**Advanced Strength of Materials**
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**Lecture - 15**

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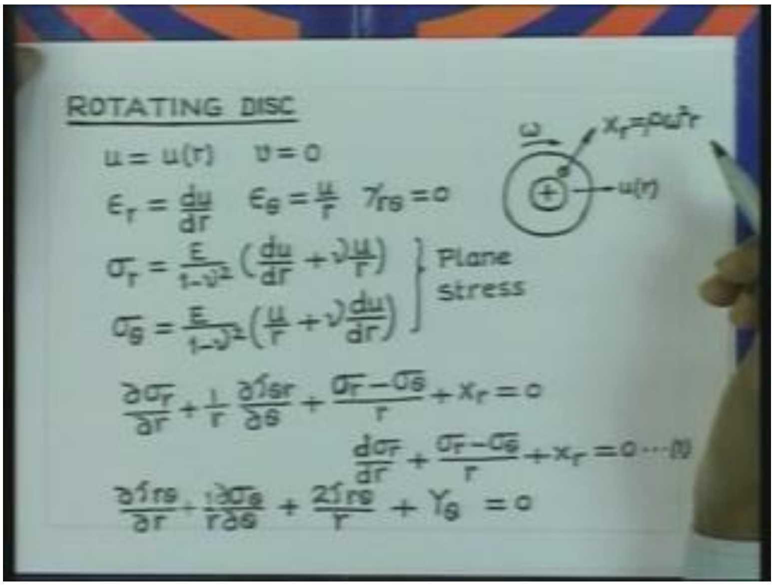


We will examine today, the distribution of stresses in a rotating disc. We have a picture like this, when it is rotating, each of the material elements is subjected to body forces, which is given by rho omega square r. It is acting in the radial direction. Where rho is the density, omega is the angular velocity, r is the radius. The loading is same at every point around the circle. And therefore, it is asymmetric problem. It is not difficult to imagine that, each of the point of the disc is going to just move radially outward.

It is a picture, which is similar to that of thick cylinder. So, therefore in this case u the radial displacement of a point is going to be function of r alone. And there is no tangential displacement. Therefore, v equal to 0. And, if we adopt the expressions, for strains that we had got in the case of thick cylinder, then we have epsilon r is equal to du d r epsilon theta is equal to u by r. And the radial, the shear strain gamma r theta equal to 0.

Again assuming a plane stress condition, to exist in the disc the radial stress is given by E by 1 minus nu square, into the radial strain d u d r plus tangent nu times tangential strain nu u by r. Along similar lines, epsilon theta is equal to E by 1 minus nu square into u by r plus nu times d u d r. This is nothing but, epsilon theta and d u d r is epsilon r. These stresses are going to satisfy the equilibrium equations.

The equilibrium equation, we have already noted in the two dimensional polar coordinate is of this form. And in this case, this shear stress is absent. So, therefore this term does not exist and sigma r is only a function of r. Therefore, this equation will get converted to d sigma r d r plus sigma r minus sigma theta by r plus x r equal to 0. So, the difference with the thick cylinder is that, we have the non zero body force here in this case.

And the equation of equilibrium in the theta direction, which is shown here, is going to be identically satisfied. Because, we do not have any force in the theta direction. This stress is absent. Sigma theta is independent of theta and again the stress is zero. So, therefore this equation is identically satisfied. If we substitute the value of x r, then the equation 1 gets converted to this form.

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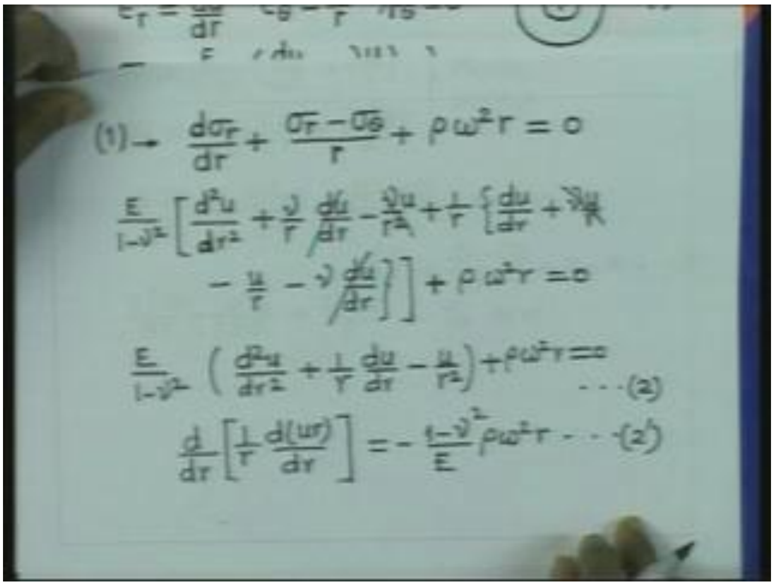


Now, we have to solve for this equation. We make use of the values of sigma r and sigma theta. So, therefore after substitution we will have the following. So, that is d sigma d r. And then we can also go on substituting the value of sigma r and sigma theta here. So, therefore we will have E by 1 minus nu square can be taken out. And we will be left with d u d r plus nu u by r minus u by r minus nu d u d r plus rho omega square r equal to 0.

So, now you see here, we are going to have some cancellation. So, this term will cancel with this term. And nu by r square, this is one u is here. So, therefore nu u by r square is going to cancel with the term here minus nu u by r square is going to cancel with this term. So, therefore we will have this canceling with this one. Finally, what you find here is that 1 minus E by 1 minus nu square, d 2 u d r square plus 1 by r d u d r minus u by r square plus rho omega square r is equal to 0.

So, this is the equation which is similar to that of thick cylinder except that we have a non zero term here. On rearrangement, we find this thing to be noting that. This expression also can be retained in this form minus 1 minus nu square by E rho omega square r. So, this is the form of 2. This is an equation, which can be easily integrated.

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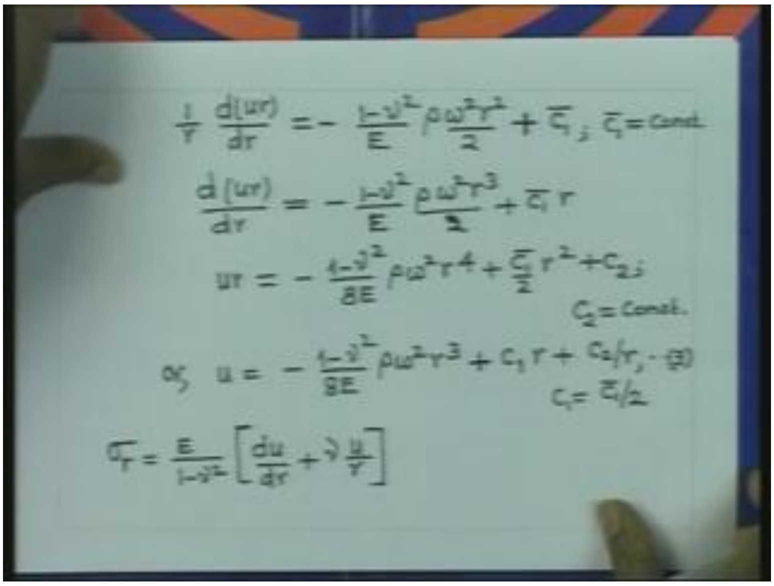


So, let us now do that 1 by r d u r d r is equal to 1 minus nu square by E rho omega square r square by 2. Let us write this thing as C 1 bar, where C 1 bar is a constant. Further, we can do d u r d r 1 minus nu square by E rho omega square plus C 1 bar into r. So, further integration will give us u r is equal to 1 minus nu square by 8 E, rho omega square r 4 plus C 1 by 2 r square. And we can bring in another constant C 2.

Or we have u is equal to 1 minus nu square 8 E rho omega square r cube. Let us write this thing as C 1 r plus C 2 by r. Where in C 1 is nothing, but C 1 bar by 2. So, this gives the expression for u. And if we now go back to the relationship, for sigma r and sigma theta we will get now values for the radial stress. So, let us now calculate this is equation number 3. So, if we now calculate sigma r, which is E by 1 minus nu square d u d r plus nu times u by r. So, let us now substitute the value of u in this relationship.

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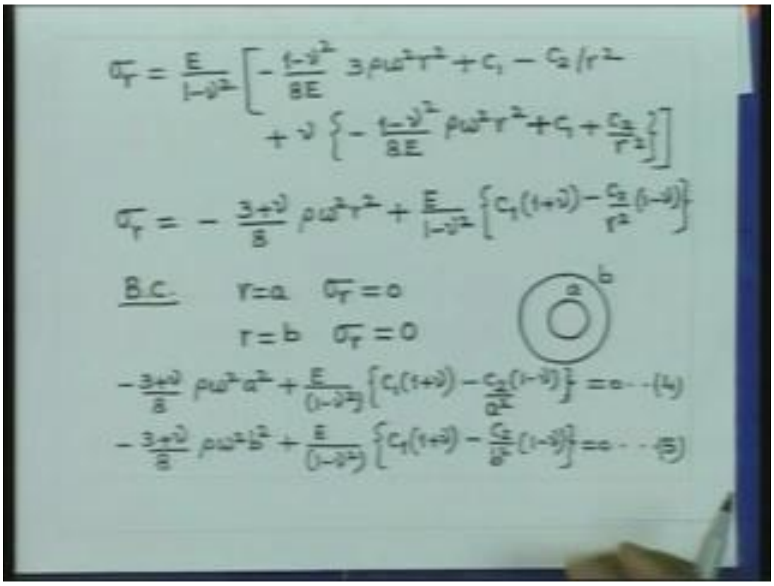


This will give E by 1 minus nu square. This is 1 minus nu square 8 E. This will be 3 times rho omega square, r square plus C 1 minus C 2 by r square plus nu times 1 minus nu square by 8 E. So, u by r if we do that subdivision, it gives us rho omega square r square plus C 1 plus C 2 by r square. This on little simplification, what we get here is that it is minus. So, we can write 3 plus nu. So, minus this term will cancels.

So, therefore it is minus 3 plus nu by 8 rho omega square r square that is a first term. Then, we are going to have E by 1 minus nu square. Then, we can add up these terms C 1 into 1 plus nu. And we will have C 2 by r square will have 1 minus nu. So, this is the expression for the radial stress. We have two arbitrary constants here. Sigma r, I beg your [FL]. C 1 and C 2 involved in this expression. Though sigma r, stress is still unknown because there are two arbitrary constants.

So, therefore we have to now solve this, solve for these two constants C 1 and C 2. How do you go about it? Let us look into the problem that, we have the disc which is rotating at an angular velocity. So, if we now try to see our problem here. So, this is the disc which is rotating.

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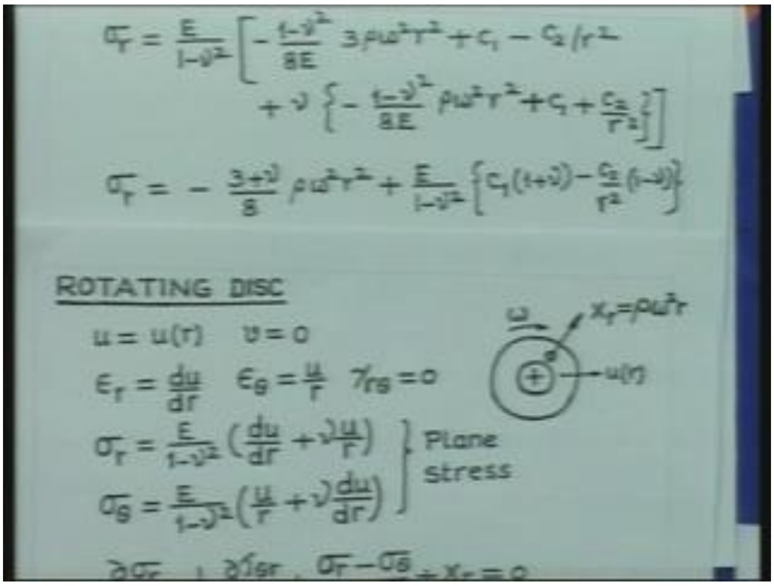


And this disc, when this disc is rotating what do you expect to be your radial stress at the inner surface, so also at the outer surface? The material which is just lying on the right of this inner surface is trying to move up, radially outward. So, therefore it will be pressing the material lying to the right of it. But for the outer surface, there is no material lying on the left of it. So, therefore there is no radial force acting the surface.

Similarly, a material point which is lying slightly inside the material layer, which is to the right of it is trying to move by away. And therefore, it will experience some radial force or radial stress. But for a material point which is right at the outer boundary. There is no material lying to the right of it. So, there is none to apply load in the radial direction. Therefore, the radial stress on the outer surface will be zero.

So, here at the inner surface radial stress is zero. So, also at the outer surface it is zero. So, the boundary conditions that we have to make use of here is nothing but, at r equal to a, we have radial stress sigma r equal to 0. Similarly, at the outer boundary r equal to b if we write the outer radius to be b. Then, that is also going to give us sigma r equal to 0. So, both are the inner radius and outer radius, the stresses are zero. So, we can now have the two equations obtained by using these two conditions.

This is equal to 0. So, this is equation number 4. Let us say. And the second equation is 1 plus nu minus C 2 by b square 1 minus nu equal to 0. So, let us consider that this is equation number 5. And wherein please note that, this is C 2 by a square. So, therefore the two equations are very similar. In this, we have a inward, in this it is b inward. So, these two equations are sufficient, so to solve for the arbitrary constants C 1 and C 2. So, from C 1 and C 2 we get the constants as follows.

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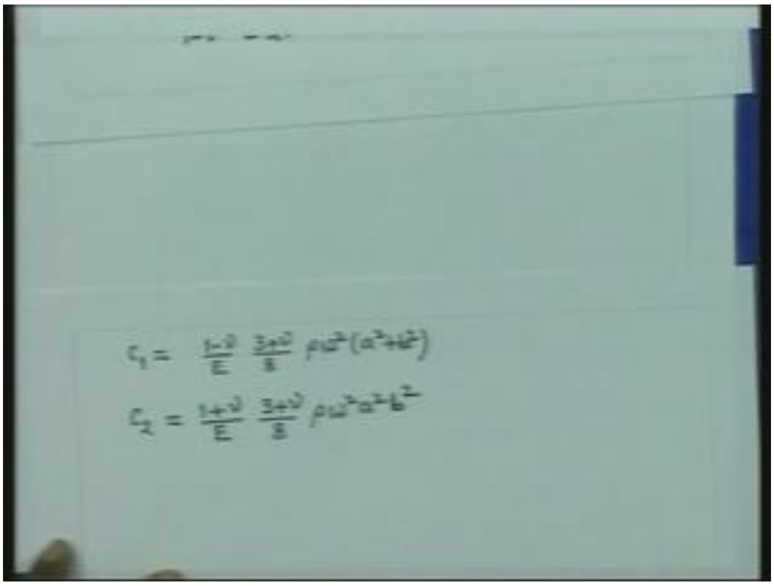


These are the two constants. And once, we go back to the expression for sigma r, we get sigma r in the form. So, once we substitute the value of C 1 and C 2, we get this sigma r stress in the following form. And if we look into the expression for sigma theta, sigma theta is nothing but, E by 1 minus nu square into nu by r plus nu times d u d r. So, therefore you can now substitute the value of u. And we will find finally, that the sigma theta stress is given by this expression.

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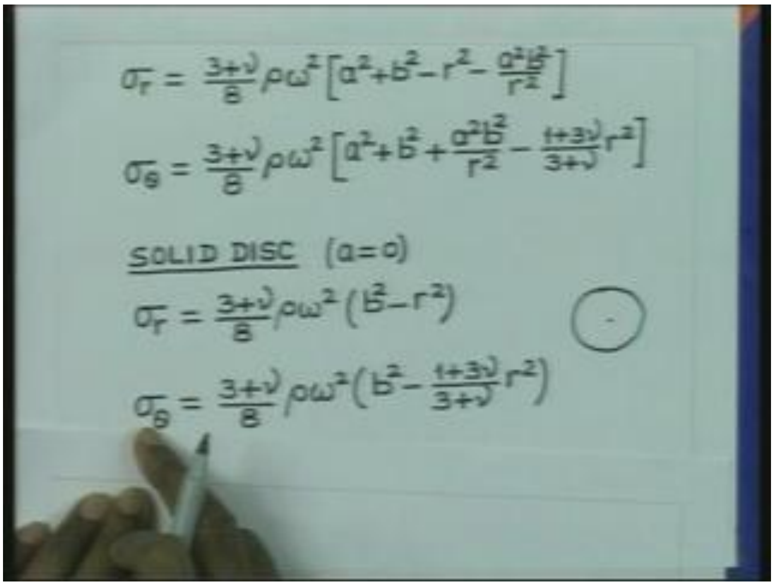


Sigma r is 3 plus nu by 8 rho omega square into a square plus b square minus r square minus a square by b square by r square. And sigma theta is equal to 3 plus nu by 8 rho omega square into a square plus b square plus a square by b square minus 1 plus 3 nu by 3 plus nu r square. So, this is the variation of stress in the case of a hollow disc. If the disc is solid, what we are saying is that a is 0. So, you can see this expression here, a is zero.

So, we will be simply left with b square minus r square here. And if a is equal to 0 in this expression, then we are going to have b square minus 1 plus 3 nu by 3 plus nu r square. So, for a solid disc the stress expression will be finally like this. Sigma r equal to 3 plus nu by 8 rho omega square b square minus r square. Sigma theta is equal to 3 plus nu by 8 rho omega square, b square minus 1 plus 3 nu by 3 plus nu r square.

So, this gives you the variation of stresses in the solid and hollow cylinder. Now, let us try to see how do the variations look like? First of all, let us consider the solid disc. So, in the case of solid disc if you put now r equal to 0, that means at the inner radius. So, what I am trying to say here. If you consider the inner centre of this disc, that is r equal to 0. Then, we find that sigma r, sigma theta both are equal.

And the stress is this, particularly radial stress if we now try to see its variation d sigma r d r. It is going to be that slope is going to be proportional to minus r. So, therefore it is a negative slope, and as r increases the slope increases. Similarly, in this case too we will find that d sigma theta d r is going to be proportional to r and it is again negative. So, therefore the slope will gradually increase in value. So, you can now see the variation of stresses in the case of solid disc. It is going to be a variation of this type.

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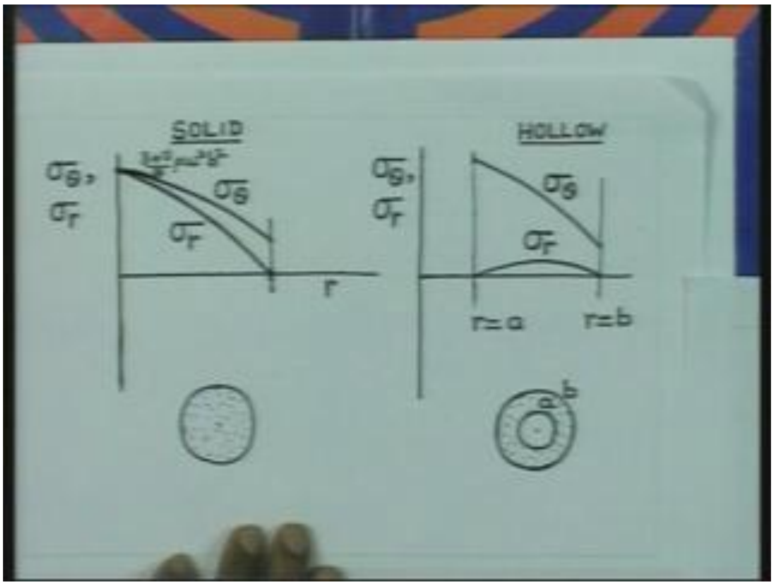


You have both sigma theta sigma r equal to 3 plus nu by 8 rho omega square into b square. So, the magnitude of the value here, at this point we can write the value. It is 3 plus nu by 8 into rho omega square into b square. That is the value of the stress. And the radial stress is zero at the outer radius. And the sigma theta stress is going to be higher than that. So, therefore this is the value here.

Look at this, the slop is going to have the value. It is going to have zero slopes. So, the variation will be such that, it is going to have a zero slope at this point. And it is gradually going to have negative slope increasing slope. So, both the variations are going to be given by this. Now, in the case of hollow cylinder the variation, which can be obtained from the expression for stresses, is going to be as shown here.

So, this is the variation of the hollow disc. Here, in the slop is going to have a finite value here. And it is negative increasing slope. So, that is the variation of sigma theta whereas, sigma 0 at the two ends. And it is going to have maximum value at the somewhere in between. So, therefore this is the variation of the stresses in the rotating disc. Now, let us try to see what is going to happen, if this radius is very very small.

Think of a situation, you get many at times shafts are rotating at very high speed particularly the turbine, gas turbine, the rotator, rpm of the order of 30000 rpm, when they are rotating for facilitating their lubrication. We try to have lubricating holes right at the centre. And this lubricating hole is going to be of very small dimension. So, that is the type of situation.

So, here it represents a typical cross section of a shaft rotating at very high speed. And let us try to see, if this radius a is tending to 0. Then, what happens to the circumferential stress. So, let us go into this, what I am going to show you that there is going to be tremendous stress amplification, due to this whole radius being shrunk to 0.

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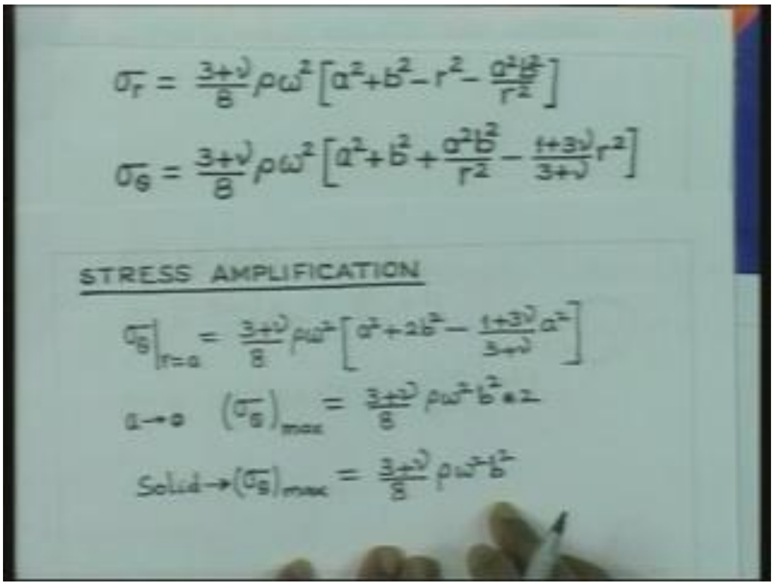


So, if you now consider this hoop stress as we have considering the stress at r equal to a. So, at r equal to a sigma theta at r equal to a is going to be given by 3 plus nu by 8 rho omega square. So, r equal to a means, we will have a square plus 2 times b square minus 1 plus 3 nu by 3 plus nu a square. So, that is the stress at the inner radius of the disc. Now, when you are trying to say that a is very very small, a is tending to 0.

Then, in that case that sigma theta the stress is really the maximum stress here. So, that maximum stress is now going to this is tending to zero. This is also going to tending to zero. So, therefore we will have a value equal to 3 plus nu by 8 rho omega square b square multiplied by 2. So, the stress is going to be this expression. Now, if we consider the discs to be solid, when a is close to 0. The disc can be considered to be almost with a radius 0.

So, it is a solid disc. So, if we substitute a equal to 0 here. Then, in that case for r equal to 0, we will have the maximum stress. So, for a solid disc, so this is the stress we get from the consideration of hollow disc. Now, if we consider the disc to be solid. Then, in that case the maximum stress we are going to get this thing, as 3 plus nu by 8 rho omega square into b square.

So, for a solid disc this maximum stress which is at the centre is this value. Now, look at the difference. So, if you just consider it to be solid disc, you are getting a stress of this value. If you are considering it to be a hollow disc with radius almost close to zero, the stress is this much. So, therefore there is an amplification of stress by almost a factor of 2 and this is very important.

So, therefore if we are trying to treat the turbine shaft with a small lubricating hole to be solid. Then, you are getting the stress almost half the actual value. And this has got to be very carefully considered. And at the same time when you have a solid disc, and you try to make a small hole, you are immediately raising the stress by a factor of 2. So, this point is very important. And this comes out only through the analysis of the stresses, in a rotating disc.

Now, I would like you to give some analogy for the understanding of the stresses in a rotating disc. We have seen that, in the rotating disc the stress radial stress and the tangential stress is equal at the centre. And radial stress is zero at the outer circumference. And the tangential stress is having value, non zero value at the outer circumference whereas, it is equal to the radial stress at the centre. So, therefore now I will give you an analogy to understand, some of these variations in the stresses in the rotating disc.

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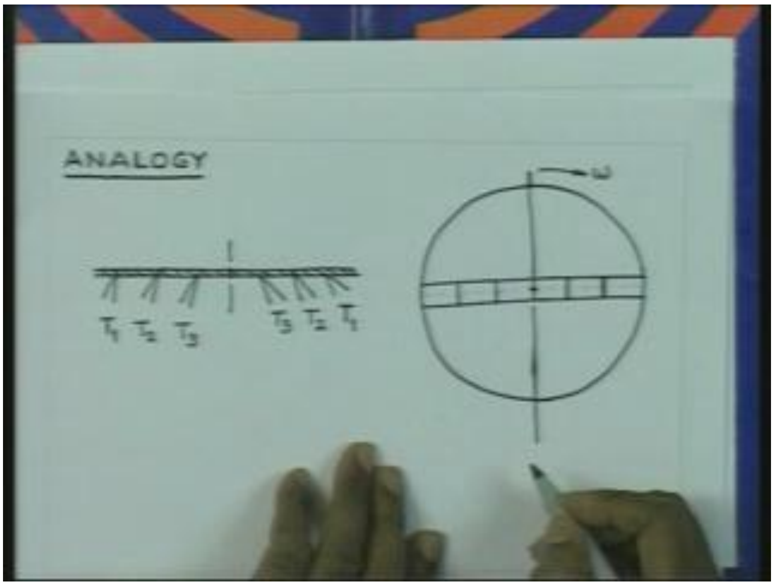


Now, think of a case that you have a tug of war situation. Let us consider that, this is the rope for the tug of war. So, this is the middle. Now, we will add the people pulling it from either side. So, let us say that, this is how we will have the. When they are balanced, let us think of it that we have. Therefore, tensions applied by the people standing here.

Let us say that, this is the person which is extremely outer is applying a tension of T 1. The next man is applying a tension T 2 and next man is applying tension T 3. Now, if we consider exactly the same situation, in a tie. Then, you are considering that this tension T 1 is applied by the outer most person. And then it is T 2. Then, it is T 3. So, here if you consider the tension over this portion of the row, there is nothing and the stress is zero.

But if you consider this to this portion, it is subjected to tension equal to T 1. And therefore, there will be some stress. If you come to this portion, the tension over this portion is going to be sum of T 1 and T 2. So, therefore the stress will increase in value over the cross section here. And as you come to the inner most, the tension is going to be sum of T 1 T 2 plus T 3. So, therefore highest tension is going to occur at the inner most regions.

So, therefore you find that the tension is highest at the centre. Note that this tension T 1, generally you try to put a strong. You try to put the strongest person at the outer end, he is like an anchor. So, therefore this is highest. Then, it is gradually it can be taught to reduce. And therefore, you see that rate of variation of the stress as you approach the center, its gradually decreasing. Now, let us see that this similar picture is existing in the case of a rotating disc.

Let us think of a rotating disc. So, this is the disc with the center here. It is rotating with an angular velocity omega. We consider a small strip of the disc, like this. And let us, divide this small strip into six equal segments. Let us say, this is one segment. This is the next. Similarly, this side we have this segment 1 2 3. Before we consider some of the problems associated with rotating disc. Let us try to go back to thick cylinder, look into some important points.

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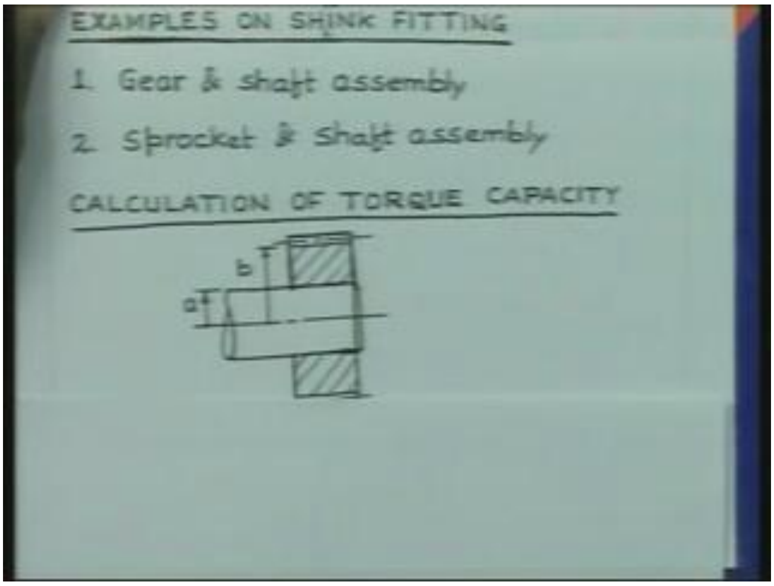


Shrink fitting is can be made use of in assembling a gear on a shaft. And when you are trying to do this, it is sometimes necessary to calculate the torque capacity. Along similar lines, some time sprockets are mounted on shaft using the shrink fitting. One advantage you get in this process is that, you can do away with the key way, the key of ways. So, you do not need to have the keying for the sprocket on the shaft or the gear on the shaft.

So, how do you calculate the torque in this case? So, here imagine that this is the shaft with external radius a and the gear is of this dimension. We can imagine the gear to be of the radius equal to the addendum radius. It should be dedendum radius, b should be equal to the dedendum radius.

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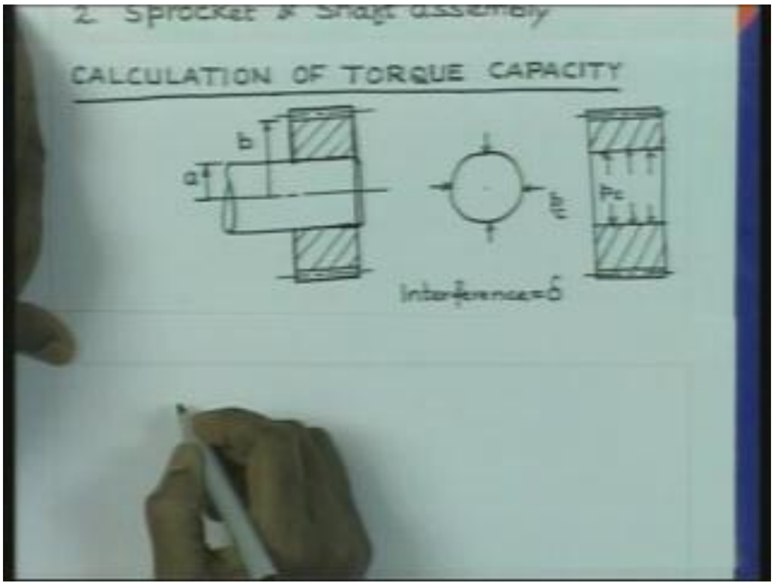


Now, under the action of the shrink fitting there will be some contact pressure acting on the shaft. Let us say that is p c. And the same p c is going to act on the surface of the inner surface of the gear hub. So, we can imagine that the gear hub is going to be subjected to the same pressure. So, this same pressure is going to act here. Depending on the initial interference, it is possible to calculate the pressure p c that you have noted already.

Now, here one typical point that you have to note is that, we have considered the both the components in the shrink fitting to be hollow. But when you have a solid shaft, then what is the distribution of stress? So, in order to calculate the interference, what you do is that you try to consider the contraction of the inner cylinder under the action of the pressure p c and the expansion of the outer cylinder under the action of same pressure.

And you sum these two and make it equal to the interference. And we do not have to calculate the contraction of the shaft at its outer radius. You need to find out the stresses in the circumferential direction and the radial direction. Obviously, the radial stress is equal to minus p c. But how do you calculate the tangential stress. Now, if you consider this is nothing, but a cylinder with external pressure.

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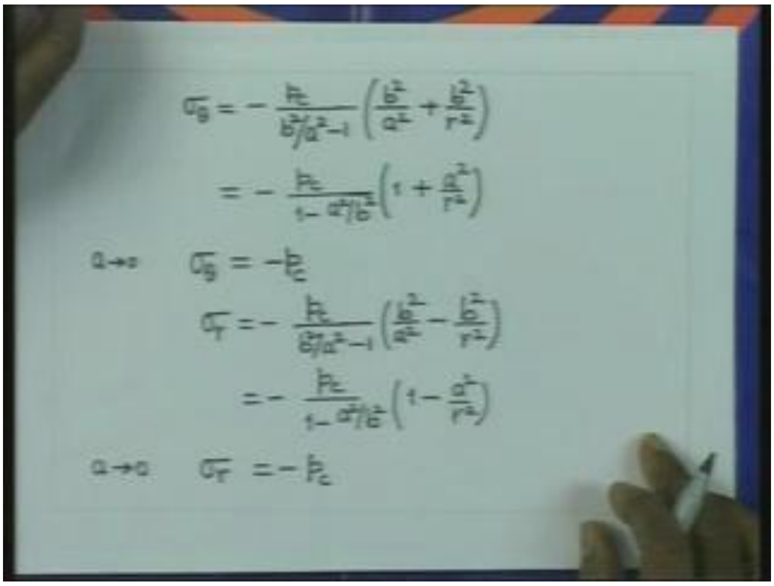


So, if we just consider now the formula that we have derived for sigma theta. Sigma theta for an externally pressurized cylinder was like this. If it is having b and a as the radii, then we have the formula like this. This is the variation and it is all compressive. Now, the shaft represents a case with a equal to 0. So, if we put a equal to 0, it will leads leads to infinity. And it is not possible to get the solution for the solid cylinder.

So, what we do is that, we try to rearrange. We try to multiply both numerator and denominator by a square by b square. So, that will give us a square by b square. And if you multiply a square by b square, this will become unity. And then a square by b square into b square by r square will give us, a square by r square. So, that is the variation of the sigma theta stress.

And this sigma theta, if you consider now a tending to 0, then you find that it comes out to be equal to minus p c. So, the stress in a thick cylinder with internal radius zero is going to be minus p c in the circumferential direction. So, therefore at a point here we are going to get the hoop stress is equal to minus p c. Now, if you also look into the expression for sigma r, sigma r was in this form p c by b square by a square minus 1.

It is b square by a square minus b square by r square. So, that is the radial stress. And again, if I go through the same procedure as we have followed in this case. Multiply both numerator and denominator by a square by b square, this gives us p c by 1 minus a square by b square 1 minus a square by r square. And then if we shrink the dimension a to 0, make it a solid shaft. Then, in that case sigma r is equal to minus p c.

So, the stress not only at the outer surface is minus p c in the radial direction. It is anywhere over the whole cylinder is going to be minus p c. So, therefore the stress in this direction is also minus p c. So, this is really a state of hydrostatic compression at all points in the solid shaft. So, we can now calculate the change in the dimension of the shaft. So, change in the dimension of the shaft if we try to calculate the change.

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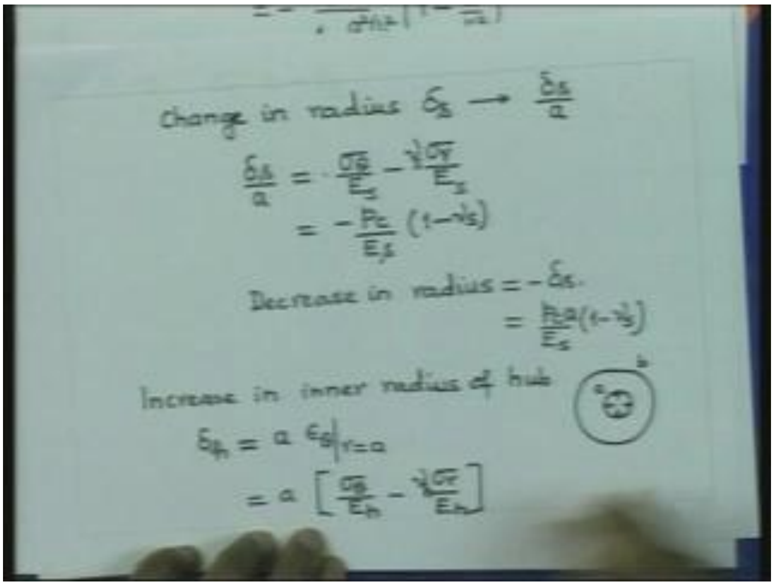


Let us say, change in radius of shaft is delta s. And therefore, this strain is going to be delta s by a. That is the strain. And therefore, this strain is equal to it is given by Hooke's law sigma theta by E minus nu times sigma bar by E. And if you would like to add the subscript for the shaft is. So, therefore we will have this value. Now, since both are minus p c. So, therefore what we have here is p c by E s into 1 minus nu s. That is the change in radius.

And you see that, decrease this delta s is really decreased. So, therefore decrease in radius is equal to minus delta s. And that is equal to p c by E s into 1 minus nu s. You can calculate the stress in the hub. So, as you consider that multiplied of course, minus delta s. And therefore, this is p c into a. That will be the decrease in radius. Similarly, you can consider the hub. The hub is of radius a and b. And it is going to have increase in the inner radius.

So, increase in inner radius of hub. Let us say, that is delta hub. And that delta hub is going to be E, that a multiplied by the strain in the circumferential direction at r equal to a. And you can write this thing as strain is nothing, but sigma theta by E of the material. Let us say E h minus nu sigma bar by E h. And we can write this thing again subscript equal to h to indicate, that is for the hub.

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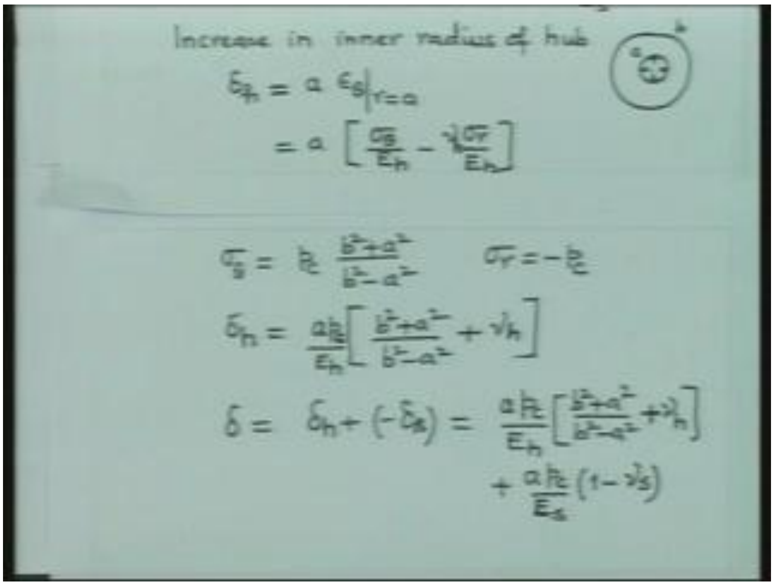


We can now see that, sigma theta is nothing, but p c is the contact pressure. Therefore, this sigma theta is equal to p c into b square plus a square by b square minus a square. And sigma r is equal to minus p c. Therefore, we will have delta h is equal to a into p c by E h b square plus a square by b square minus a square. And this is going to be minus gamma h. So, we have both got both increase in the radius of the hub and also the decrease in the radius of the shaft.

And therefore, we can now write the compatibility condition that, delta is equal to delta hub plus delta shaft. And therefore, we have a into p c by E h into b square plus a square by b square minus a square into plus gamma h plus a into p c by E of the shaft, multiplied by 1 minus nu s. So, this gives us the relationship to calculate the p c for any delta. Now, once you have got this p c from this relationship, this p c is going to act at the inner radius. And therefore, you can now find out the friction stress.

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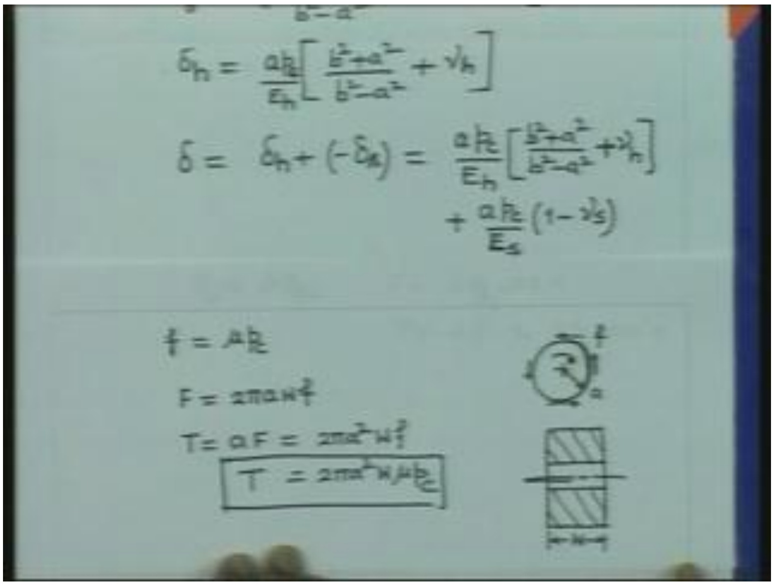

So, friction stress let us say f s. So, you have got now contact pressure p c. If you want to calculate the friction stress f, that f is going to be given by coefficient of friction. Let us say mu multiplied by p c. So, look at the shaft. That shaft is rotating in this direction. And the resistance to the rotation will come from the hub. And this is going to the friction stress, acting all over the circumference and if you consider, the hub of the gear who had a width.

So, let us say that this is the, this dimension is equal to w and this radius of the shaft is equal to a. So, therefore the friction force total is going to be acting over an area of 2 pi a multiplied by w into f. And the moment of this friction force about the centre, is going to be torque. So, therefore it is a into f and therefore it is 2 pi a square w into f.

So, if we now substitute the value it is 2 pi a square w mu into p c. So, this is really the formula for calculating the torque capacity of a shaft and hub assembly. So, this if you wanted to see the section, this is how we will have. So, this finally to sum up the torque is given by 2 pi a square w mu into p c, where in the dimensions are a is the shaft radius, mu is the friction coefficient, w is the width of the gear hub and p c is the contact pressure.