Advanced Strength of Materials

Prof. S. K. Maiti
Department of Mechanical Engineering
Indian Institute of Technology, Bombay

Lecture – 18

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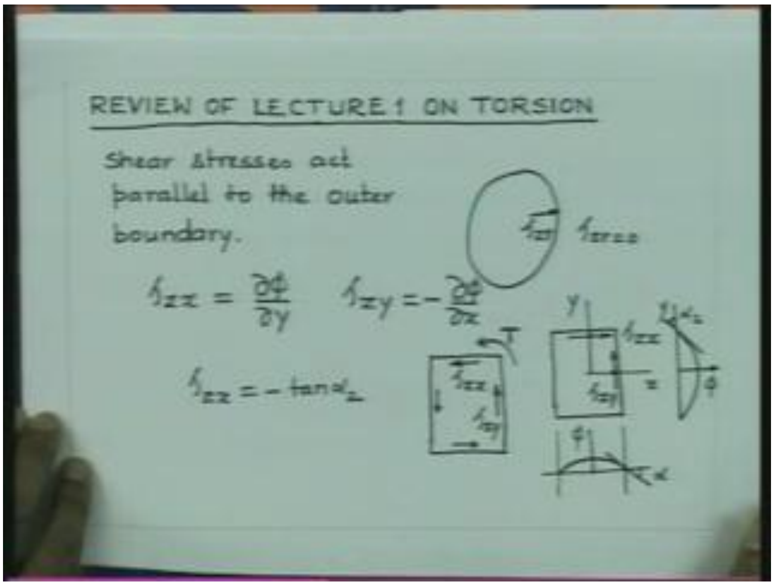

Let us first review our earlier lecture. That is lecture 1 on torsion of non-circular shafts. We have noted few things, that while applying torsion, each section is going to have warping. And that warping is a solution of Laplace equation. And we have noted that the section is subjected to shear stresses, which will have a typical feature. That shear stress cannot be acting normal to the boundary.

If you have the section like this, what we have found that at a point in the boundary. The stress, which is normal to the boundary. It acts in the radial direction. So, therefore, this tau z r space is 0. So, tau z r is 0. So, therefore, all along the boundary, we will find that stress is going to act only parallel to the boundary. This is a very important point, in the case of torsion of this type of component, shear stresses act parallel to the outer boundary.

We have introduced the Prandtl's stress function. And this Prandtl's stress function, phi it was defined that tau z x stress is given by the slope of the phi surface, with respect to in the y direction. Similarly, tau z y is negative of the slope of the phi surface, in the x direction at a point. So, this something again important, that if you are trying to think of a section like this. Suppose, you are interested in finding out the shear stress at this point, it is going to be parallel to the boundary.

So, therefore, there can be only shear stress tau z y. And if this shear stress is to be calculated, what we need to do is, we need to take a section of the phi surface, with a vertical plane containing x axis. Suppose, I take that section of the phi surface with the plane containing x axis, this is the shape of the phi surface. So, we take a section passing through this line. And therefore, this will be phi surface. And therefore, if you now try to take the slope of this phi surface.

So, this slope, let us say alpha, tan alpha. That tan alpha will give us the stress tau z y. And you see that, they are the slope is negative. And up to this stress will be positive. And it is really acting in the positive direction. Now, think of it, if you are interested in finding out the stress at this point; and this stress could act only in the direction. If it is a positive stress, it will be directed like this. So, therefore, it is going to be tau z x stress, here tau z y will be 0.

And now if we have to find out this, then what we need is that, we have to take the section of the high surface considering a vertical plane, containing the y axis. So, you have to take a vertical plane, containing the y axis. And then take this section, suppose we take this section and draw it here. Let us say that the phi surface, intersection is like this. Wherein, I mean that we have the plotting of phi like this and y is here, then at this point if I take the slope here.

Slope of the phi surface, that slope would be phi surface, this angle alpha 2. Tangent of that angle will be giving us the magnitude of the stress tau z x. And if the slope is negative, then obviously this minus minus will become plus. And therefore, we are going to get the stress to be directed in the opposite direction. So, therefore, if I have this angle, this angle is negative, minus tan alpha. Then, we are going to get stress to be again acting in opposite direction. That means, it will be acting in the direction like this.

No, this tau z x is given by delta phi delta y. So, if this angle is negative here. So, therefore, this stress will become negative. So, therefore, what it means is that, tau z x is nothing but minus tan alpha 2. And therefore, this is negative stress, thereby meaning that this stress will be directed in the opposite direction. So, finally, what we find is that, if we have the surface to be like that. Then, the stress here will be directed like this, stress here will be directed like this.

So, therefore, this is tau z y stress and this is tau z x stress. And these stresses, it will act along the boundary, in the same sense as the applied torque. Or applied torque, you remember that it was like this. And therefore, this shear stresses are really oriented in the same sense as the torque. So, all along the boundary, you can now draw the direction of the shear stress like this. Now, let us try to solve some problems. And get to know, where the stresses are going to be maximum; and how this vary over the whole section. So, we will consider some examples.

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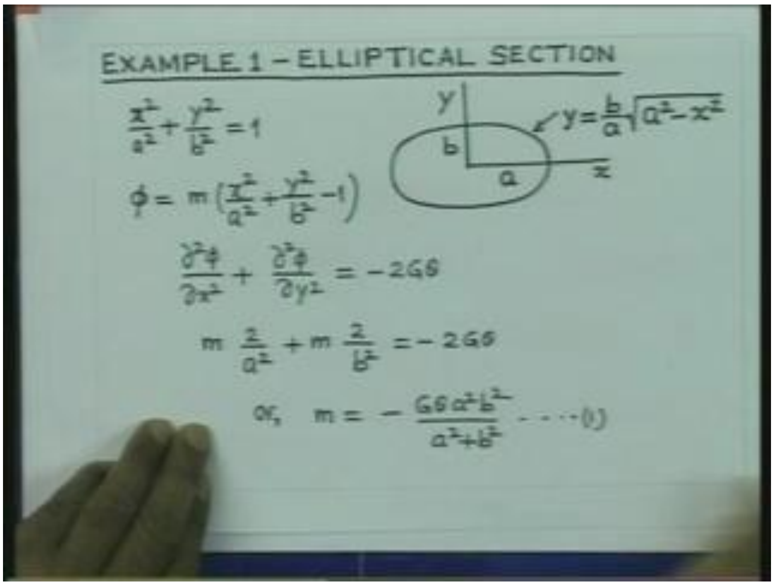

Let us first consider the example of elliptical section. Let us say that the major axis, semi major axis is a, semi minor axis is b. And therefore, this curve is nothing but y equal to b by a root over a square minus x square. The equation of the boundary is given by x square by a square y square by b square is equal to 1. So, the boundary equation is nothing but curve is nothing but x square by a square plus y square by b square minus 1 equal to 0.

Now, we can assume the phi function. In this form m into x square by a square y square by b square minus 1, this ensures that the phi function is 0 all over the boundary. You remember that, Prandtl stress function phi have to be 0 at the boundary. So, therefore, that is ensured and we have just brought in one constant m here. Now, to determine this constant, we can write the equation of the, equation satisfied by this or equation that is satisfied by the Prandtl's function.

That was delta 2 phi delta x square plus delta 2 phi delta y square is equal to minus 2 G theta, where theta is the angle of twist per unit length. Now, if we substitute the value of phi here, it gives us minus 2 G theta. And therefore, this constant m is equal to minus G theta a square b square by a square plus b square. So, let us write that equation number 1. So, therefore, we have got the value of constant m, in terms of angle of twist per unit length and the cross sectional dimensions.

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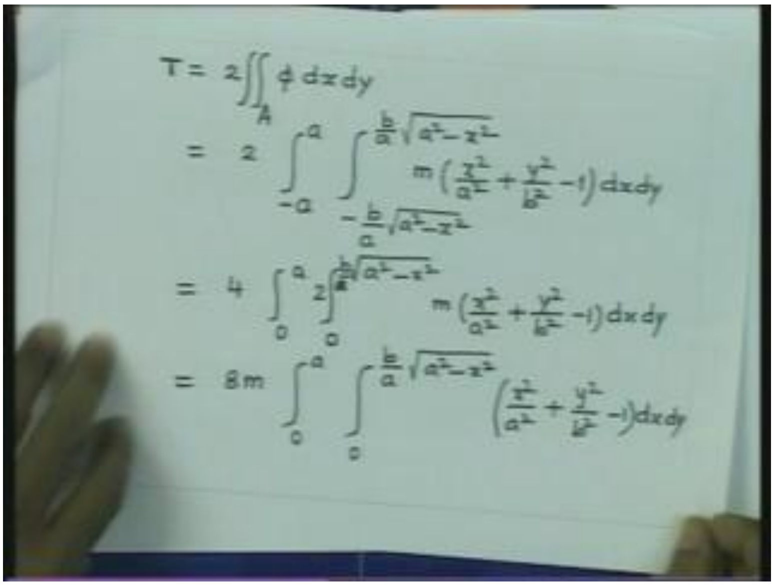

Now, we would like to go in for the utilization of another relationship. That the torque applied is related to the phi surface by this relationship. Integration of the phi function, over the whole area. This is nothing but integration we can now consider that, this integration x is varying from minus a to a. And y is varying from minus b by a, a square minus x square to b by a a square minus x square. And phi is, you can now write the value of phi.

If we write in terms of m, m into x square by a square y square by b square minus 1 d x d y. Now, since this section is symmetric, we can simplify this. This will take the form, that is 4 times 0 to a. You can write this also 2 times 0 to b by a square root of a square minus x square m x square by a square plus y square by b square minus 1 d x d y. So, this gives 8 times m 0 to a 0 to b by a a square minus x square x square by a square plus y square by b square minus 1 d x d y. Now, first we will integrate with respect to y.

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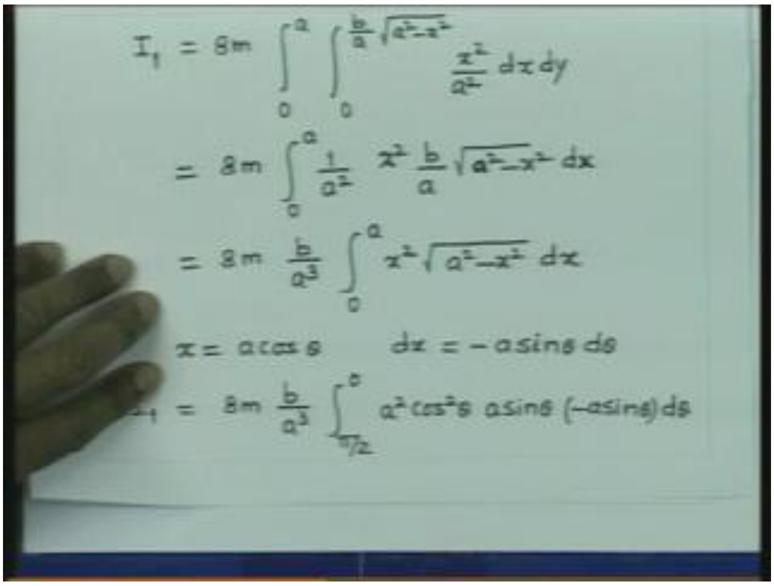

So, if we do that, 0 to a we can take only one term, we will just concentrate on this term, x a square by a square. So, if I consider that this integral, let us say that this is I 1 b by a a square x square x square by a square d x d y. So, this I am concentrating only the first term. So, this gives us 8 times m 0 to a, it is 1 by a square. And now this is going to be x square b by a square root a square minus x square d x. So, just integration with respect to y. So, this gives us 8 m, it is b by a cube 0 to a x square a square minus x square d x.

Now, if you make the substitution, let us make the substitution x is equal to a cos theta. So, if we make the substitution x is equal to a cos theta. Then, we have d x is equal to minus a sine theta d theta. And the limits of integration will then become, this is a cos theta 0 means pi by 2, and this is going to be 1 means 0. So, therefore, our integral I 1 is going to look like this, 8 m b by a cube pi by 2 to 0. And this is a square cos square theta. And this is going to be a sine theta and this quantity is minus a sine theta d theta. That is nothing but 8 m b by a cube. And we can now write the limits 0 to pi by 2, so this minus will be canceling.

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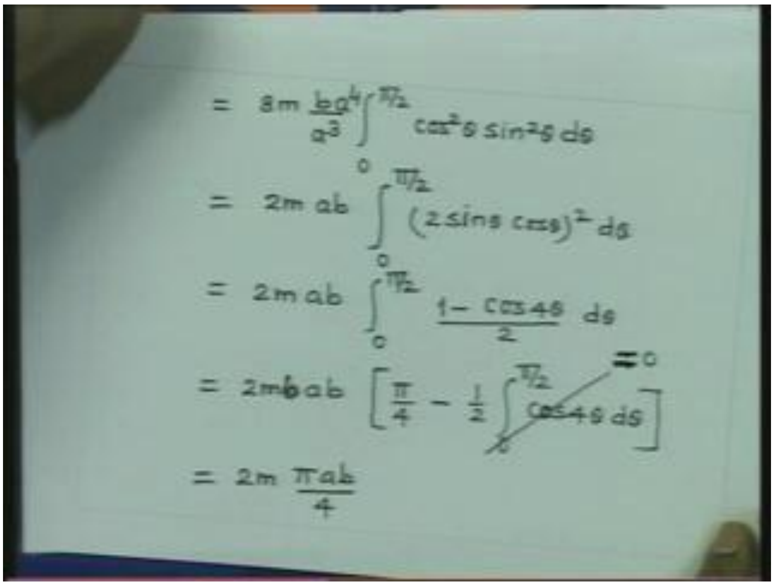

So, therefore, this will become 0 to pi by 2, we have now a 4. So, a 4 can be taken out. We have cos square theta sine square theta d theta. And this means, it is you can write that 2 m into a b 0 to pi by 2 will have 4 cos square theta sine square theta; which can be written as 2 sine theta cos theta whole square d theta. And that is equal to 2 times m a b 0 to pi by 2 this is nothing but sine 2 theta square. So, square 2 theta, and that can be written as one minus cos 4 theta by 2.

So, we can write this thing as 1 minus cos 4 theta by 2 d theta. This gives us 2 times m a b. And this is going to give us, first term integration is going to be, pi by 2 into half is pi by 4 pi by 4 minus, it is half 0 to pi by 2 cosine 4 theta d theta. You know that this integral is going to be 0, so therefore this is 0. And we can now write this thing as 2 times m pi a b by 4.

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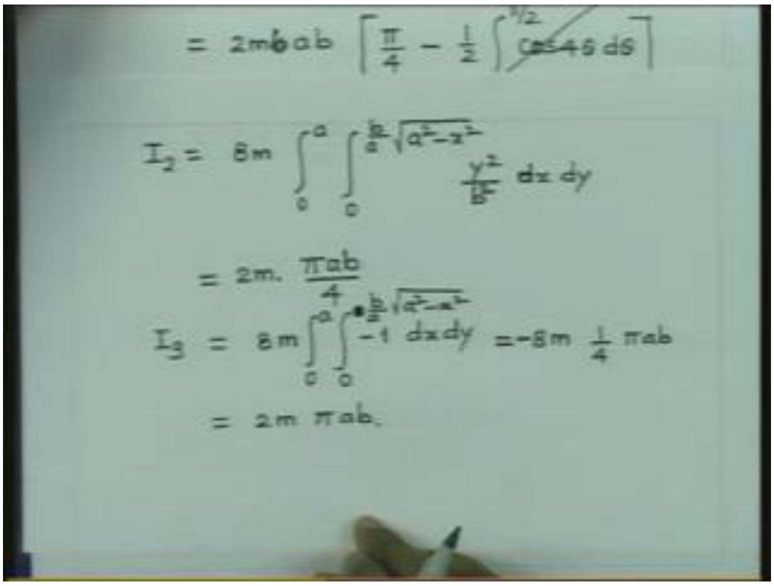

Similarly, the integration of the second term. If we consider the second term I 2, it is going to be 8 m 0 to a 0 to b by a a square minus x square. And we will have y square by b square d x d y. So, here also we can proceed in a similar manner. And this is going to give us 2 times m again pi a b by 4. So, the detail you can work out. The third term I 3 is nothing but it is 8 m integration of minus 1 d x d y over the region 0 to a. And can probably write here 0 to a and this is 0 to b by a a square minus x square. In fact, this integration is nothing but the quarter area of the ellipse. So, therefore, you can write now, this is 8 m, quarter area of ellipse is nothing but pi a b. So, this gives us 2 m pi a b and this is of course, negative.

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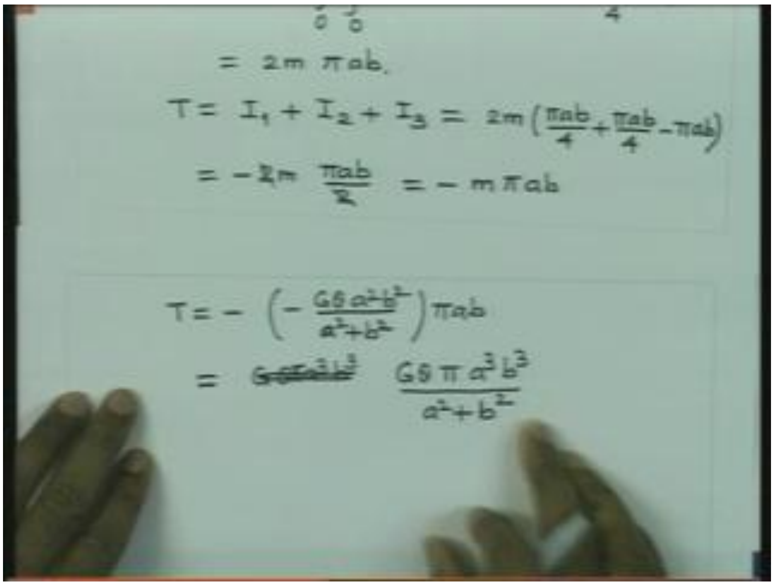

So, finally, if we look into all this, if we look into all this T is equal to I 1 plus I 2 plus I 3. So, we will try to get the value which is nothing but twice m pi a b by 4 pi a b by 4 minus pi a b. So, this gives us torque is equal to minus 2 m pi a b by 2, which is nothing but minus m pi a b. Already we have obtained the value of m earlier. So, if we substitute the value of m, m was G theta a square b square by a square plus b square with a negative sign.

So, therefore, T is equal to minus G theta a square b square by a square plus b square, that is m pi a b. So, this finally, gives us it is G theta a cube b cube. We have of course, pi here, let me rewrite G theta pi a cube by a square plus b square. So, this is the relationship between the angle of twist per unit length. And the cross sectional dimension, with the applied torque. You can see the units in perfect match, you have G given in Newton per meter square. And dimensions will be in meter. So, therefore, we are going to get Newton per meter square into meter square. And therefore, it will give us Newton meter, so torque will be Newton meter.

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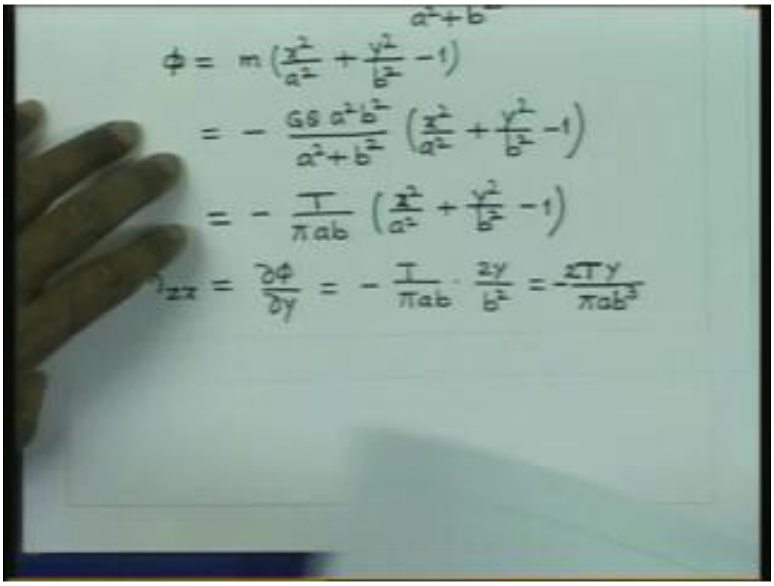

Now, if I consider the phi function, phi is going to be m x square by a square y square by b square minus 1. Let us substitute the value of m of G theta a square b square divided by a square plus b square into x square by a square plus y square by b square minus 1. Now, if we write this G theta, in terms of T. Then, I can now write here that, this is nothing but minus T by pi a b x square plus y square b square minus 1. Let us calculate the stresses now, in the cross section. Since, you have got the phi, it is possible to calculate now the stresses, over the whole cross section.

We can now write tau z x, which is given by delta phi delta y this is nothing but T by pi a b 2 y by b square. So, this is T, 2 times T into y by pi a b cube. Again you can check that the stress, this will become Newton meter square. And this is going to be meter 4, therefore we are going to get Newton per meter square, the stress unit. Look at this relationship the stress tau z x, is going to vary linearly with y. So, we can get the value of the maximum shear stress tau z x occurring, when value of y is maximum. So, for the cross section that we are trying to talk about, we certainly have the value of y maximum, at the extremity of the minor axis, that is y equal to v.

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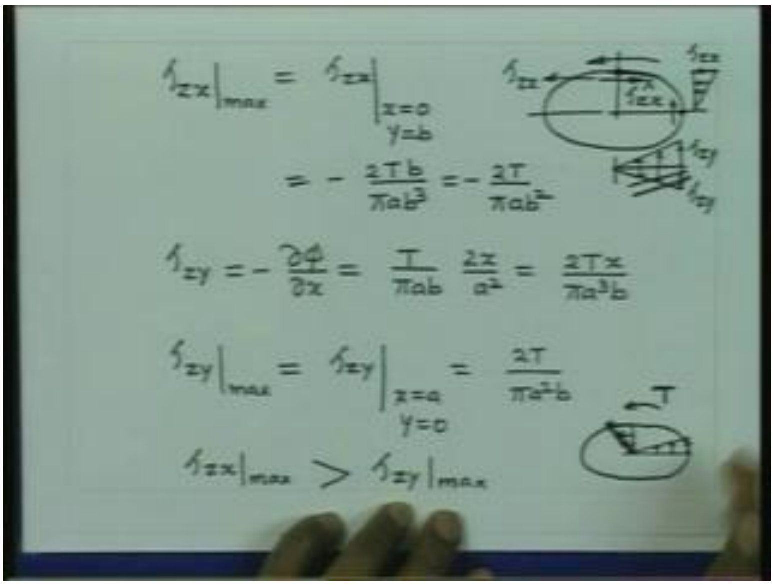

So, the maximum stress tau z x, it is going to come up here, it is independent of x. So, therefore, for the cross section you look into the cross section. So, it depends only on the distance of the point from the x axis. So, therefore, the maximum distance is here. So, it is going to be varying linearly with y. And it will have maximum value at this point. And this is now, that x equal to 0 y equal to b. And the value is given by minus 2 times T into b by pi a b cube; and that is 2 times T by pi a b square and this stress is negative.

Now, remember that, we have been considering the torque to be applied in the anticlockwise direction. So, therefore, the torque is acting like this. The positive stress will be acting in this direction. So, that is your positive stress, but this relationship shows that the maximum stress is going to be negative. So, therefore, actual direction of the stress, is going to be in the opposite direction. So, therefore, this is really the direction of the stress tau z x.

So, therefore, this is not the direction. So, this direction is not really correct. And it is in conformity with the torque direction. That it should be in the direction towards negative x. That is the point you must know. What we have got here, that the shear stress is varying linearly along the y axis. So, therefore, it is varying linearly along the y axis like this. For a particular x the variation is linear, and it is maximum at that point. Now, let us see what happens to the other component of stress tau z y. Tau z y is nothing but minus delta phi delta x.

So, if we now write it is going to be T by pi a b. And this differentiation is going to be 2 x by a square. So, we will have 2 T x by pi a cube into b. So, it says that tau z y stress is proportional to x, it does not depend on y location at all. So, therefore, we will have the maximum value of x coming up at the end of the major axis. So, therefore, the stress is going to vary, if you consider the x axis, which is going to vary from origin to this point linearly. And it will have maximum value.

And you see that, the direction of this stress is going to be positive, means this direction. And it is really show and it is again in confirmative with the applied torque direction. So, the variation of stress, let us draw it here. So, this is your tau z y and this is tau z x varies. Now, tau z y maximum, is occurring at x is equal to a y equal to 0; and if we write that, it comes out to be twice T by pi a square b. So, we have seen the distribution of the stresses, over the cross section.

In this case, the stress is varying linearly over the minor axis. So, also over the major axis. If you would like to show the stresses in the same sense, as the torque, then this will be the direction. So, you can show them to be tau z y, rather than this. Now, the magnitude of the maximum stress here is given by 2 T by pi a b square. And the magnitude of the maximum stress here is given by 2 T by pi a square b. Look at this, the value of the stress at this point is higher.

So, therefore, the section would have the highest stress, at the end of the minor axis. So, wherever the distance is lower, there you get the maximum stress. So, therefore, the tau z x, z x maximum is higher than tau z y maximum. So, in this cross section, you can certainly show the stresses to be like this. And you have shown in the case of circular cross section, it is going to be like this here. And it is going to be varying like this. So, the stresses are, this is how the stress pattern will be given, provided the torque is acting like this. Now, let us consider another cross section, which is triangular, which is equilateral triangle.

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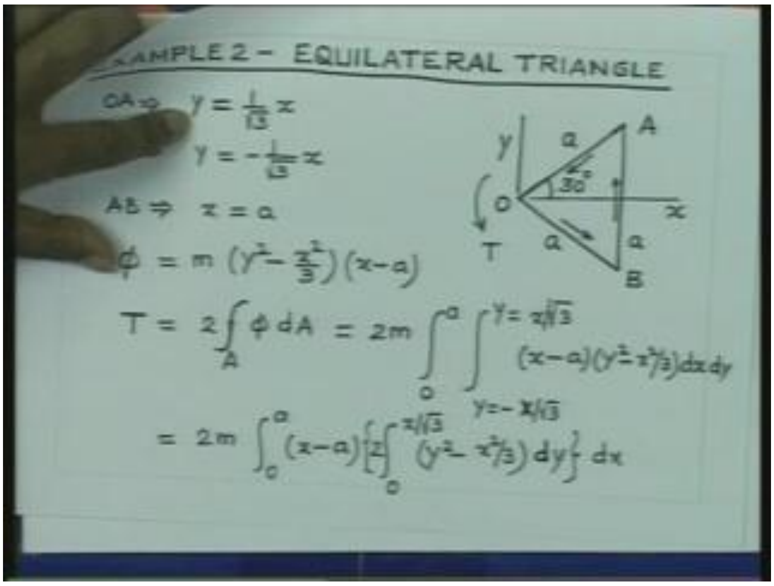

And the dimension is given by a here, the torque is acting in the anticlockwise direction like this. Let us find out, where is the maximum stress. In this case, we expect as usual the stresses to be acting parallel to the boundary. So, we can show the direction of the stresses to be there, like that. The equation of the line O A is nothing but y tan 30 x 1 by root 3 x equation of O B is y equal to minus 1 by root 3 x and A B x is equal to a.

Making use of the equations of the boundary, we can write a phi function like this, Y square minus x square by 3 into x minus a. This is the equation. Now, if we relate the torque to the phi function, we have phi d A over the whole area. And therefore, this is 2 m integration of the quantity 0 to a. And we can have now y root 3 y equal to x by root 3 x minus a y square x square by 3 d x d y.

So, if you integrate with respect to y first, x minus a 0 to... We can now, this is symmetric, so therefore you can really write this thing at 0 to x by root 3. And integration of y square x square by 3 d y d x. 2 m 0 to a 2 x minus a, this is going to give us y square, this is going to be x cube by 3 root 3 minus x cube by 3 into root 3 into d x. So, this gives us 4 0 to a x minus a into minus 2 x cube by 9 into root 3. Yes, I think we have some error introduced here ((Refer Time: 41:48)). The equation of the line a, b, it should be root 3 by 2 a. So, therefore, this quantity will come here root 3 by 2, at all root 3 by 2, here also root 3 by 2, root 3 by 2.

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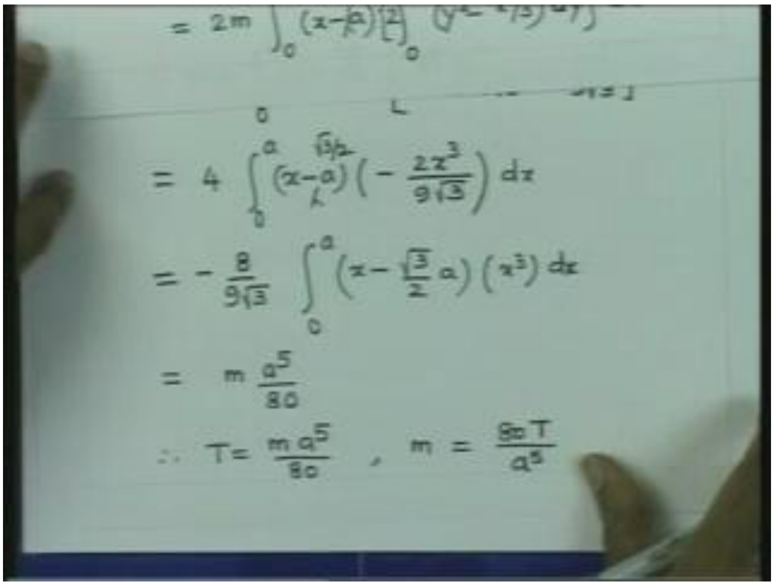

So, this gives us minus 8 by 9 into root 3 0 to a x minus root 3 by 2 a into x cube d x of course, of integration is there. Once you do all these, now the integration is straight forward. So, I would like to carry on, bring it for you. So, this will give us finally, it is m a to the power of 5 divided by 80. So, further integration will give you this quantity. Therefore, T is equal to m a 5 80 and hence, m is equal to 80 T by a raised to the power 5. Let us now get back to the expression for the phi function.

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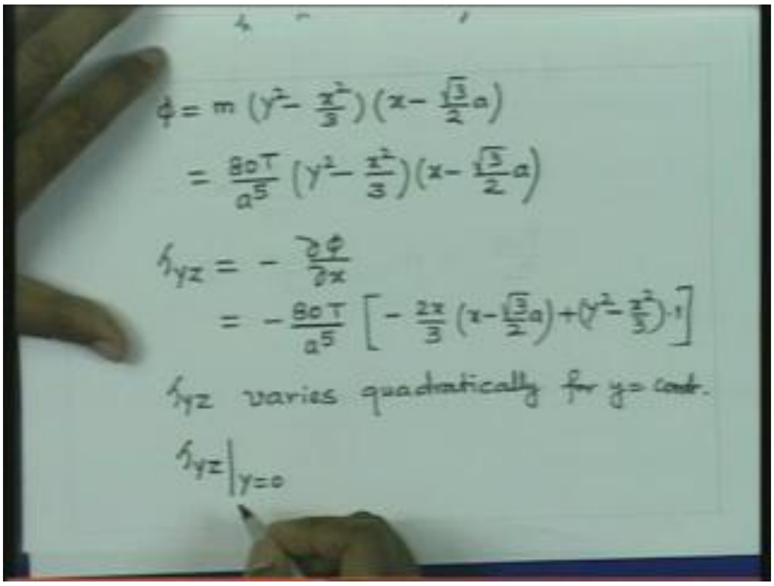

Phi was given as m into y square minus x square by 3 x minus root 3 by 2 into a. Now, if you substitute the value here, 80 T by a 5 x minus root 3 by 2 a. Let us calculate the stress, tau y z which is given by delta phi minus delta phi delta x. So, this will give us minus 80 T by a 5 minus 2 x by 3 x minus root 3 by 2 a plus y square plus x square by 3 into 1. So, that is the variation; and you can see that the stress varies quadratically, quadratically for a particular value of y.

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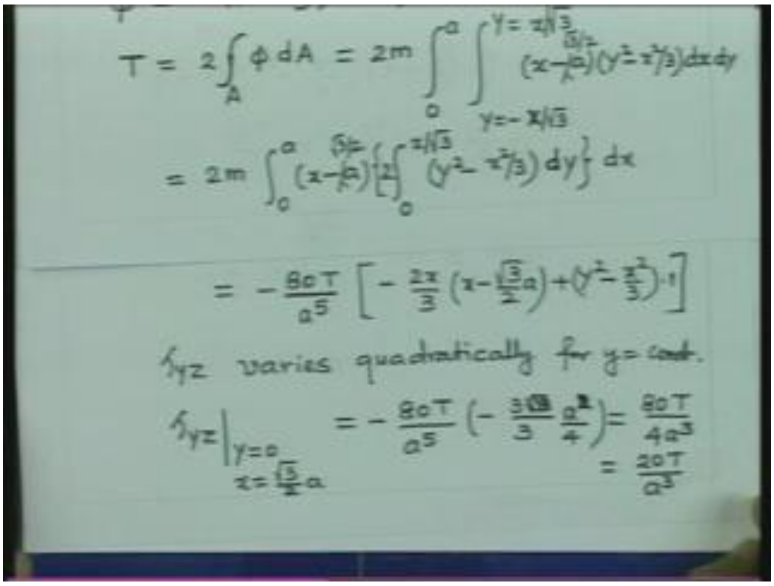

So, tau y z varies quadratically for y constant. Typically if you like to consider tau y z for y equal to 0. And x is equal to root 3 by 2. Let us get back to our ((Refer Time 46:55)) extreme point. So, extreme point is x is equal to root 3 by 2 a. So, if you substitute that, then it comes out to be minus 80 T by a 5. Here, this is going to cancel out and we are going to have minus, it is 3 into root 3 by 3; it is going to be 3 by 3 4 a square.

So, this gives us 80 T by 4 a cube which is nothing but 20 T by a cube. So, the stress which is going to be occurring at the end point of the x axis. So, if you consider ((Refer Time: 48:25)) this point, at this point the stress is going to be 20 T by a cube. So, we have got the maximum stress. If you would like to consider, again a point like this. You will find that, these are all going to have same stresses. The stress position here and here, they are identical. And you will find that the stress is going to have magnitude of this value 20 T by a cube.

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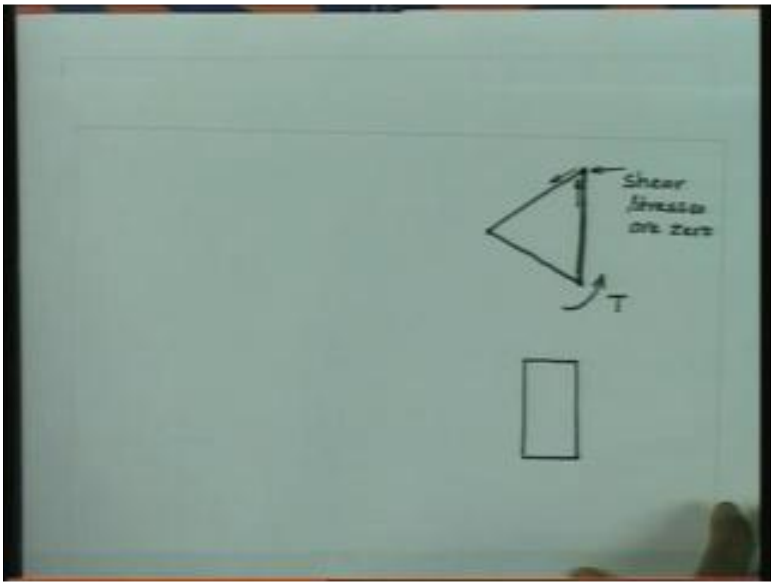

Now, let us try to consider, what happens to the corner. So, we have the section like this. If we consider that the stress here, at this boundary is directed like this. For the applied torque this way, then the stress along the boundary is going to be directed like this. Now, at this point it is a common point. So, the stress at this point is going to have duality, in terms of direction. So, therefore, there cannot be duality of stresses at the corner point. And therefore, the stress at this corner is going to be 0. So, this cannot be a point with stress. So, shear stresses are 0 at this corner.

Same logic will apply to this corner and this corner. So, if you have any corner, occurring in the cross section, we will find that the stress is 0. We will consider cross section, which is little easier to understand. We will consider now a section like this, rectangular cross section.

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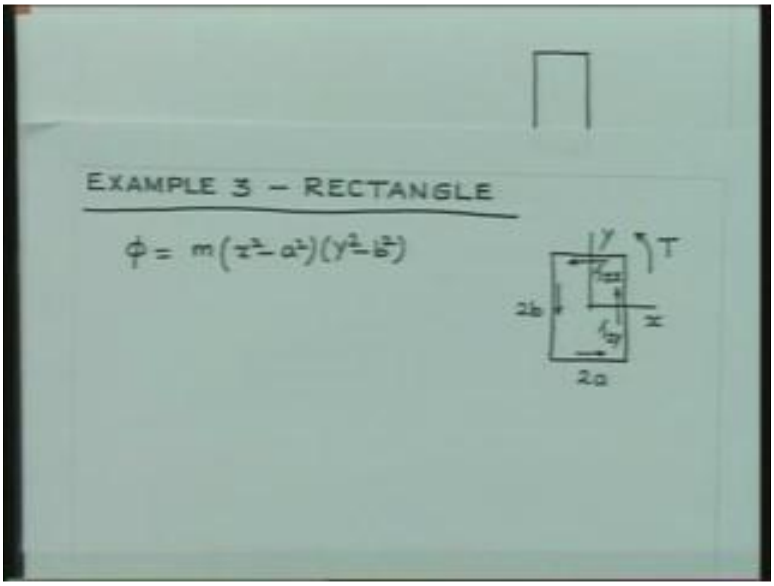

So, I would like you to think about it. And we will continue with this example, in our next presentation. Let us select the coordinates like this. This height of the section is 2 b and the width is 2 a. What should be the form of the Prandtl stress function here? Please try to guess, you have the two boundary which are vertical, which are located at x equal to minus a and plus a. And you will have two horizontal boundary, which are located at y equal to minus b and plus b.

So, therefore, in this case without any difficulty. We can make an approximation about the phi function in this form, m into x square minus a square y square minus b square. So, this is the ((Refer Time: 52:24)). Now, in this case, what do you expect to be the stresses, just guess that you have to have a phi surface, and looking into the analysis of elliptical section. Try to guess, what is going to be the stress variation.

And at the same time, where do you expect the maximum stress to occur. The stress directions are going to be parallel to the boundary like this. This is the direction of torque and the stress here is nothing but tau z y. And the stress at the end of the y axis is tau z x. So, we would like to find out these stresses, in the next class. It will be good idea for you to, just find out the stresses starting with the Prandtl's stress function.