this. We can draw this parallel to y axis and if we consider point to the D2 since these angle. Let us, say this gamma 2 is very very small we can now approximate A dash B dash by A dash D2.

So, it is A dash D2 minus AD by AD and since DD1 is known and this displacement v is known DD1 is nothing but v plus delta y delta y delta y. Then, you can write this thing as suppose this is I think you should write this thing approximate not in fact equal to and therefore, which is now substitute the value we get limit delta y tending to 0 and this is delta y plus A dash del this will be DD1 minus v minus the original length is delta y divided by delta y. And this DD1 is equal to v plus delta v delta y del y.

So, this v and this v will cancel this delta y this delta y will cancel therefore, DD1 only this part will be remaining. And on simplification what we get is that, del epsilon y is equal to limit delta y tending to 0 it is delta v delta y del y by del y and therefore, this is equal to delta v delta y. So, the strain in the y direction is nothing but partial derivative of v in the y dash of slope of the v surface in the y direction.

Similarly, your strain in the x direction which is nothing but slope of the u surface in the x direction. Now, you would like to consider the shear strain shear strain is defined as the change in the angle between 2 orthogonal directions under the action of external loading. Here in, we had AD and A AB and AD 2 orthogonal directions and upper deformation they have become A dash B dash and A dash D dash. And therefore, had been a change in the angle with original 90 degree by the amount gamma 1 and gamma 2.

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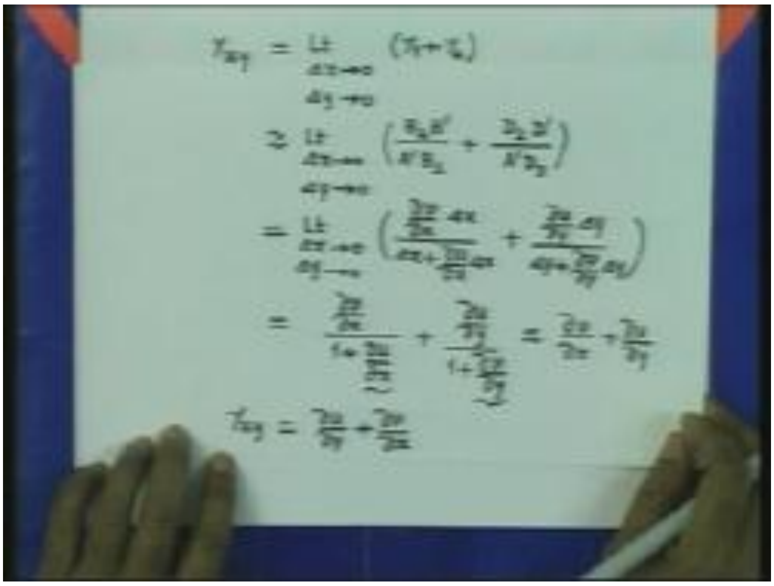

And the shear strain in 2 dimension gamma xy is defined by limit delta x tending to 0 so as this length of the element is shrunk to 0 the limit gamma 1 plus gamma 2 that gives us the value of the shear strain. Now, we can calculate the shear strain as follows you can now write that this is approximately equal to limit delta x tending to 0 delta y tending to 0. That this angle I can approximate it by, that this is B2 B dash divided by this term A2 B2.

So this is nothing but B2 B dash divided by A dash B2 that is gamma 1 plus we have here this gamma 2 is nothing but, D2 D dash divided by the length A dash D2. So, this is what is the shear strain? Now, if we see that B2 B dash this whole distance this whole distance B2 B dash is nothing but, it is going to be v plus it is nothing but we have this distance is given by B1 B dash minus B1 B2 which is nothing but, v therefore this distance is nothing but delta v delta x del x.

So therefore, this distance is nothing but delta v delta x del x, therefore this is this distance. And similarly, here this distance we can write this distance which is nothing but this total distance which is D1 D dash minus this distance u therefore, it is delta u delta y del y. So therefore, this is this distance is nothing but, delta u delta y del u.

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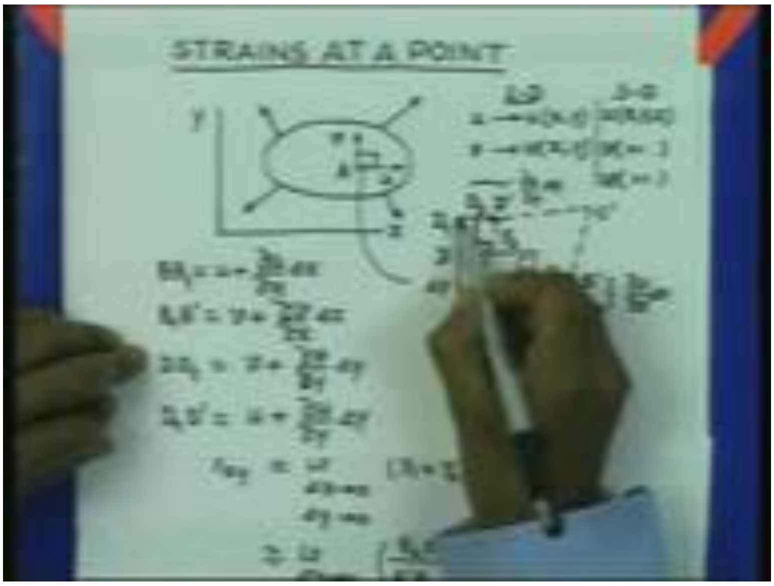

So, if you consider that this distance is nothing but this total distance minus this 1 and which is therefore, is given by delta u delta y del y. So therefore, if we substitute all this what we find now what we get now, is this that limit delta x tending to 0 delta y tending to 0 this is delta v delta x del x divided by this distance A dash B2 is nothing but delta x plus delta u delta x del x plus this D2 D dash is nothing but delta u delta y del y divided by delta y plus delta v delta y plus del y.

So, this distance you can also consider to be this distance as original length plus rate of change of v with respect to y multiplied by delta y. Similarly, this distance to be equal to original length plus rate of change of u with respect to x, so that gives us the length of the sides a dash B2 and A dash A2 A dash D2.

So now, if you simplify all this it gives us now delta v delta x by 1 plus delta u delta x and this is delta u delta y 1 plus delta v delta y. In fact, the strains are very small compare to unity therefore, we can neglect these quantities. And therefore, we can write this thing for small deformation this is nothing but delta v delta x plus delta u delta y. So, this quantities can be neglected and we get the strains finally, it will be given by gamma xy is equal to delta u delta y plus delta v delta x. So, it is the sum of the cross derivatives that gives you the shear strain at the point A that will have been considered. Now, the this shear strain is known as Engineering Shear Strain.

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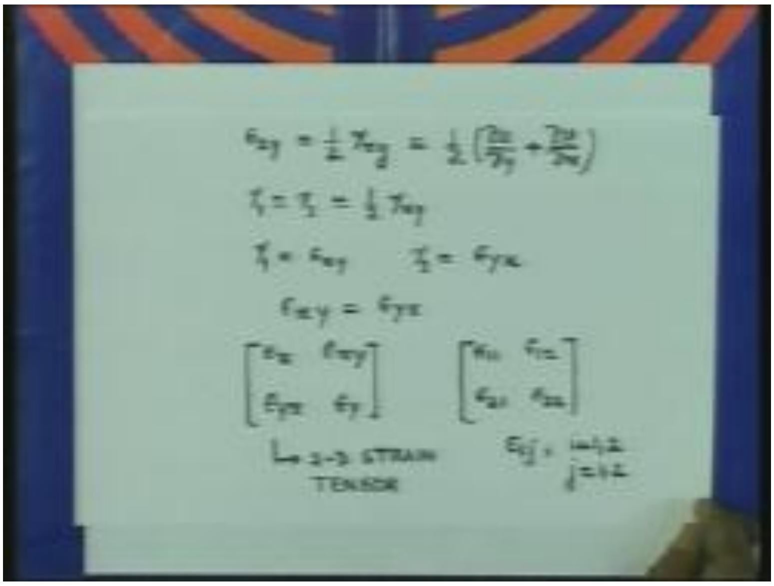

We define the tonsorial shear strain like this, that epsilon xy is equal to half of engineering shear strain and therefore, this is equal to half delta u delta y plus delta v delta x. So, this is tonsorial shear strain I want you to note 1 point carefully that when we are trying to consider the element, which we have started with you see here the this side rotates by gamma 1 and this side is rotating in the opposite direction anti clockwise direction.

So, this is rotate this is rotating in the clockwise direction this is rotate in the anti-clockwise direction and this gamma 1 and this is gamma 2 we generally take it to be gamma 1 equal to gamma 2 and that is equal to half of gamma xy. And this gamma 1 is the rotation of the x direction in the x axis in the y direction. So this gamma 1 is nothing but rotation of x axis in the y direction.

Similarly, this gamma 2 can be considered to be rotation of the y axis in the x direction and therefore, we represent them by symbol gamma 1 is nothing but epsilon xy rotation of x axis in the y direction and gamma 2 to be rotation of y axis in the x direction. So, these are the tonsorial shear strain and in fact since gamma 1 equal to gamma 2 we have shear strains epsilon xy is equal to epsilon yx.

Finally therefore, got a point we have distance like epsilon x epsilon xy and epsilon yx epsilon y. So, these are the strain associated with the x directions; these are the strains associated with the y direction this is known as 2-D Strain Tensor. And in the initial notation we can write this thing as epsilon 11 epsilon 12 epsilon 21 epsilon 22. So, in short we indicate this thing as, epsilon ij with the implicit assumption that I takes up value 1 to 2 and j takes up value 1 to 2. So therefore, this is the 2 dimensional strain tensor epsilon ij.

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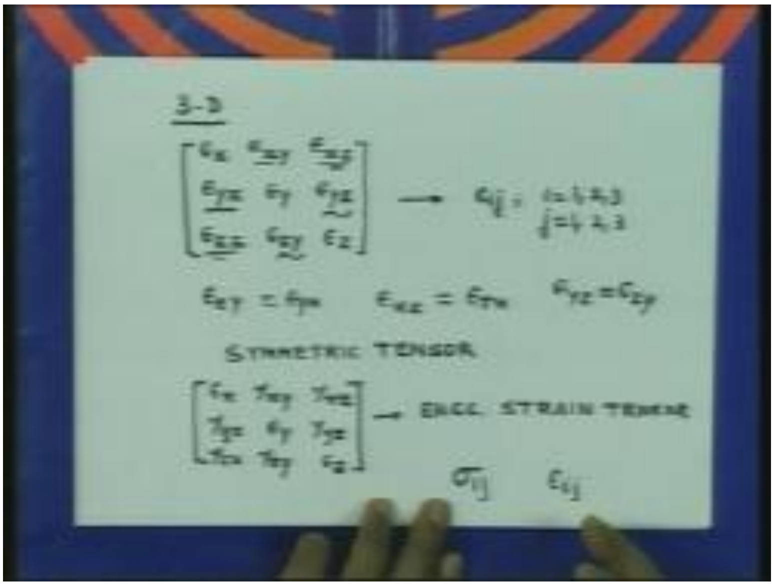

We can extend this consideration to 3 dimensions and in 3 dimensions we will have the strains like epsilon x epsilon x axis rotating in the y direction, epsilon xz x axis rotating in the z direction similarly, will have epsilon y axis rotating in the x direction direct strain and then we have epsilon y z y axis rotating in the z direction.

Similarly, epsilon zx epsilon zy and epsilon z, so these are the 9 components of strain tensor in 3 dimensions and in initial notation. We can again write, and then you can also write in sort epsilon ij with the implicit assumption that I takes up value 1 to 3 j takes up value 1 to 3. So therefore, it is a 3 dimensional stress tensor and you also note, that we have the condition that epsilon xy is equal to epsilon yx similarly, epsilon xz is equal to epsilon zx and then epsilon yz is equal to epsilon zy.

So therefore, these shear strains components are equal and therefore, we have this equal to this; this component is equal to this component and again this component is equal to this component. So therefore, the strain tensor is also a symmetric tensor. Now, we have got engineering strain tensor which is like this in the case of 3 dimension it will be epsilon x and it will be gamma xy gamma xz and this is gamma yx epsilon y gamma yz gamma zx gamma zy epsilon z.

So, this is engineering strain tensor. Herein the difference is that, the shear stains are double of the tensor shear strain. And the this is also symmetric; the tonsorial definition is very important because, both stress and strain we have symbol for stress is sigma ij, symbol for strain is epsilon ij. Now, this is going to the helpful in engineering to this tensor definition of shear strain will be helpful. Because, whatever relations we write for sigma ij can be directly adopted for epsilon ij, so this tensor definition is that way very useful.

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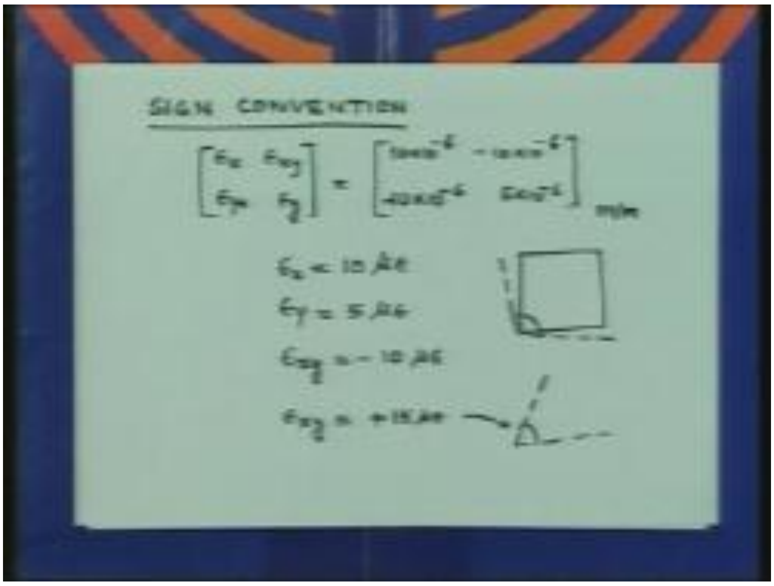

Now, convention for positive strain and negative strain I would like to give sign convention. If we have let us say, that the strains that this strain at a point is given by value like 10 10 to the power minus meter per meter and this is let us say 10 10 to the power minus 6 and this is 10 10 to the power minus 6 and this will 5 into ten to the power minus 6 and these are strains which the unit like meter per meter.

So, there dimensions so meter per meter now what it means, that this strain is positive epsilon x is equal to 10 you sometime write it as micro strain. So this strain micro strain it is positive therefore, it is causing elongation, so also epsilon y is equal to 5 micro strain it is causing elongation in the y direction whereas, this strain is causing epsilon xy is equal to minus 10 micro strain it is causing a change in the angle.

So, this must be understood that if we have an element like this; this particular definition is such that this angle in the first quadrant is going to increase. So this, is really the angle has now increased therefore, this means that the angle is increasing that deformation is different. If it is positive let us say, epsilon xy is equal to let us say 15 micro strain then the deformation would be like this the deformation will loo give rise to the angle of this sort.

In the first quadrant therefore, this is going to be the final. So, please note that this epsilon xy when it is negative it is going to be increasing the angle over 90 degree and this is positive means it is going to indicate. So, these are the conventions which are followed for strains. You must be wondering why this epsilon xy and epsilon yx they are same there is some physical reason behind back.

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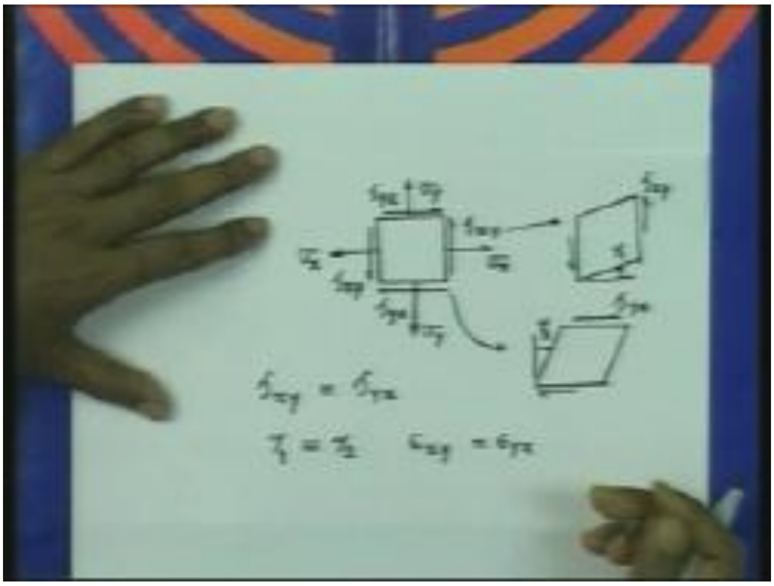

Look at the stresses, so if you consider the element to be subjected to stresses sigma x sigma y and the shear stresses. So this is tow xy, this is tow yx, this is sigma x, this is tow xy tow yx and then of course, we have also the sigma y stress here. This tow xy, this this spear is trying to rotate the element like this, so it this spear causes the rotation of the element like this. So, in fact that is responsible for this rotation what we indicated as gamma 1 and these 2 components tow yx they are responsible for the rotation of the element like this. So, in fact these components are responsible for rotating the elements like this. So, it is responsible for the rotation of the y axis in the x direction.

So therefore, you see that this rotation is nothing but gamma 2 so these are the stresses tow xy and these are the stresses tow yx the element is square in shape. And therefore, see the rotation due to these stresses will have some magnitude it will depend on the moment that is produced by them. Similarly, the rotation gamma 2 is going to be caused by this stresses and moment produced by them, therefore it will depend on the magnitude.

Since the dimension is the same which is square therefore, these 2 moments are going to depend on the magnitude of these 2 stresses and you can see that stress is a symmetric tensor that there by indicating the tow xy is equal to tow yx. Therefore, the moment produced by these 2 pairs is same as the moment produced by this 2 plus. And hence there is nothing to see that these 2 rotations are going to be different; they must be the same.

And hence we have since these stress tensor is symmetric we are going to see that, this gamma xy is equal to gamma yx; rotation of the x axis in the y direction is equal to rotation of y axis in the x direction. So, that is why the 2 angles or shear strains are considered to be the same. So, we have now been able to quantify the stresses and strains at a point.

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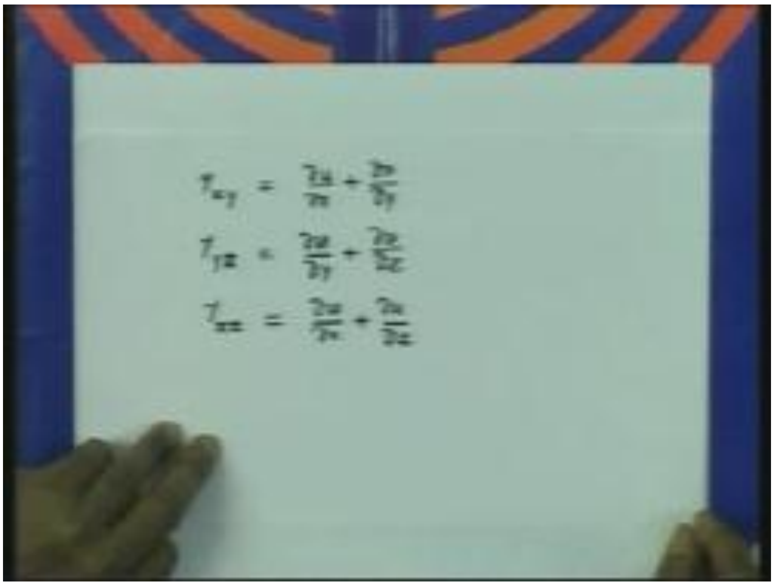

It is also important to note, that the shear strain expressions are that gamma xy is equal to delta u delta x plus delta v delta y. So herein, these 2 directions and the definition of shear strain involves the 2 displacement involved these 2 di displacement have in that 2 Cartesian directions that we have here; by similar argument you can say that delta gamma yz shear strain is nothing but will have displacement v and w and there will be cross derivative.

Therefore, it is delta w delta y plus delta v delta z and gamma xz is equal to delta w delta x plus delta u delta z, so these are the shear strain in 3 dimensions. Now, we would like to consider, if the stresses are given at a point by sigma ij we are interested in trying to see what is going to be the traction on any arbitrary plane.

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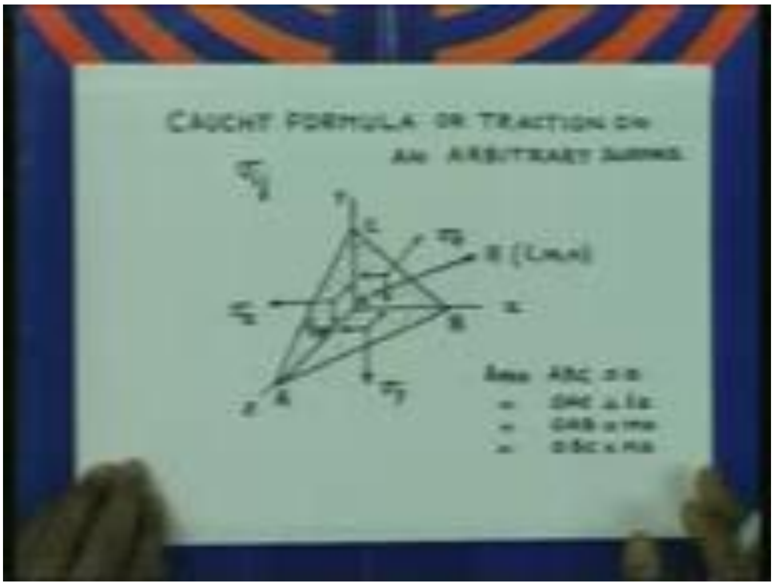

Now, this stresses you can show it to be like this let us consider the Cartesian direction as it is and then you try to consider an arbitrary surface. Let us, consider an arbitrary surface like this. So let us, say that this is my origin o this is let us, say that this plane is A B C. Now, the stresses at the point O is given by sigma ij so we can represent the stresses by showing the stress component on the phases OCA, OAB and OBC.

So these are going to be like this that will have the stress components here it is sigma x and then its negative phase so if you consider that this is your x direction this phase is negative similarly, let us axis xyz. So, this is sigma x and now the shear stress is going to be directed like this; this is tow xy and the other component is going to be like this; this is tow xj.

Similarly, will have the components like this this is sigma y and the other components are shear stress components are like this; this side we are going to have the normal components like this sigma z and the shear stress components are going to be this and this. Now, we are interested in finding out the traction on this area ABC arising out of the stresses at the point O, so this is the area.

Let us, say area ABC is equal to a and let us also consider that the outer normal to the plane ABC is directed like this; this is the outer normal. This outer normal would have some direction cosine its inclination with the x y z axis will cosine of the angle will indicate by the direction cosines L, M and N these are the direction cosines. Then, in that case it is clear that area OAC is nothing but L into a area OAB is nothing but M into a similarly, area OBC is equal to N into a, so these are strength of the geometry.

Now, we are interested in considering the traction on this phase ABC. So, if we consider that the traction force path unit area on this plane ABC in the x direction it will have 3 components. Suppose, we will have 3 components 1 in the x direction will represent that by Sx similarly, the component in the y direction will represent by Sy and the component in the z direction this is Sz. So, this is Sz, this is Sy so these are 3 components.

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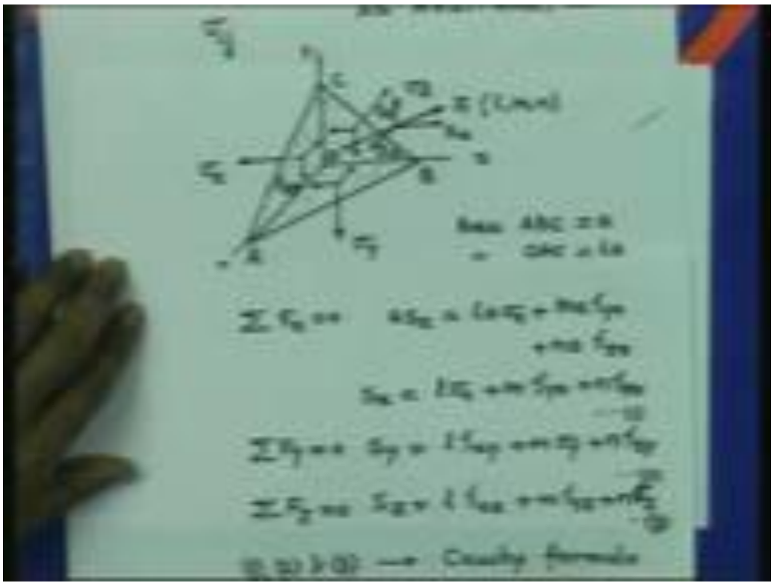

Now, if we try to consider the equilibrium in the x direction fx equal to 0 the force in the x direction on this area is a into sx acting in the positive x direction. And now, we are going to have forces in the negative x direction arising out of sigma x acting on this area which is nothing but L into a into sigma x. Then, we will have forces coming from this stress which is nothing but tow zx and also this is tow yx.

So therefore, this force arising out of this component acting on an area is nothing but N into a multiplied by tow yx plus this component which is acting on an area is equal to n into a it is tow zx. So therefore, this is what is the equilibrium equation in the x direction and we can now write these simplified it is Sx is equal to N into sigma x plus N into tow yx plus N into tow zx. I can write these equilibrium equations in the y direction.

So, that will give me by similar com consideration it is very simple you can find out the force in this y direction, it is force in this y direction is given by this traction. And it is balanced by the traction or forces coming up due to this stress, it is coming out due to the shear stress this 1 and the shear stress this 1. And if you simplify it will look like this it is going to be equal to l into tow xy plus m into sigma y plus n into tow zy.

Similarly, if we consider the equilibrium in the z direction it will give you Sz is equal to l tow this is xz plus m into tow yz plus n into sigma z. So, these are the 3 relations this 1, 2 and 3 these are known as Cauchy formula. So, to calculate the traction on any arbitrary plane if you know the direction cosines of the plane and the stresses at the point p then you can calculate the tractions by this formula.

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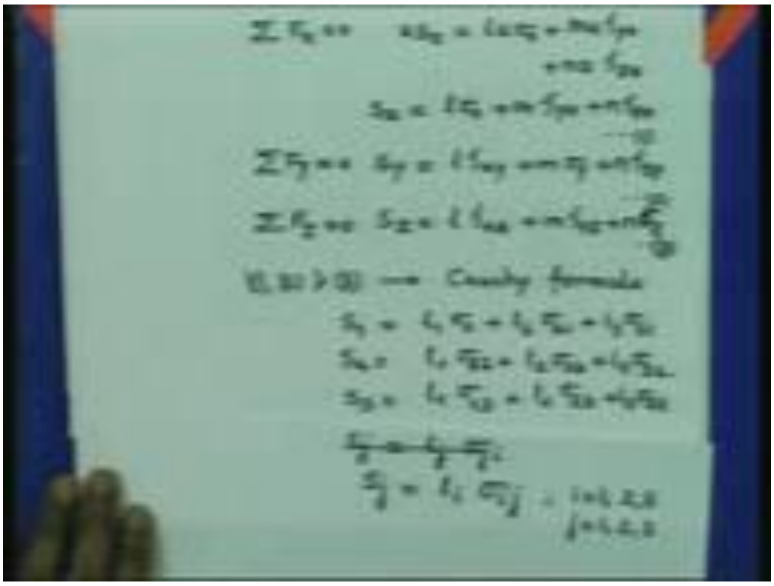

And in initial notations, if you make use of the initial notations we can write that s 1 is equal to it is l1 into sigma 11 plus l2 sigma 21 plus l3 into sigma 31 S2 is equal to l1 sigma 12 plus l2 sigma 22 plus l3 sigma 32 and S3 is equal to l1 sigma 13 plus l2 sigma 23 plus l3 sigma 33. In short, in tensor notation these 3 relations can be written like this is Sj equal to lj sigma ji wherein, Sj is equal to li sigma ij. So, this is an initial notation and with the implication that I takes up value 1, 2, 3, so also j takes up value 1, 2, 3 so we get the traction 3 traction.