Advanced Strength of Materials
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Lecture – 19

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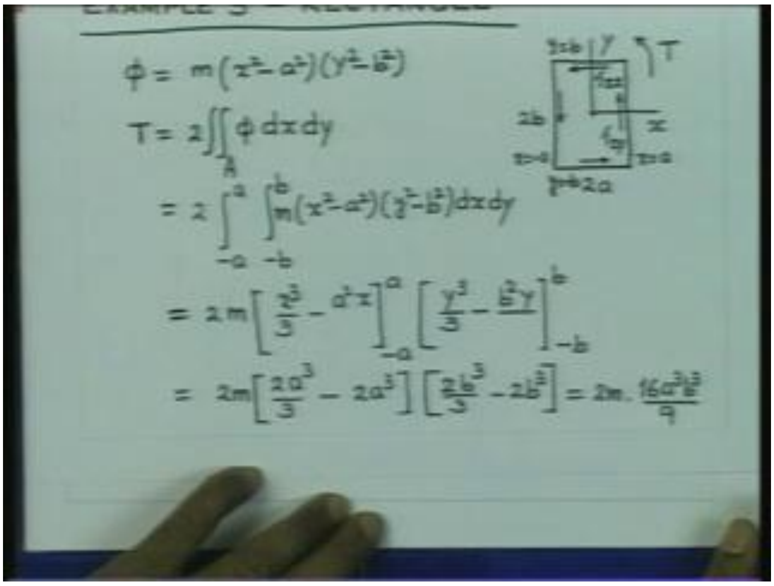

Let us get started with example on rectangular section. Last time I would mentioned that, the boundaries of the cross section, can be expressed in terms of this boundaries x is equal to a. The left boundary is x is equal to minus a. Similarly, this boundary y equal to minus b; and the top boundary is y equal to b. So, noting that, we can write a function phi, as x square minus a square into y square minus b square. And we had one constant m this ensures that the phi function is 0, all over the boundary.

Now, I would like to take up the steps, to find out the constant m. T is given by 2 times integration of the function phi over the domain. Here, the limits of integration are minus a to plus a minus b to plus b x square minus a square into y square minus b square d x d y. Since, the limits of integration are constant. We can write, we can do the integration of this function with respect to x, this function with respect to y.

And we get x cube by 3 a square x minus a 2 plus a. Similarly, y cube by 3 b square y by minus b to plus b. So, this will give us a cube by 3, this will give a cube. So, that will give us 2 a cube by 3 minus 2 a cube 2 b cube by 3 minus 2 b cube. So, this will give us minus 4 a cube by 3 and this will give minus 4 b cube by 3. So, that makes it of course, we have left out the constant here m. That m is carried forward. So, we have 2 m into 16 a cube b cube by 9.

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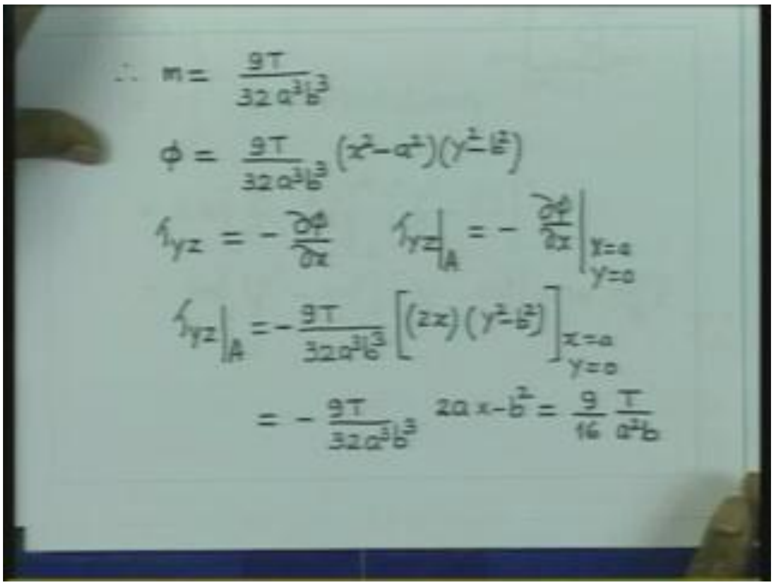

Therefore, constant m is equal to 9 T by 32 a cube by b cube. So, if you substitute back, the value of m phi is obtained. This is 9 T by 32 a cube b cube into x square minus a square into y square minus b square. Let us now calculate the stress, which is going to act at the boundary. Particularly, we will be interested in calculating the stress at the end of the x axis. So, that stress is going to be tau y z; and this is given by minus delta phi delta x.

So, we are interested in calculating this stress, at the end of the x axis here, which is nothing but x is equal to a y equal to 0. So, let us calculate, if we indicate ((Refer Time: 06:25)) this point as A. So, tau y z at point A is delta phi delta x, for x is equal to a y equal to 0. So, let us see tau y z at a is equal to 9 T 32 a cube by b cube. And when we differentiate this, we get 2 x y square by b square. And we have to substitute the value x is equal to a, y equal to 0. When we do that, 9 T 32 a cube b cube, this is 2 a and this is minus b square. So, what we get is 9 by 16 T a square b. So, the stress at the end of the x axis is 9 by 16 T a square b.

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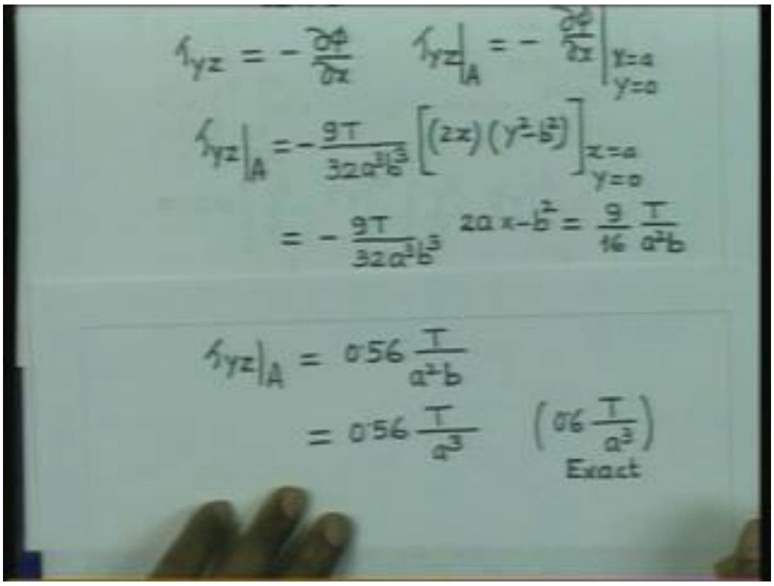

And this is approximately equal to 0.56, 0.56 T by a square b, when a equal to b. Then, we have a square cross section. So, the stress at square section is 0.56 T by a cube. And it is known from exact and arises of this problem, the answer is 0.6 T by a cube. So, this is the exact solution. So, you can see that, you have got the results quite accurately, by simple analysis. Exact analysis has been done following the energy principles. We are going to talk about it. But, I would like you to note some of the results, which you may find useful later.

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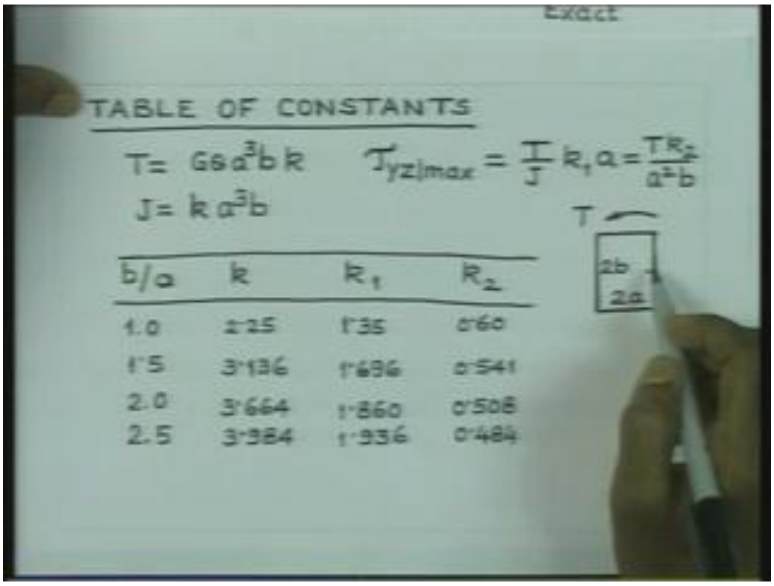

Here the T is expressed in the form, G theta a cube b into a constant k. Similarly, the stress at this point which is nothing but tau y z section, which is really maximum stress there. It can be written as T by j into k 1 into a. And that is equal to, sometimes written as T into another constant k 2 by a square b. And this j is given in the form k a cube b. For various rectangular cross sections with b by a varying like this, the constants are going to change, which are shown here.

So, b equal to a this constant k 2, which is here is equal to 0.6. And as the section become longer in the vertical direction, you are going to get this constant gradually reduced. And for infinity, this constant is equal to 3 by 8 ((Refer Time: 11:17)). We would like to use this result later. One point from what noting here is that, the stress that we have obtained tau y z stress, it is proportional to x. So, therefore, as the distance from the origin or the centre of the cross section increases.

So, if we consider the section here, as the distance increases the stress increases. And it is going to be highest at the outer periphery. You will find similar picture that, if you calculate tau x z stress, it is going to be highest there. And it is going to vary linearly from this point to that point. And the maximum stress is acting at this point. However, this maximum is lower than the maximum here, that is why this is the most critically loaded point.

We will try to look into more of physics, little later. Let us introduce some analogy to understand the torsion problem. We will known as membrane analogy, it was introduced by Prandtl. You can do experiment to understand the, to determine the stresses in a section, which is not circular; and which is subjected to torsion.

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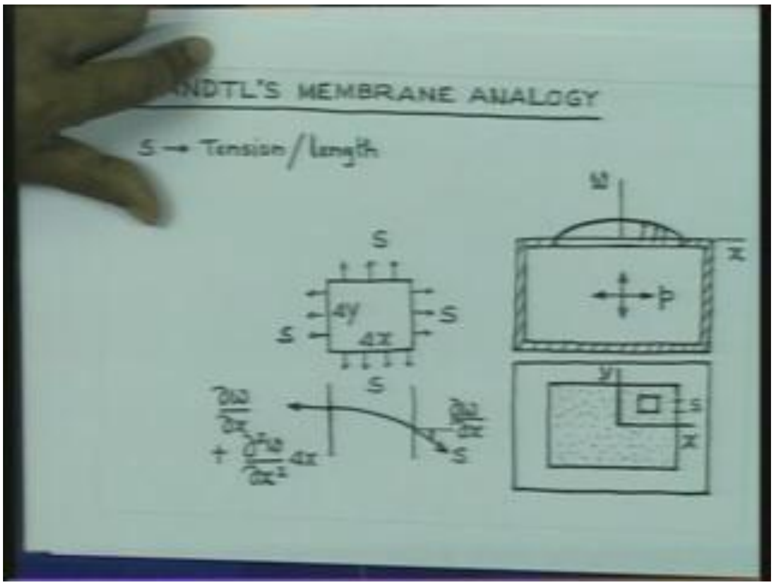

In the Prandtl's membrane analogy, let us consider a box, closed box. And in one of the face, it has a cut out here. So, this is the cut out; and on this cut out it is closed by a membrane. It could be a thin balloon material, sheet of balloon material or it could be a film. Think of it, that we have closed this cut out by a membrane or a soft ((Refer Time: 13:45)). So, it is very thin membrane indeed If you now consider, therefore the box has become now shield, or here tight.

Now, if we try to increase the pressure here, the membrane is going to deflect. It is going to be deflected like this. If you look from this side, it is going to have a shape like that. There is deflection of this membrane, from the undeformed state. Now, this deflection of the membrane, will try to derive what sort of equation is satisfied by this deflection. To understand the or to get the governing equation, which involves this w displacement.

Let us concentrate on a small element here, which is delta x by delta y. I would like to now, draw this element in larger scale. So, this is delta x, this is delta y, this membrane is stretched along the boundary. So, it is a touch to this boundary and it is also stretched. And that stretching per unit length of the boundary, let us say that is equal to capital S. So, the tension per unit length of the boundary. So, S is tension per unit length. Obviously, the membrane is going to be under pressure. And the pressure will act normally to the surface.

And the deflection of the membrane is very small. So, therefore, we can assume that, this pressure is almost acting vertically. Therefore, the picture that we have here, we have tension all over the boundary. And since, the deflection is very very small, you can consider that this tension is unchanged per unit length. If we now consider the forces in vertical direction. Let us also try to write one more information.

That the, if I look from the bottom to top direction or from origin towards y axis. Then, the deflected membranes shape is going to be like this. So, therefore, we have the attraction or the tension S is acting like this, here it is acting like this. Now, let us say that the slope of this, this is the slope here. And this is delta w delta x, the slope and now as we move to this one, there is a change in slope. And let us consider, that this slope is equal to delta w delta x plus the rate of change of delta w delta x with x. So, delta 2 w delta x square multiplied by delta x. So, therefore, the slope here, let us say that this slope is like this.

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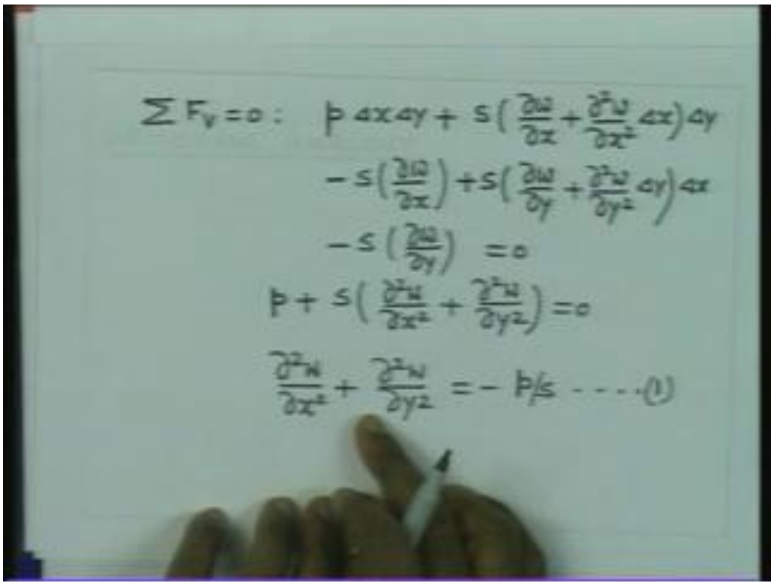

So, now if we try to write the forces in the vertical direction. Let us say, that is in the vertical direction equal to 0. What we have is the pressure is acting vertically upward, and that is nothing but p into delta x delta y. P is the pressure, and now this component, which is acting on this part of boundary delta y.

So, this attraction is acting on this part of the boundary, which is delta y. So, therefore, we can write now that S. And it is acting at an angle of this much. So, therefore, ten component of this will be approximately this. So, this multiplied by delta w delta x plus 2 w delta x square multiplied by delta x into delta y is the length of that span. So, therefore, that is the attraction acting vertically or the force, which is acting vertically upward. Minus S into this one, this is nothing but delta w delta x.

Similarly, if we consider the other side, here we will have some forces coming due to tension, per unit length. And here this will be also acting. So, therefore, if we consider that the slope here is delta w delta y. And the slope here is, delta w delta y plus delta 2 w delta y square into delta y, then we can write from the other side that S. If you would like to draw a sketch for it, you can do it that way. We have this, then you draw it here ((Refer Time: 20:57)).

So, we have delta w, this is S delta w delta y. And the angle here, this angle is going to be... So, here this angle is going to be delta w delta y plus delta 2 w delta y square into delta y. That this has become little complicated, delta w delta y plus delta w delta y square into delta y. So, now we can, this is acting on the face of length equal to delta y. So, therefore, you can write S into delta w delta y plus delta 2 w delta y square into delta y. And this is acting over the face of length delta x.

So, therefore, delta x minus S delta w delta y is equal to 0. So, therefore, I want you to note this, that this slope is delta w delta y. And this slope here, delta w delta y plus delta 2 w delta y square into delta y. So, once you simplify this, it gives us p plus S into delta 2 w delta x square plus delta 2 w delta y square is equal to 0. On rearrangement, what we have is delta 2 w delta x square plus delta 2 w delta y square is equal to minus p by S. Let us say that, this is equation number 1.

So, therefore, the deflection of the membrane, that we were looking into deflection of the membrane under pressure. You are going to have the deflection satisfying the function w, which stands for deflection is satisfied by the Laplace equation, because p is a constant so and so. S which is tension per unit length along the boundary is also constant. Therefore, this is going to be the equation satisfied. And it is nothing but it is cos theta equation.

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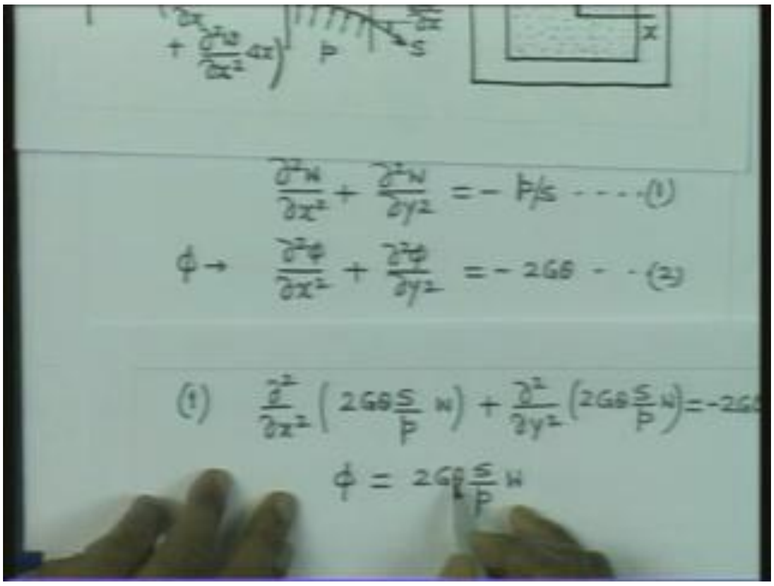

If you remember the Prandtl stress function phi, also satisfies an equation of the type delta 2 phi delta x square, delta 2 phi delta y square is equal to minus 2 G theta. So, this two equations are analogous. And therefore, it is possible to get an idea, about the function phi. By looking into the deflection of the membrane, provided the geometry of the membrane boundary, corresponds to that of the shaft that one is interested in.

Just I would like to work on this equation number 1. And we can write from 1, delta w delta delta 2 w, added one delta 2 delta x square. If we write this, twice G theta S by p w plus delta 2 delta y square twice G theta S by p w is equal to minus 2 G theta. So, herein what it means is that, phi is equal to twice G theta S by p w. Herein, all are constants, G is a constant, theta is a constant, S is a constant, p is a constant. So, therefore, this phi is linearly related to w.

By taking a membrane, having boundary say corresponding to that of a section 1 is interested in. It is possible to get the idea about the function phi and hence, one can calculate the stresses also. In fact the stresses delta phi stresses tau x z is given by delta phi delta y. So, therefore, it is also going to be related to delta w delta y. Similarly, tau y z is minus delta phi delta y. So, therefore, it is related to delta w delta x. So, membrane shape will give us the idea about the stress. We will now like to solve some problem and then get the little more a quittance with this membrane analogy.

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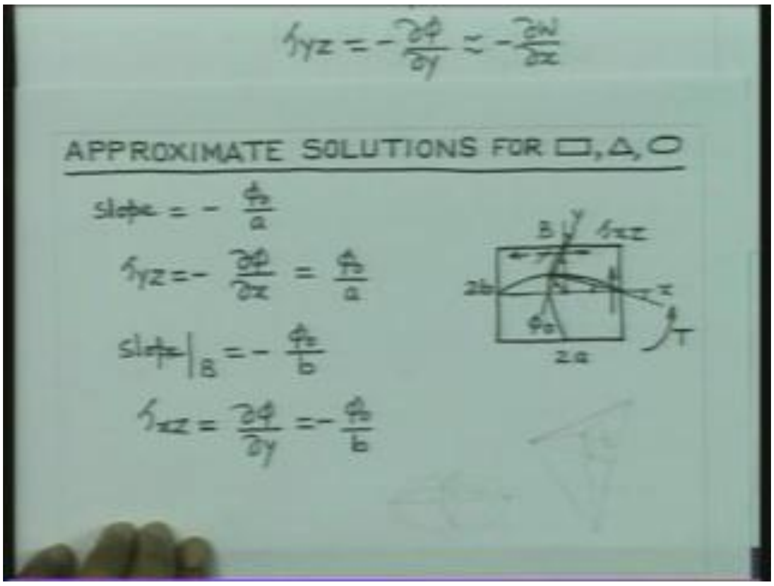

So, let us consider by this analogy. Let us try to tackle the problem of rectangular section, triangular section and also elliptical section. If you consider now a rectangular section, let us say this is the section. So, therefore, membrane is going to have this boundary. And if we now imagine the membrane to deflect, obviously the membrane is going to have zero deflection, along the boundary. And it is going to have maximum deflection at the center.

And that shape you can imagine, it is a doubly curved surface. So, you can imagine that, this is the shape of the surface. And we will certainly have the maximum deflection at the center, if you are trying to consider this direction. Let us say that is x direction. And let us consider the y direction to be like this. It is not difficult to see that, this is really the maximum deflection. And let us say, that this maximum deflection is equal to phi 0. If we approximate this curved surface by a straight line, if we just approximate it.

So, therefore, the slope here is going to be given by this angle. If you consider that, this side is equal to 2 a; and this side is equal to 2 b. Then, this slope, slope is equal to this really negative slope. So, this is the slope and this slope is equal to minus phi 0 by a. So, this is the phi 0, it is proportional to w, we have written directly phi. So, therefore, this is phi 0 by a that is the slope. And now we know, that the shear stress, which is going to act along the boundary which is nothing but tau y z is equal to minus delta phi delta x.

And we get that delta phi delta x is nothing but this slope. And therefore, it is equal to phi 0 by a. This is the value of the stress, which is going to act parallel to the boundary. Now, if you consider the curve here, the slope that we are going to get by joining the tip and this point, this is the slope. So, therefore, this is the slope, and hence the slope at the end of y axis. Let us consider that, this is point number b slope; slope at point B is equal to minus phi 0 by b, because this height is equal to, this distance is equal to b.

The stress which is going to act here, is parallel to the boundary and it is nothing but tau x z stress. So, tau x z is equal to delta phi delta y. And we have found that, this slope which is given by this angle is nothing but minus phi 0 by b. What it means is that, the stress cannot act in this direction. It is going to act only in this direction only. So, therefore, that is not the direction which is correct. And the if torque on the shaft is acting like this.

We know for shear, that this stress here should be acting in that direction. And the stress here should be acting in this direction. And we have got the values correctly. So, there herein, without any difficulty you can understand that, if b is less than a. We are going to get this slope to be higher than this one. And hence, higher stress is going to act at this point. And the stress is going to vary linearly. In this case, you see that slope is maximum there and it is gradually becoming 0 at the center.

So, therefore, the stress is going to vary gradually from the... I mean in fact, it is a quadratic curve, it is going to vary linearly up to this center point to 0. Similarly, here the stress is maximum; and it is going to vary linearly to 0 at the center.

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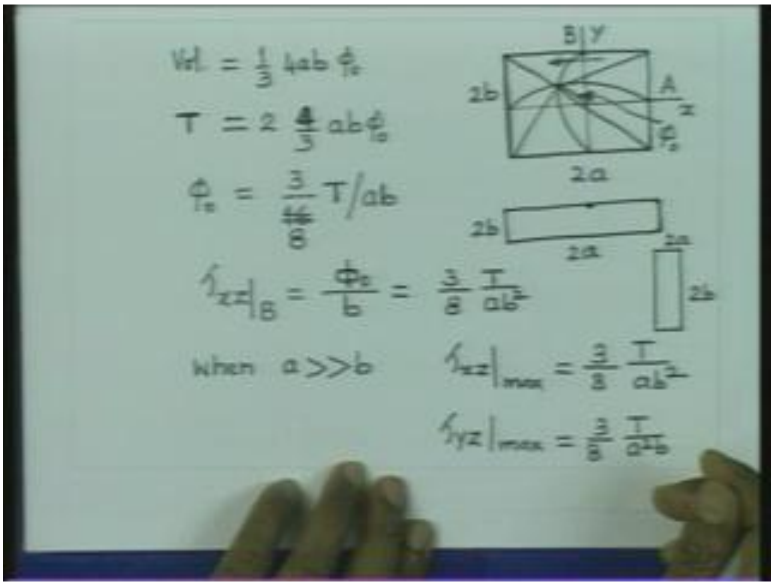

So, we found that the stress is going to be maximum, at this point B. Let us say that this point is A, herein higher stress is going to act, at the point which is closer to the center. It is different from the circular cross section, wherein you have found that the stress is going to be the highest at the farthest point. We would like to do little more calculations, to see the value of the stresses, in terms of the applied torque. So, this doubly curved surface that we were talking about, let us approximate by a pyramid.

So, let us approximate this doubly curved surface by the pyramid, whose faces are this triangle. Then, this and this and finally this triangle, at this height, we have indicated this thing as phi 0, let us consider that height is equal to phi 0. The volume bounded by the approximate surface, and the cross section. Volume is nothing but it is 1 3rd the base is 4 a b height is phi 0. We have already seen that, the torque capacity is nothing but twice the volume bounded, by the surface phi surface and the base.

So, therefore, it is 8 by 3 a b phi 0, so we have phi 0 is equal to 3 by 16 T by a b, it is 4 by 3. So, this will become 3 by 8 T by a b. Now, the stress at the point b, we have got that, tau x z at b is recalled to, it is T by... Now, that if I am applying this stress to be acting like this, it is positive stress. And therefore, this stress is equal to T by, that is nothing but phi 0 by b. So, if we do calculations, it becomes 3 by 8 T by a b square.

When a is very large, a is very large, in fact the stress when you are trying to think of that this cross section is elongated in this direction. It becomes narrow cross section, like this. So, herein a is very large compared to the 2 b. That you are going to get the maximum stress at this location. And that is this value tau x z maximum is equal to 3 by 8 T by a b square, and if you have the cross section, the other way that this is the cross section.

So, here 2 b is larger than this. Then, the stress will become tau y z maximum. And that is just going to be phi 0 by a. And that will give us 3 by 8 T a square b. And in fact, the exact value is 3 by 8 T by a square b. So, therefore, we have got really, the value correctly obtained. I have given you little while ago, that the maximum stress in sub cross section is going to occur here. And it is magnitude is 3 by T a square by b.

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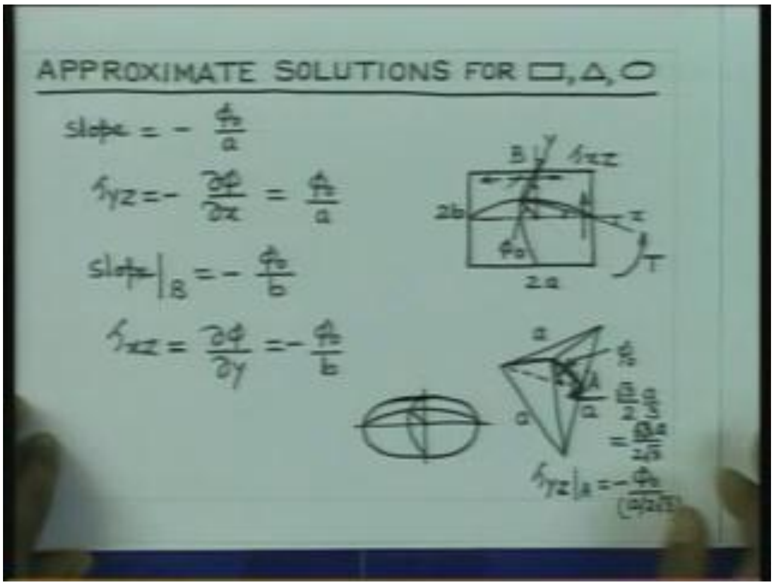

I would like you to look into triangular cross section; and elliptical cross section. Triangular cross section if you consider, so your membrane boundary is going to have the triangular shape as the shape of the bar cross section. And then we will find that the membrane surface is going to have doubly curved surface. And it can be approximated by a pyramid again, for fast approximation.

So, this is the surface. So, we have approximated really the surface, which is like this, by this pyramid. Therefore, you can now immediately get the feeling that this is the C G of the triangle. And therefore, this the maximum height is equal to phi 0 here. So, this is the maximum height, and now if you consider, this height of the triangle. The sides are let us say a. So, this is nothing but root 3 by 2 a, root 3 by 2 a. And this distance you can calculate, it is going to be 1 3rd of that. And the slope here is going to be easily available, by considering this distance divided by this distance.

And this distance is nothing but this distance here, it is root 3 by 2 a by 3. So, that is nothing but root 3 by 2 into root, 2 into root 3 a by 2 root 3. And you can calculate now the stress, parallel to the boundary at this point a. So, if you consider that, this is your y direction. So, tau y z at A is nothing but minus phi 0, by the distance a by 2 into root 3. And you can obtain this phi 0 by considering the fact, the torque capacity of the shaft is given by, twice the volume bounded by this approximate pyramid.

Twice the volume of the approximate pyramid and hence, you can get this phi 0. And you can also obtain the solution approximately. For the elliptical section, if you consider the elliptical section, the surface of the membrane is going to be again doubly curved. And if you just see the surface, it is going to look like this. That is the membrane surface. And it has the maximum height here. And therefore, the stress is going to be higher, because the slope of the surface is higher at this point.

And therefore, this slope is going to be higher. And it is lower in this point. In this case, you again know that, this is the end of the minor axis. And it is closer to the center the stress is, higher at this point. And in the case of triangle, the stress is higher at this point. And now, we would like to look into, you can really go on considering the problem of various types, various cross sections, by this analogy. And you can immediately comment upon the point, where the stress is going to be the maximum.

Let us try to look in some points. I indicated earlier that, for a rectangular cross section or a triangular cross section, the stress at the corner is going to be 0. Let us understand why that is shown, the answer to this problem can be obtained three different ways.

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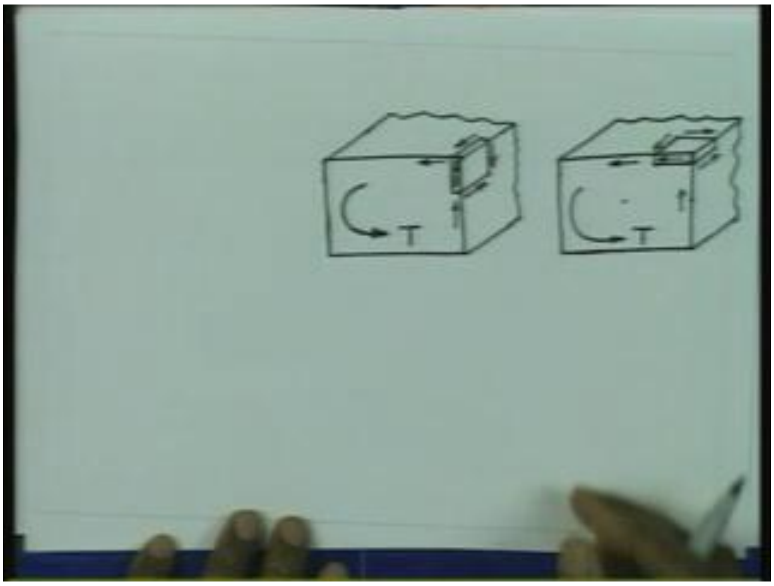

First of all, let us try to consider this rectangular section. And let us concentrate at this corner. So, torque is like this, this is the axis of the shaft. If you consider this point, the stress is going to act parallel to the boundary, in the same sense as the torque. So, therefore, we will have stress acting like this, close to this boundary. And if you consider this boundary, stress is going to be acting like this, close to this boundary.

Now, at the corner, which is a common point it has therefore, the possibility of stress acting in this direction and also in this direction. It is a single point. and it cannot have two direction from the shear stress acting. And therefore, from the uniqueness of the stress, the possibility is that, this stress has got to be 0. Both the stresses are going to be 0, that is from the consideration of the, uniqueness of the stress.

Now, I would like to give you the answer, from the consideration of the membrane analogy. Then, if you now think of the membrane a touch to this boundary. And it is inflated from inside. The boundary deflection of the membrane is going to be 0 here. Now, boundary deflection of the membrane will be 0 there, in the direction out of the cross section. We know for sure, that if the stress is going to act perpendicular to this boundary, this stress which is acting.

So, this concentrate on this element, that located here. The stress acting on this area is going to be directed vertically upward. And that is again related to the slope of the phi, surface in this direction. And since phi is 0 all along this slope is 0, therefore this slope is

0. Similarly, if you now consider the stress, let us draw the element for clarity. So, let us say that this is the element.

If the stress is going to act at this point, parallel to the boundary, it has got to be directed like this. Now, this stress it is obtainable from the slope of the phi surface in this direction, in the perpendicular direction. Since, phi is 0 along this boundary, therefore again the slope is 0. Therefore, the stress is 0. So, this is from the Prandtl's membrane analogy.

So, you can find the answer, that the corner stress is 0, from the uniqueness of the stress from the corner point, from the membrane analogy. Now, I will give you the third way of finding out the stress to be 0. Let us concentrate again on this element. If the stress is going to be acting on this element, on this face like this. Then, the stress which is going to act on this stress, the opposite stress is going to be like that.

And the complementary shear stress will develop on the top surface, it is direction has got to be like this. Similarly, on the opposite face, the stress direction is going to be like this. So, if the stress is to nonzero on this face, close to this boundary, then this stress has got to be nonzero. But, we know that the surfaces, these surfaces of the shaft is free of any external loading. Therefore, this is 0 and if this is 0, this is 0, so also this is 0, this is 0.

So, therefore, this stress is 0, by similar type of argument, a nonzero stress is going to act at this point in this direction. Then, on the opposite face, we will have stress acting like that. And the complementary shear stresses are going to act like this, on this face, outer face. Since, this outer face is free of any external loading, this stress is 0. Therefore, this is 0, so also this and this is 0. Therefore, you will find that, the stresses are going to be 0.

So, corner stresses the same type of argument, can also be considered for the triangular cross section. We will find that, for that matter any corner is going to be free of any shear stress, in a torsion problem. So, note that the stress for this point, which is the corner is 0. In the case of the circular shaft, we have found that the stress is going to be maximum at the ((Refer Time: 50:06)). But, in such cross section, it is different the stress at the, for this point it is, a carnality 0.

And if it is a point here, somewhere there, the stress is maximum. But then higher stress occurs at a point, which is not the farthest. It can be at a point here which is at a distance shorter than this distance. So, the stress acting parallel to this boundary. If this is longer side than this one. Then, this is going to be higher than the stress acting at this point.

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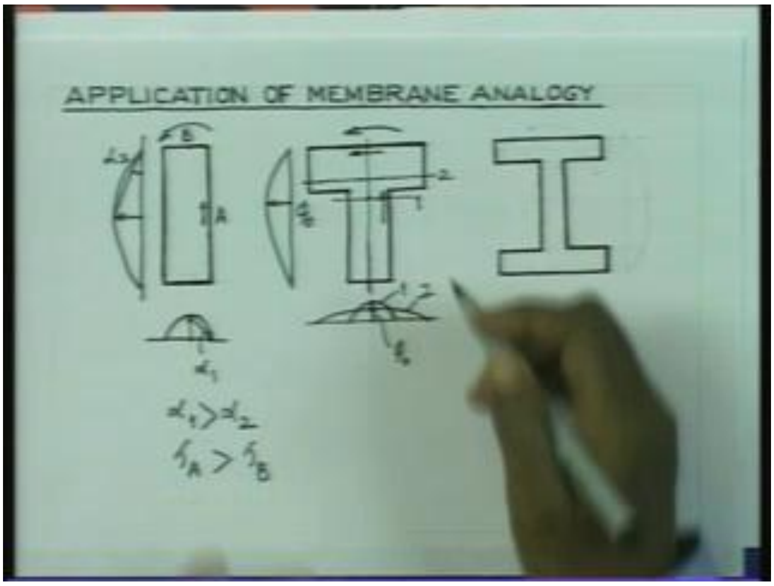

I would like you to now, see some of the cross sections, which are shown here. Think of this case, already we have got the idea that, the stress is going to be highest at this point. And from the membrane analogy, you certainly can conclude on that. If you look into the membrane surface for this sort of boundary, we will have the surface looking from this direction. And the surface looking from this direction. In the horizontal direction, it will look like this.

So, that this is the membrane surface. So, the height here and here is the same. And since, this distance is small, we are going to have higher slope. This slope is higher and therefore, this stress is going to higher at this point. Then, this point where the slope is given by this angle. So, if you consider that, this is let us say alpha 1, this is alpha 2. Since, alpha 1 is greater than alpha 2, you are going to have this stress at point A.

So, stress at A is going to be higher than the stress at the... If of course, the direction of the shear stress is like this, provided we have a torque like that. Now, concentrate on a cross section like this. So, if you have the T cross section, imagine that the membrane is deflecting, we have the torque direction like this. So, the where do we expect the highest deflection of the membrane to occur. Obviously, it is going to be somewhere over here. And if you now try to look at the section, let us take a section here of the membrane surface.

So, at this point, it may typically loop like this. That is the membrane surface. And if you consider a section there, let us say one the membrane is going to have deflection like this. So, therefore, more deflection at this point, so this is for 2 and this is for 1. Similarly, if you look in this direction, if you take a intersection of the membrane surface with the vertical plane passing through this line, it may look like this. So, you have the maximum deflection here, this is phi 0. And here it is again phi 0. And immediately you can decide, that the stress is going to be acting like this, at this point and this is highest.

And obviously, the stress at this point is lower than the stress at this point. And it is going to act like this. And we will find also the corner stresses are all 0s. I leave it to you for consideration. And also look into a cross section like this, I cross section. And try think about the stresses, that are going to come up at various points on the boundary.