Advanced Strength of Materials
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Lecture – 17

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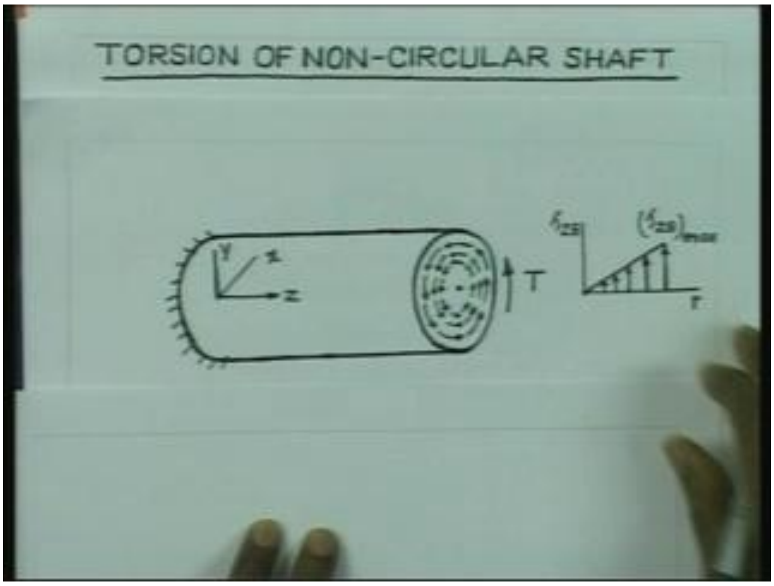

Today, we are going to study torsion of non circular shaft. You have already studied torsion of circular shaft, where you have seen when a circular shaft is subjected to twist. The shaft of uniform cross section will undergo deformation. And a typical cross section of the shaft will give rise to shear stresses, which are going to be distributed like this. The shear stress has a variation with radius, which varies from 0 to maximum value at the outer radius.

And this shear stress is going to act parallel to the boundary. Or in other words, at every point the shear stress is acting in the tangential direction. Now, the question is, if the shaft cross section is not circular. It could be elliptical, it could be rectangular, it could be in the shape of a often I in the shape of T. So, if such cross sections are subjected to twist, then how the stresses are going to be distributed over the whole cross section. Then, that will be our object of study. And we would like to start with, let us consider that we have shaft of uniform cross section all over length.

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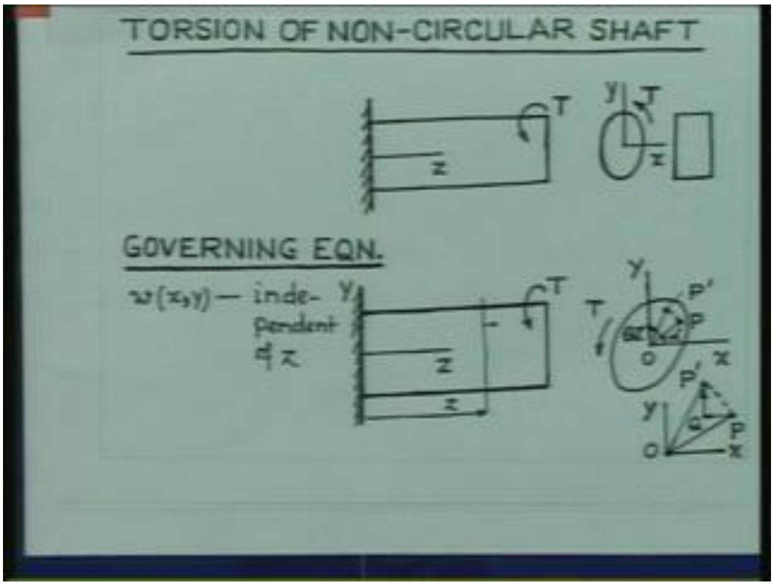

The section as I said it could be elliptical or rectangular. Now, another action of this torsion, the deformation of the shaft will take place. And this deformation will give rise to stresses, at each and every cross section. Now, if you look into a typical cross section of the shaft. Let us consider that, this is the typical cross section and ((Refer Time: 02:33)) apply angle of twist. Angle of ((Refer Time: 02:37)).

If we apply the torsion a typical point P on the cross section is going to move to a new position P dash. And this displacement is going to give rise some motion in the x direction. Some motion in the y direction. And now the difference that arise, between a circular cross section and a torsion of a non circular cross section is this, that under the action of angle of twist any typical cross section, will not remain plane. In the case of a circular section, this section will remain plane.

But, in the case of non circular section, this will not remain plane. And if we assume, that the shaft cross section is uniform all over. Then in that case, we can assume that the deformation of any cross section will remain the same along the length of the shaft. So, we will consider that the section deformation, which is typically known as warping of the cross section, will indicate that this warping or the displacement of the, of a point on the section in the z direction, will simply be a function of it is position.

It is not going to depend on where it is located, along the length of the shaft. So, we would like to consider, that the displacement in the z direction of any point on the section, which is w. This w is a function of x and y only. It does not depend on. So, it is independent of z of the location of the section. Now, if we get back to the typical point P, which was located at an angular orientation of alpha with the x axis.

Now, under the action of torsion T, this radial position O P is going to rotate by some angle. And if you consider that the angle of twist per unit length is equal to theta. And this distance from the origin is equal to z. Then the total angle of twist will be theta into j. And therefore, this rotation of the P O P dash is theta into z. Now, we can calculate from this the displacements. So, if you now try to calculate the displacements of the point P, you can now see that this angle. This angle is alpha and this angle is theta into z. So, therefore, this angle is theta into z. And as the point P is shifted to P dash, there is a displacement in the x direction of magnitude P Q. And there is a displacement of magnitude Q P dash in the y direction.

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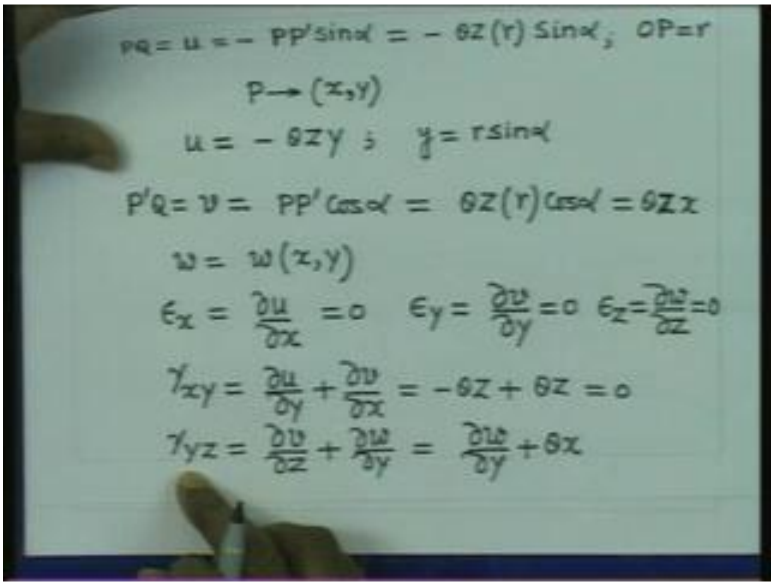

So, therefore now, we can write that the displacement P Q, which is nothing but u. And this u is minus its nothing but P P dash. If this angle is alpha, we can take approximately, this angle to be also alpha. So, therefore, it is nothing but P P dash sine alpha. And what is P P dash. P P dash is approximately equal to theta z into O P. So, this is nothing but theta z. And if we represent O P by r, so this is r. Then, in that case, we have theta z r into sine alpha wherein of course, we have taken O P is equal to r.

Now, ((Refer Time: 07:59)) if we write the coordinate of point P. So, P to have coordinates x and y. Then, we can write r sine alpha is nothing but y. So, we can have now, u is equal to minus theta z into y. y is equal to r sine alpha. Similarly, by similar configuration you can find out the distance Q P dash. So, let us write now, P dash Q and that is the v displacement. So, that v displacement is P P dash into cosine alpha. And again we can write, which is nothing but theta z into r cosine alpha.

And since r cosine alpha is equal to x. So, this becomes theta z, theta z into x just the displacement P dash Q. And displacement of the point P in the z direction, which is w. So, w is nothing but we have w as function of x and y. So, therefore we have now, the displacement u v and w of a typical point. Now, since we know the displacement it is possible to calculate, the strains. And then by using the hook's law we can calculate the stresses.

So, let us now, look into the calculations of strains first. So, if we now consider the strains Epsilon x. Epsilon x is delta u delta x. And since, u is independent of x, so that becomes 0. Similarly, Epsilon y, which is nothing but delta v, delta y. v is independent of y. So, therefore, again delta v delta y is equal to 0. And epsilon z the strain in the z direction, which is delta w delta z. This is a function, which is independent of z and therefore, this is equal to 0.

So, what we find is this that, the deformation is taking place in such a manner. That the normal strains, three normal strains are 0. Now, let us consider the shear strains, gamma x y is nothing but delta u delta y plus delta v delta x. Now, delta u delta y, see the expression here, it is minus theta into z. And delta v delta x is theta z. So, therefore native 0. So, therefore there is no shear strain. Again, if you calculate gamma y z, gamma y z is nothing but delta v delta z plus delta w delta y.

So, what we have delta v delta z is nothing but theta x. So, therefore, you can write now, this is delta w delta y plus theta x, that is the strain shear strain. And it is non 0. So, let us now calculate the other shear strain gamma x z.

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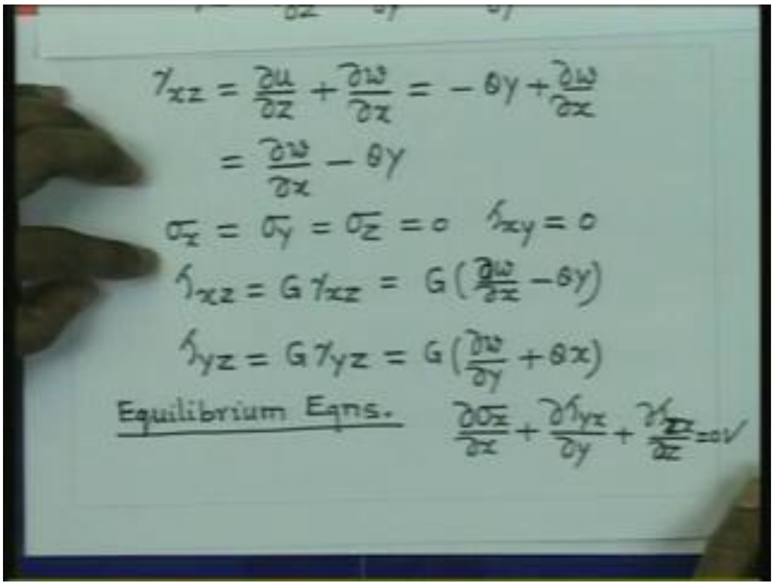

So, let us calculate gamma x z, that is going to be delta u delta z plus delta w delta x. Now, u is nothing but minus theta z y. So, therefore, delta u delta z will become minus theta y. So, this is minus theta y plus delta w delta x. So, if you write this, it is delta w delta x minus theta into y. So, please note that, in this particular problem out of the 6 components of strain. We have Epsilon x 0, Epsilon y 0, Epsilon z also 0. And at the same time gamma x y is 0. And the two nonzero components of strains are gamma y z and gamma x z.

Note that, the cross section is lying in the plane x y. So, therefore, the cross sectional plane is x y. And z is perpendicular to the section. Since, you have got the strains, now you can calculate the stresses. Stresses would be we will have all the stresses going to be 0. So, we have sigma x 0. So, also sigma y 0. So, also sigma z all these are 0. And since, gamma x y is 0 here. Therefore, we will have this tau, which is nothing but rigidity modulus multiplied by gamma x y, that is also equal to 0.

And the stresses taw x z is equal to G into gamma x z. So, therefore, it should be G into delta w delta x minus theta y, that is the stress x z. And then tau y z is equal to G into gamma y z. And again when you substitute the value, so tau y z is equal to G gamma y z. And when you substitute the value, it gives us G delta w delta y plus theta x. So, these are the only two nonzero components of stresses, in a problem of torsion of non circular sections.

Our objective is to determine this two nonzero components of stresses and their distribution. How do you go about it? If you consider that the stresses at a point in a body must satisfy the equilibrium equations. So, therefore, we must go back to the equilibrium equations. The equations are delta sigma x delta x delta tau y x delta y delta tau z x by delta z equal to 0. Now, note that sigma x is 0, tau y x is 0 and tau z x, which is tau z x it is not dependent on z.

So, therefore, ((Refer Time: 17:18)) this equation is identically satisfied. Be whatever the value of stresses it is going to satisfy this equation. So, the first equation is going to be satisfied.

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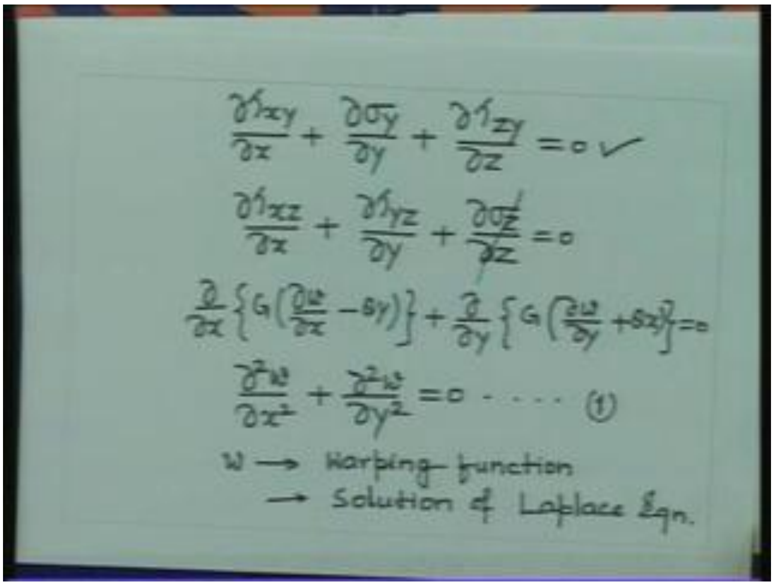

Now, let us look into the second equation. Second equation is tau x y tau x delta sigma y delta y delta tau z y delta z equal to 0. Of course, we have assumed that there is no body force. And therefore, the body force is absent here. Now, tau x y is 0 therefore, this is satisfied this particular derivative does not exist. Sigma y does not exist therefore, this is 0. And tau z y, if you see the expression for tau z y, we had tau y z independent of z. So, therefore, the derivative with respect to z is 0. So, therefore, this equation is again identically satisfied.

Third equation is delta tau x z delta x plus delta tau y z delta y delta sigma y z delta z is equal to 0. So, this equation sigma z is absent. So, therefore, we do not have this term and the derivative of the stresses tau x z with respect to x, plus derivative of the stress with respect to y. Some of these two is equal to 0. So, therefore, now let us substitute the value of stresses tau x z and tau y z. And then see, what this equation gives us.

So, we will substitute the value. So, we will have delta, delta x G delta w delta x minus theta y delta, delta y G delta w delta y plus theta x is equal to 0. If we simplify, this gives us delta 2 w, delta x square plus delta 2 w, delta y square is equal to 0. You remember that, this w is nothing but the warping function. Deformation distortion of the section, so this is known as warping function. And the whole of torsion is dependent on the warping functions stress tau x z depends on w.

Tau y z depends on w. And of course, the angle of twist will depend on the torque. So, the stresses are dependent on the warping function. And that, warping function, if you look into the equation it is a solution of Laplace equation. So, therefore, this warping function satisfies the Laplace equation. It is a solution of Laplace equation. Now, let us look into some other typicality's of the torsion problem and the stress distribution.

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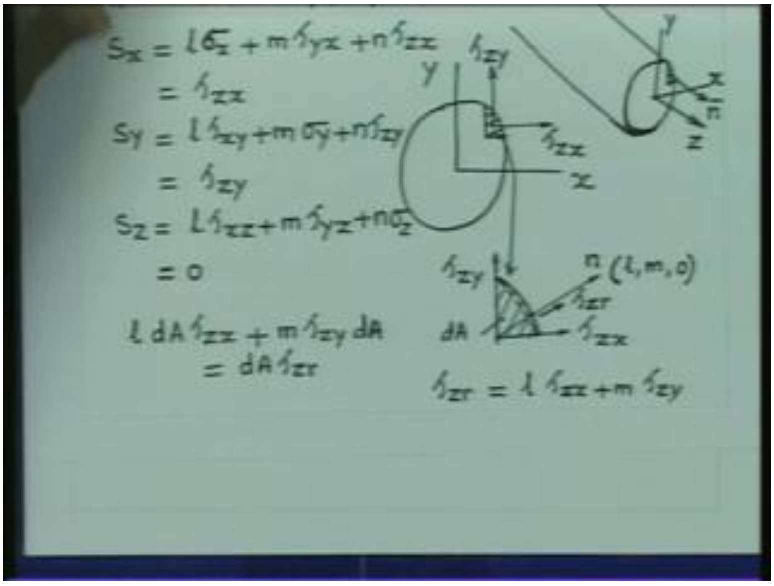


Let us consider, thus one of the typical section. Let us, say we have the typical section. We would like to concentrate on a zone here. And z axis is oriented along the axis of the shaft. This is our x axis, this is y axis. On this section, we are going to have stresses only nonzero stresses on the section. Or if you try to larger scale, let me draw it to a larger scale. So, this is my, this is x, this is y and what I am trying to concentrate upon is a small portion here. Like this and on this section which is perpendicular to positive z axis.

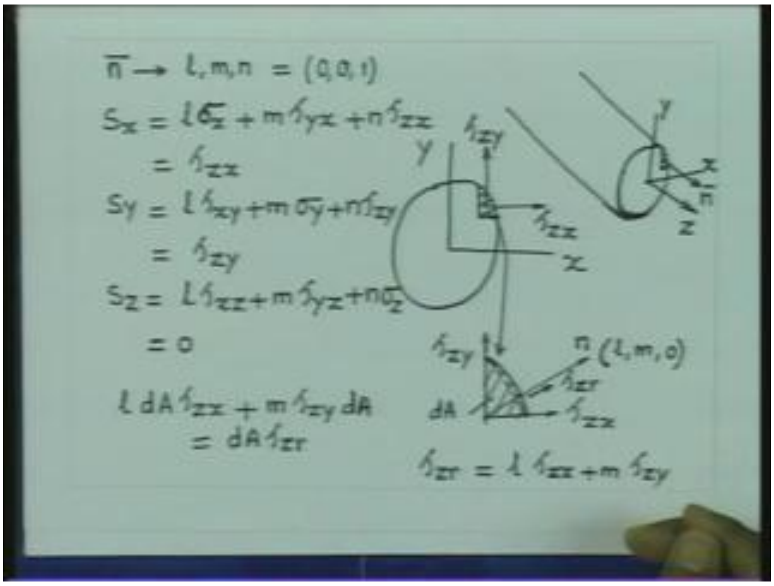
We are going to get the stresses to be tau z x, which is same as tau x z. Similarly, the stress in the y direction is going to be tau z y. And this section or this element it is outer normal is oriented in this direction. So, this is the outer normal on the section. So, therefore, this n bar outer normal to this element would have direction cosines. Let us say, l m and n. Since this outer normal is perpendicular to both x and y. Therefore, we are going to have this direction cosine to be nothing but 0, 0 and 1.

Now, the attraction that is going to come up on this area, you can write now S x. S x is nothing but l into sigma x plus m into tau y x plus n into tau z x. Now since, stress sigma x is 0, tau y x is 0 and this n is equal to 1. Therefore, this s x is nothing but given by the stress tau z x. Similarly, s y is nothing but l tau x y m into sigma y plus n into tau z y. Here again, since l is 0, m is 0 and n is 1. So, therefore, this attraction on this area is nothing but is tau z y.

Coming to the torque component of attractions S z. That is tau x z plus m tau y z plus n sigma z. So, here l is 0, m is 0 n is 1, but sigma z is 0. So, therefore, the net force in the z direction on this plane is 0. Now, let us look into this relationship little more carefully. Now, if we just again get back to our area. That area we have concentrated upon it is nothing but this. If I now consider, the stress here is tau z x. And stress here is tau z y let us say, that the outer normal to this.

This is the outer normal you can approximate it by straight line. This boundary you can approximate, it by a straight line. So, therefore, this is the outer normal to the boundary. And let us say that, this outer normal this one has the direction cosine l m and n each 0. Now, the stress, if I consider this area to be d A. This whole area to be d A. Then, we can write l d A into tau z x. So, the component of this force, due to z x in this direction that is going to be the component in this direction.

Similarly, if we try now to consider the component due to tau z y. And it is the force due to tau z y. And it is component along this direction and sum it up. So, it is going to be m into tau z y into d A. That is the component, which is now that is equal to and then of course, we will have these two components adding up together. So, that is equal to, if we say that, the stress in this direction is given by tau z r. Then, that must be equal to d A into tau z r. So, finally, tau z r is equal to l tau z x plus m tau z y.



So, this component is going to be like this. I want to note this point very carefully that this tau z r is the stress acting in this direction. And it is really acting perpendicular to the boundary. So, this stress acting perpendicular to the boundary, this is given by this magnitude. And if you now consider, consider an area let us say that, we consider an area like this.

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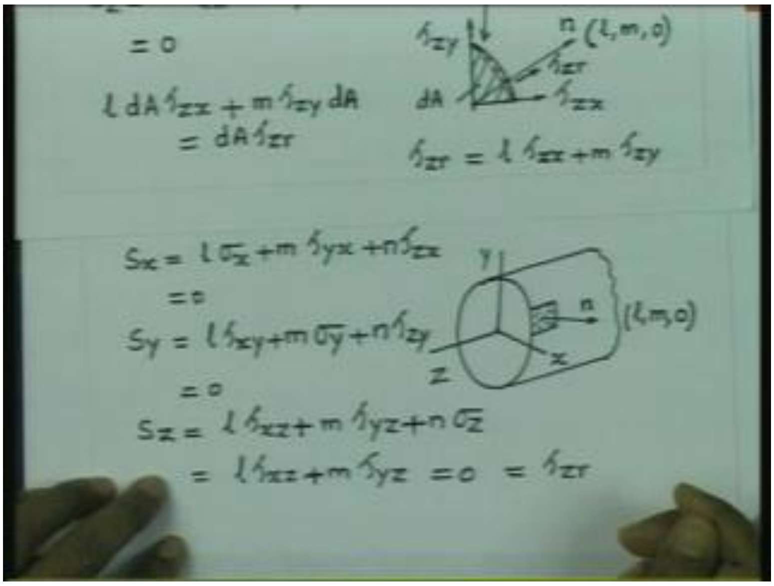

So, this is x axis, this is y axis and this is the one z axis. Now, let us consider an area here. And this area it is outer normal. Let us say, that it is equal to n. And this n could have direction cosine. Let us say l m and since this is in the plane of x and y. This outer normal is in a plane parallel to the plane x and y it is component. The direction cosine will be z axis should be 0. So, therefore, now you see, if we just calculate the attraction on this area.

Since this, area is free of any loading, we will find that we will have S x. This is p of any external loading and therefore, there is no attraction in the x direction or y direction or z direction. Now, you may look into s x component. So, therefore, it will give us, l into sigma x plus m into tau y x plus n into tau z x. So, therefore, this is now l sigma x is 0, this is 0 and this n is 0. So, therefore, this is 0. And we find sigma S y the attraction in the y direction is l tau x y plus m sigma y plus n tau z y.

Here again, this is 0, this is 0. So, therefore, this is 0 and S z is equal to l into tau x z plus m into tau y z plus n into sigma z. So here, this S z is again we find that, it is 0. And then this n since n is 0, sigma z is also 0. So, what we have is that, this l tau x z plus m tau y z is equal to 0. So, since the surface is free of any loading this s z is 0. And if you try to now combine these two relations that we have derived. Here in this relation and this relation. They are indicating that S z is 0.

And this relation is indicating that, tau z are 0. So, therefore, what it means is that, this is equivalent to saying that tau z r is equal to 0. So, therefore, this stress which is going to act perpendicular to the boundary is going to be 0. So, i would like to show you a sketch to illustrate this point.

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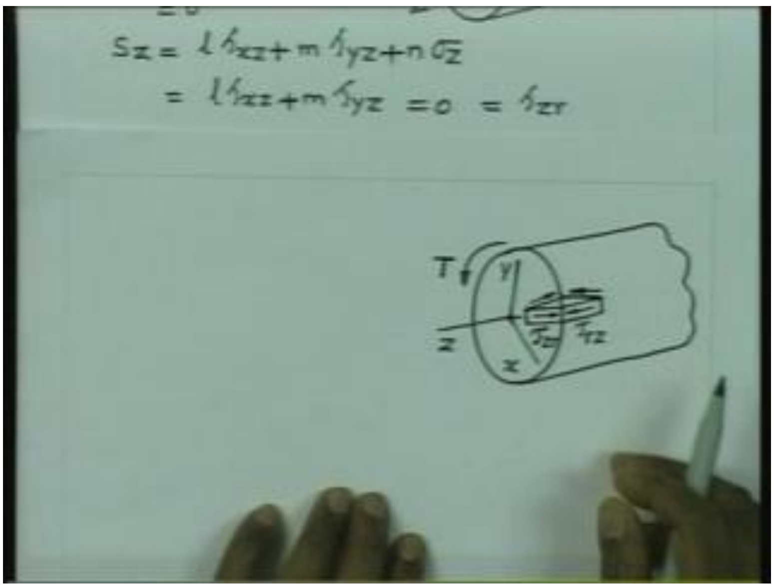

So, I would like you to look into this sketch we are considering this element. It is oriented in the r z plane. So, this is your r direction. And this is your z direction. The z r stress is going to act on this plane. Similarly, the pair of this is going to be directed towards the center. And the complementary shear stress, if you are going to come up on this stress it will be oriented like this. And on the other phase it is going to be oriented like this.

So, now this r z space is going to act on the outer surface. And this cannot happen, this is not there. In fact, this is 0. So, this is 0, this is 0. Similarly, the complementary shear stress tau z that is 0. So, this is 0. So, therefore, since this carved surface of the shaft is free of any external loading, you have tau r z 0. Therefore, this tau z r is also 0. So, that is the meaning here that earlier we derived, that s z is 0 means, tau z r is 0.

Let us, look into, how to get the solution for the warping function. This warping function was introduced by saint venant. And later on, a function was introduced by Prandtl, which is convenient for obtaining solutions to different geometries. So, we would like to consider, this channel stress function approach.

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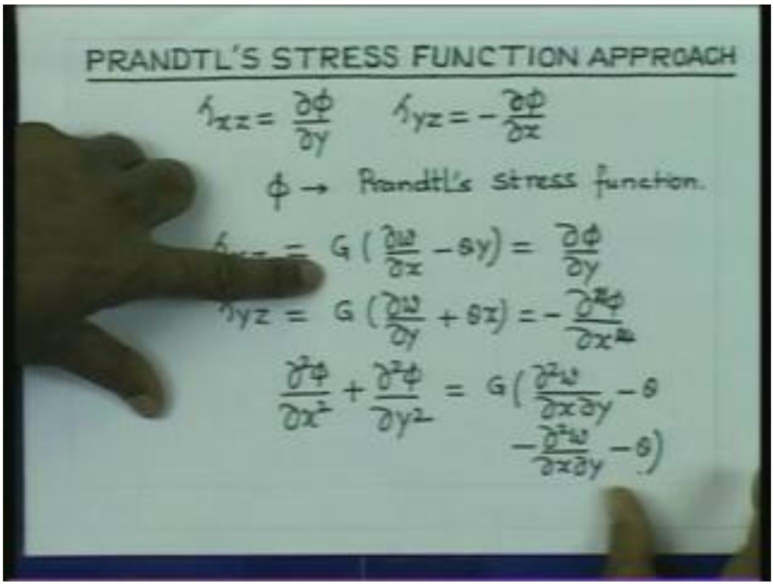

Prandtl introduced one stress function. And he defined the stresses, we had two nonzero component of stresses tau x z. And he defined that, this stress is given by delta phi delta y, where phi is the Prandtl stress function. Similarly, tau y z is given by delta phi delta x. So, phi is Prandtl stress function. Now, we can get some relationship between this

Prandtl stress function. And also the saint venant's work ((Refer Slide Time: 38:43)) function.

And it is also possible to also comment on the properties of the Prandtl stress function, looking into the physics of the problem. We have already seen that, tau x z was given by G delta w delta x minus theta y. And that is equal to delta phi delta y. And tau y z, G delta w delta y plus theta x is equal to minus delta phi, delta x. Now, if we differentiate the first relation with respect to y. And second relation with respect to x.

So, if you differentiate this with respect to y. And this with respect to x. And subtract the second equation from the first. Then we have, delta 2 phi, delta x square plus delta 2 phi delta y square is equal to G delta 2 phi delta x delta y minus theta. Then delta 2 w, delta x delta y minus theta.

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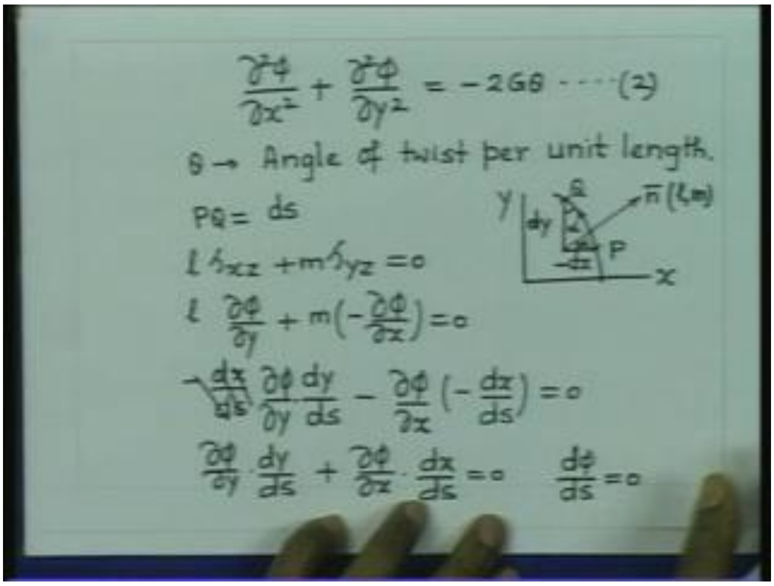

So, delta 2 phi delta x square plus delta 2 phi delta y square is equal to minus 2 G theta. This is an important equation of torsion. So therefore, the Prandtl's stress function is such that, it satisfies the Poisson type equation. Note that, G is a material constant and theta. Theta is nothing but angle of twist per unit length. We will derive certain characteristics satisfied by demonstrated by this Prandtl's stress function. We have already seen that, if I consider the part of the boundary.

Suppose, you consider this is the outer normal the part of the boundary let say this is n bar. And this portion is lying in the x y plane. And it has direction cosine l m n. If I say P Q are the two points that we are trying to talk about. Let us say, P Q is equal to d s and as we move from P to Q. We move in the x direction, that is going to be minus d x. And we move in the y direction, which is going to be d y. And now, we consider that this angle is equal to alpha.

So, this angle is alpha therefore, this angle is alpha, therefore, this angle is also alpha. We have already noticed, that for a segment for a boundary like P Q. What we had is that, l tau x z, m tau y z equal to 0. The radial stress the shear stress in the radial direction is 0. That stress cannot act perpendicular to the boundary. And now, if i substitute this value delta to be substituted for tau x z, tau x z is nothing but delta phi delta y. And this is m, this is nothing but delta phi delta x equal to 0.

If you look into value of l, l is nothing but that is cos alpha cosine alpha is nothing but minus d x by d s, P Q is d s. So therefore, you can write now, instead of l we can write, minus d x by d s, l is cosine alpha, cosine alpha is nothing but d y by d s. So, therefore, this is let us, write again l is d y by d s. So, we have delta phi delta y into d y d s, where d s is l. And then we have minus delta phi delta x, m is nothing but sine alpha, sine alpha is minus d x by d s.

So, therefore, this is nothing but minus d x by d s. So, that is equal to 0. And now, if you can see this is nothing but delta phi delta y d y d s plus delta phi delta x d x d s equal to 0. And this is nothing but left hand side is nothing but derivative of phi with respect to s, that is equal to 0. What does it mean, that d phi d s is 0. Derivative of phi along the boundary G 0, thereby it means that phi is constant along the boundary.

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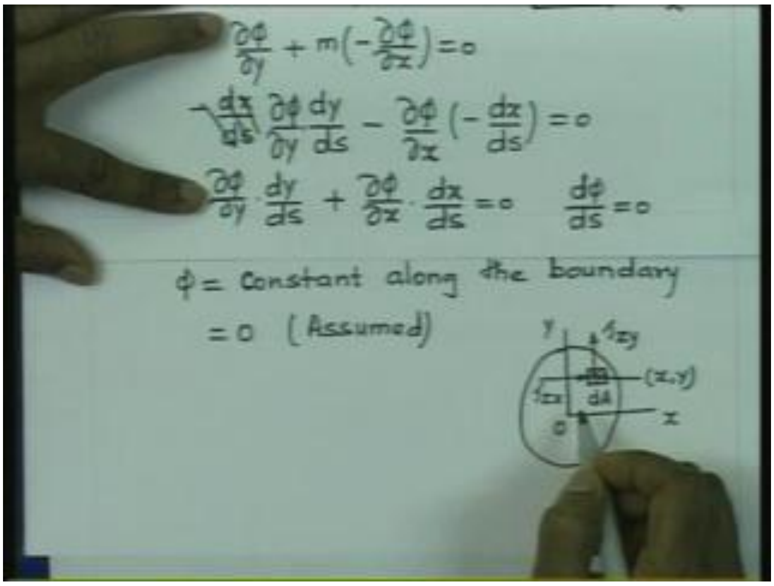

So, therefore, phi is the constant, phi is constant along the boundary. And it is customarily to take this boundary value of phi at 0. So, generally phi is assumed to be 0, along the boundary. We will also make to see, there is another relationship which is satisfied by this Prandtl's stress function phi. So, we have the typical section. If i concentrate on an element, which is locate at a distance x, y from the origin. Now, on this section, we are going to have stresses tau z y and we are also going to get stresses tau z x. And let us the area d is equal to d A.

Now, if you try to calculate the moment of this, moment of the forces arising out of the stresses about this point about the origin.

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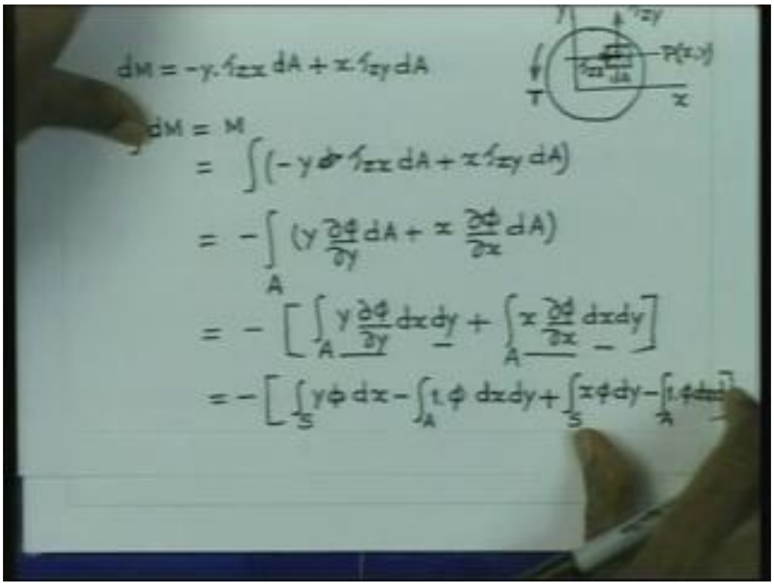

d M is equal to tau z x into d A multiplied by the distance y. So, this is going to be ((Refer Time: 49:47)) y tau z x into d A. That is the moment and this moment is going to be clockwise. So, we are going to write it negative. Then the moment due to tau z y into d A. That is going to be anticlockwise and the moment is x. So, therefore, we will have x tau z y into d A.

So, this is the elemental moment wherein it is going to be in the direction of the applied torque, which is considered to be positive in the anticlockwise direction. So, the total moment we can get by integrating this. And therefore, that is equal to M and that is equal to integration of minus y tau z x into d A plus x tau z y into d A. Now, if you substitute the value for tau z x, which is nothing but delta phi delta y. And tau y is delta phi minus phi delta x.

So, therefore, you would have this integration y delta phi delta y d A plus x delta phi delta x d A. And we could consider that this area is A. So, therefore, this integration is implied to be over the total area. We can write this thing as, integration of delta phi delta y over the area therefore, this is d x d y.

So, also we have integration of this quantity x delta phi delta x d x d y. It is a product of two function here y and delta phi delta y. So, we can integrate this function with respect to y ((Refer Time: 52:38)). Similarly, here it is the product of x and delta phi delta x, two functions are involved and we would like to integrate with respect to d x. So therefore, if we do it. We get, so integration of delta phi delta y with respect to y will give us, y phi and this integration is still to be done with respect to x.

This is now what is going to be the boundary here minus. Integration of this quantity is phi differentiation of this quantity is 1. And now, we have finished the integration with respect to y. So, therefore, we have to have this product again integrated with respect to y. So, therefore, we will have d x d y remaining here and this is over the area A. By the same consideration, we can write this thing as S x phi d y minus 1 phi d x d y. And this is also over the area A.

Since, the function phi is 0 over the boundary. Therefore, this product will always be 0. Similarly, this product is always going to be 0 over the boundary. So, therefore, they sum up the whole boundary will be 0. And finally, we will be left with integration of phi over the area A, integration of phi over the area A.

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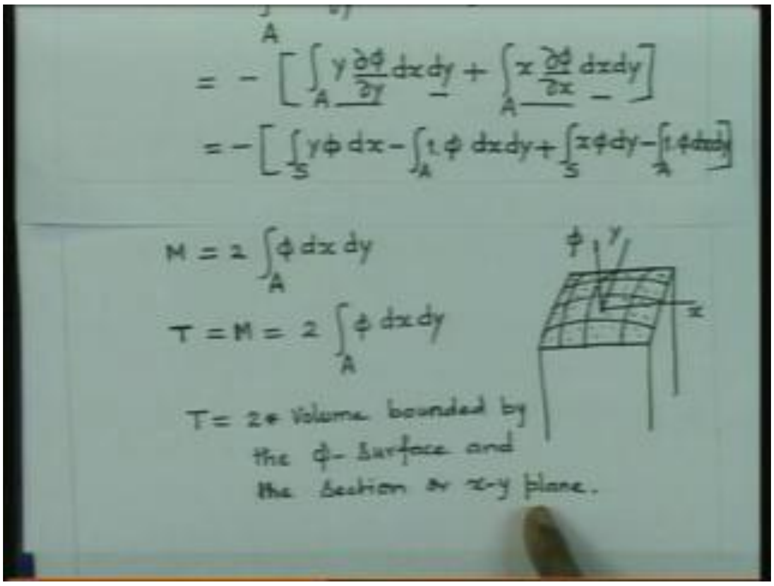

So, therefore, we will have now, m as 2 times integration of this function d x d y over the area. So, please note that, I have made use of M. T is nothing but finally T is equal to M, which is the external torque and therefore, this is 2 times A phi d x d y. Now, if we plot the phi function. Let us, assume that we have the cross section like this. Let us consider also, that this is our x direction, this is y direction. And if we plot phi in the vertical direction. We are going to get this plot this phi in this direction.

For values of phi over the whole domain, we are going to get a surface. And that surface will look like this. So, the surface is really going to be a surface shown here. And this integral can be interpreted as, if we consider this cross section to be the base or x y plane. And phi is in the vertical direction. Therefore this torque is nothing but equal to twice the volume bounded by the surface phi. And the base which is the x y plane. So, therefore, torque capacity is equal to 2 times the volume bounded by phi surface and section or x y plane.

So, this is a very important information summation that the torque capacity of this shaft is obtainable from the volume bounded by the high surface. And the section or x y plane, when of course, we are plotting this phi in the vertically upward direction.