Advanced Strength of Materials
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Lecture – 13

Let us now consider determination of stresses in thick cylinder and rotating disc. You are already acquainted with the stresses in thin cylinder. When the thickness divided by the internal radius of a cylinder is less than about 0.1, it is called thin cylinder.

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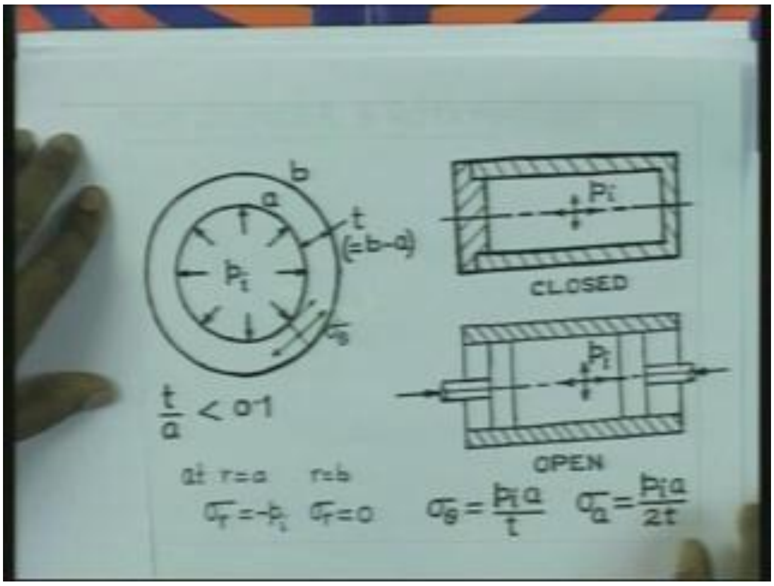

So, if we consider a situation like this, internal pressure is pi internal radius is a, external radius is b. When this wall thickness divided by the internal radius is less than about 0.1 it is called thin cylinder. And you know that when you have such thin cylinder, they are subjected to circumferential stress or hoop stress, given by pi into a by t. And the axial stress is given by pi a by 2 t.

When, the cylinder is closed then we have both these two stresses present. But, if the cylinder is open like this, then there is no axial stress only the circumferential or hoop stress exists. You already have the experience of inflating a balloon. When you inflate a balloon it goes on expanding in the circumferential direction and also in the axial direction. Therefore, you see that there is stretching in the circumferential direction. Therefore there is stress in that direction.

Similarly, there is also stretching in the axial direction. That is why there is also stress in the axial direction. That is a typical example of a closed cylinder like this. Now, why we have considered the stress distribution, in the thin cylinder, we have assumed that the hoop stress is uniformly distributed over the thickness. So, therefore, if you consider this is the thickness. Then the hoop stress which acts in this direction. So, that is the hoop stress at a point. And that hoop stress is going to be the same, at all points over this thickness.

So, therefore, it has the same magnitude over the thickness. Similarly, the axial stress is also uniform over the thickness. Now, if you consider the loading of this cylinder. It is subjected to uniform pressure in the radial direction. And that is why there is expansion in the circumferential direction. And since the pressure is uniform, you get uniform expansion of the cylinder and there is axisymmetric loading and the deformation is also axisymmetric.

It is not difficult for you to appreciate that the stress in the radial direction at the inner radius is nothing but, minus p i. That means at r equal to a, you have the radial stress sigma r is equal to minus p i. And at r equal to b, you have a radial stress, if there is no loading at the outer surface it is going to be 0. So, we have not considered the existence of this radial stress at all in the thin cylinder. But, there is no doubt that the radial stress is maximum at the inner radius. And it is 0 at the outer radius, in the case of internal pressure loading.

Now, if the cylinder is very thick, where the thickness of the wall is comparable to dimension a. Then the question that will arise, does the distribution of hoop stress remain uniform. Does the distribution of axial stress remain uniform. And what could be the distribution of the radial stress. And we can rephrase these queries also, what would be the error in the thin cylinder formula, if we are considering a particular ratio of t by a, ehere would be the maximum hoop stress, where is the maximum axial stress. So, all this questions are very important from the point of view of design of thick cylinder. And these are some of the issues, we would like to look into. Now, the example of thick cylinder, there are lot of situations, where you come across thick cylinder.

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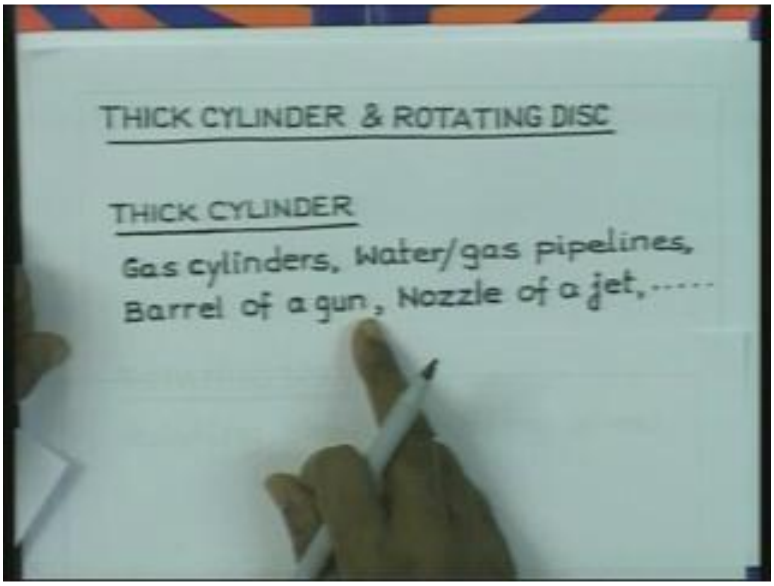

Particularly, if you consider the domestic gas cylinders, oxygen gas cylinders, which are used in the hospitals water or gas pipelines, barrel of a gun, nozzle of a jet. Or even the submarine whose body can be considered to be more or less cylindrical. When it is at the bottom of the sea, it is subjected to high pressure loading due to the water head. And the stress calculations in the submarine body has got to be also very similar. Or it is sometimes calculated, considering the loading to be that of a thick cylinder under external pressure loading.

Now, what is the connection between thick cylinder and rotating disc. If you consider the grinding wheel or a shaft of a gas turbine, they rotate particularly the gas turbine shaft rotate at speed goes to about 30,000 RPM. Similarly, the grinding wheel also operates at high speed. Now, when they are rotating, look at the loading that comes up on the material of the body.

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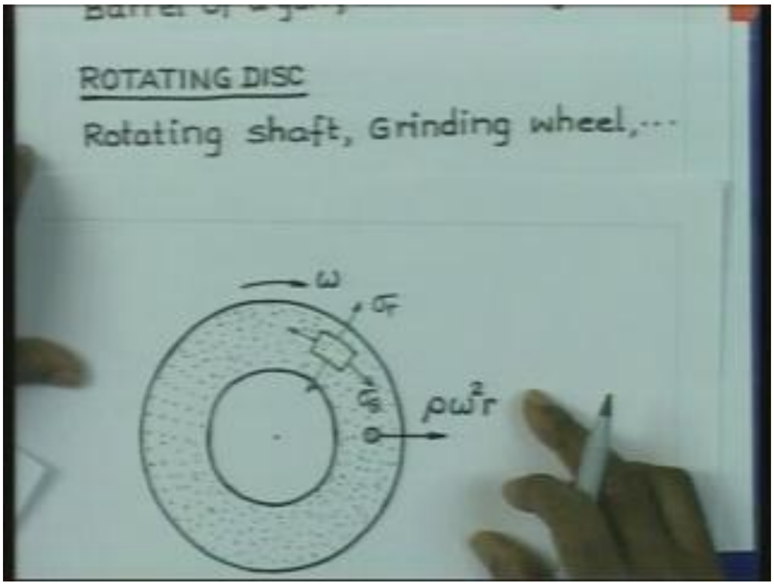

So, you think of a situation like this, that you have a grinding wheel which is mounted on a shaft, it is rotating at some angular speed omega. Then each point of the rotating disc is subjected to centrifugal force. Whose intensity is going to be given by rho omega square into r; where rho is the density of the material, omega is the angular velocity and r is the distance of the point from the centre of the disc.

So, here each point is trying to move away from the centre. And therefore, you have a loading, which is again axisymmetric. And under the action of the loading like this, you are going to have deformation of the disc and there will be stresses developed in the circumferential direction. So, if you consider an element, consider an element oriented like this. Then in that case, you will find that there is going to be stresses developing in the radial direction.

So, we will indicate that stress by sigma r and there is also going to be a stress at each and every point in the circumferential direction. So, there be there will be hoop stress. So, each point is subjected to both radial and circumferential stress. The difference between this case and the cylinder under internal pressure or thick cylinder under internal pressure is that in this case. The loading is like a body force, whereas in the case of internal or external pressure, it is pressure loading. That is coming on the outer boundary.

So, this is a case wherein the loading is due to body forces. And therefore, the loading is different, but then the analysis. Since in this case also you have axisymmetric deformation. So, also in the case of thick cylinder we have axisymmetric deformation, the two things can be considered along similar lines. Obviously I have given you some examples of the rotating disc, it could be a rotating shaft or a grinding wheel.

In the case of shaft, you are already aware that there could be stresses due to the torque acting, bending moment acting. Now, you must realize that if the shaft is rotating at high speed, there should be also some extra stresses due to the rotation. So, to calculate the total stresses in the shaft, one must also calculate the stresses due to rotation. In the case of grinding wheel, the basic particles are held in position by means of some bonding material.

So, the strength of the bonding material must be able to hold the, hold against or must be able to resist the stresses due to the rotation. And therefore, one can calculate the stresses due to rotation and then select the appropriate bonding material So, therefore, the calculation of the stresses due to rotation are also important from the point of view of design. Whether it is a rotating shaft or a grinding wheel, there is a need to calculate the stresses. So, therefore, we would like to look into these problems.

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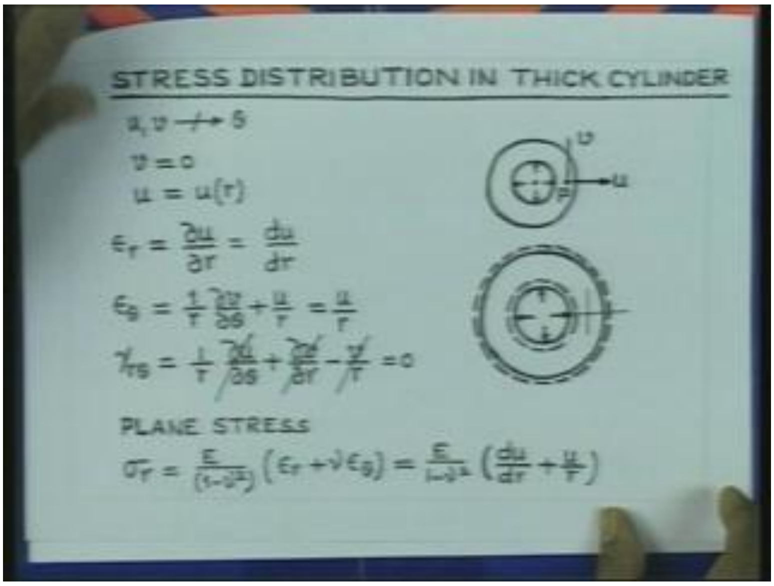

Now, as I have mentioned that while you are trying to consider the stressing of a thick cylinder. The deformation is axisymmetric thereby meaning that a point in the wall is going to move under the action of the pressure loading radially outward. If it is external loading, external pressure loading then in that case it is going to move radially inward.

There is no displacement of the point in the circumferential direction. Therefore this is a typical case, where we find that the two displacements, first of all they are independent of the angular position of the point.

So, therefore, u and v they are independent of v are independent of theta. So, there is no dependency on theta. And then what do we find in this case is that v is equal to 0. And what do we have finally, is that u is a function of only radius. So, therefore, u is a function of radius alone in this case. Now, with this deformation pattern, to find out the stresses or stress distribution in the thick cylinder. We can get started with this displacement.

Let me just tell that this displacement is related to the strains, strains are related to the stresses by Hooke's law. Therefore if we are in a position to find out the displacement then in that case, we are in a position to calculate the stress distribution in the thick cylinder. So, therefore, the problem of stress distribution in thick cylinder is really a problem of determination of the distribution of this displacement in the thickness direction.

Now, let us consider the strains. So, if we consider now epsilon r, you remember that epsilon r in the polar coordinates it was given by delta u delta r. I think you must also see the point that the since geometry is circular here, it is good to going for polar coordinates and therefore, we will also have stresses and strains in polar coordinate. So, the strain in the radial direction is delta u delta r. Now, since u is only a function of radius, I can write this thing as du dr. So, it is total derivative rather than partial derivative.

Now, strain in the circumferential direction, epsilon theta it was given by 1 by r delta v delta theta plus u by r. So, the strain in the circumferential direction is given by the partial derivative of v with respect to theta plus u by r. Now, since v is 0 therefore, we have the circumferential strain is given by u by r. Now, shear strain gamma r theta is nothing, but 1 by r delta u delta theta plus delta v delta r minus v by r.

Now, u is independent of theta. So, therefore, this term is 0, v does not exist therefore, this is also 0 and so also v is 0 this is 0. So, therefore, the shear strain is 0. So, therefore, in this case, what you find is that the shear strain does not exist there is no deformation of the cross section. There is another thing, which you can also see from the physical deformation that as you load the cylinder. Let us say this is the internal radius and this is the external radius as you pressurize it, the internal radius is increasing in size. So, also the external radius or any radius for that matter is just increasing in magnitude.

Now, any two orthogonal directions, if you consider the radial directions this one and the circumferential direction perpendicular to this, the two directions before deformation at this point was orthogonal after deformation they are again orthogonal. So, therefore, there is no change in the right angle and hence the shear strain is 0. So, therefore, from the physical deformation also we expect, the shear strain to be 0 mathematically also we have obtained that to be 0.

Now, we have got the strains in terms of the displacement u, which is a function of radius. Now, let us see if I make use of the Hooke's law and if few we assume thus stet of stress to be plane stress. So, assume plane stress. So, if we are assume this disc or the cylinder to be thin and loaded by internal pressure loading, then in that case we can make use of the Hooke's law and the relationship between the stresses and strains are sigma r is going to be given by E by 1 minus nu square multiplied by epsilon r plus nu times epsilon theta.

So, that is the stress in the radial direction and now if we substitute the value of the strains then the stress is going to be given by 1 minus nu square du dr plus u by r. So, therefore, this is the expression for the radial stress.

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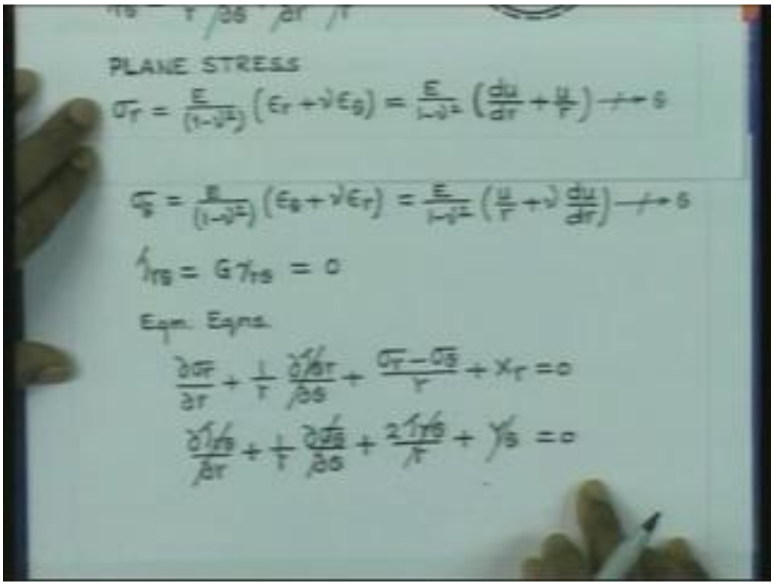

If we calculate the circumferential stress, it is going to be given by sigma theta is equal to E by 1 minus nu square multiplied by epsilon theta plus nu times epsilon r. Again if we substitute the two strain components, in terms of displacement then we have E by 1 minus nu square u by r plus nu times du dr. So, these are the two stresses. Now, the shear stress tau r theta is equal to modulus of rigidity G multiplied by the shear strain gamma r theta, since gam gamma r theta is 0 therefore, the shear stress is 0.

So, therefore, what you find in this case, the two non 0 stresses at any point on the wall are sigma r and sigma theta. Now, how do we determine these two stresses. So, we can consider, now that these two stresses at each and every point of the cylinder must satisfy the equilibrium equations. So, if we now write the equilibrium equations, the equilibrium equations in the radial direction is nothing, but delta sigma r delta r plus 1 by r delta tau theta r delta theta plus sigma r minus sigma theta by r plus body force in the radial direction is equal to 0.

Similarly, in the circumstantial direction, we had delta tau r theta delta r plus 1 by r delta sigma theta delta theta plus twice tau r theta by r plus y theta equal to 0. So, these are the two equilibrium equations. You have already seen that the shear stress is absent. So, therefore, we do not have this term existing. So, also this term is not there and we see that, sigma theta is related to the displacement which is a function of radius alone. So, therefore, this stress is not really a function of theta.

Similarly, sigma r stress is related to displacement, which is a function of only radius. So, therefore, it is not a function of theta again. So, therefore, what we find is that this term is nonexistent, shear stress is not there and in the case of thick cylinder subjected to pressure loading there is no body force. So, therefore, this equation in the theta direction is identically satisfied. Now, if you consider the equation in the radial direction, here in we find that this partial derivative, since sigma r is a function of r alone, therefore we can write this thing as now total derivative rather than partial derivative.

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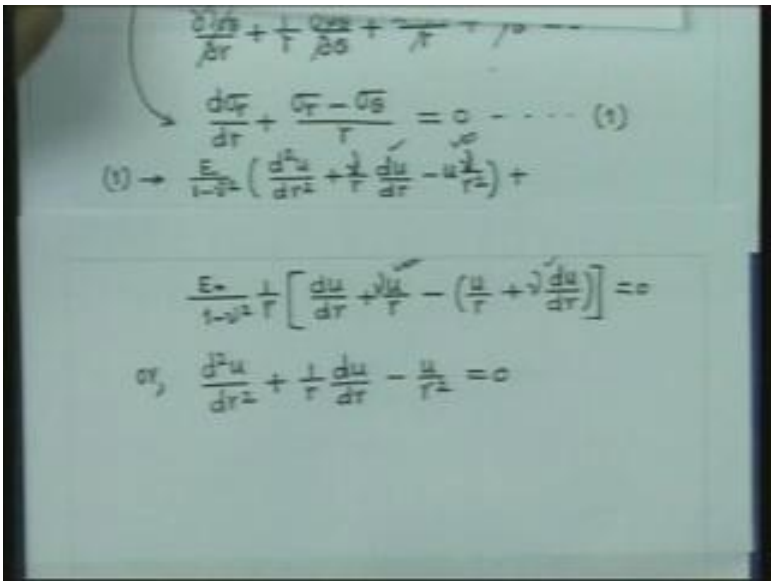

So, this equation we can now write that this is d sigma r dr plus sigma r minus sigma theta by r. And there is no body force present in this case. So, therefore, this simply becomes this sum equal to 0. So, that is the equilibrium equation or that is the governing equation of thick cylinder. You can easily understand that sigma r sigma theta r functions of u. So, when we substitute the value of the stresses in terms of displacement, this gives an equation involving the u displacement and that gives the basis to find out the unknown function u.

So, we will continue determination of u, now starting from this equation. So, we would like to now substitute the value of the stresses in terms of u. So, already we can substitute the value of sigma r here. So, therefore, if we substituting. So, from 1 we can now get this is E by 1 minus nu square and then we have to take the differentiation of this. So, if I take the differentiation it will be d 2 u d r square and then we will have 1 by r du dr.

So, considering this to be constant, then we have du dr then we will have another term u is constant and if you differentiate 1 by r will gives you minus 1 by r square. So, that is the du dr then we are also going to have the other contributions coming from these two stresses, so we can now write. So, we will have plus we are going to get now E by 1 minus nu square, we have 1 by r and then we have sigma r that will give us du dr plus u by r and due to sigma theta we can make use of this expression here.

So, therefore, it is going to be. So, already E by 1 minus nu square I have taken common. So, therefore, it is going to be now u by r plus nu times du dr, so that is equal to 0. So, therefore, what do we have is finally, is this and if we simplify E by 1 minus nu square is common. So, therefore, we will be having d 2 u dr square that is the first term. And then we are going to, I have not actually made use of one more term here, we have nu is missing here. So, therefore, this expression here is in the expression for sigma r we must have nu there.

So, therefore, we will have nu also occurring in this expression. So, this is nu in this expression here also it is nu occurring here. And in the second expression, we are also going to get nu coming up here and this. So, now if I try to simplify we are going to have d 2 u dr square and here it is nu du dr, this is going to cancel with this term. So, therefore, this term will cancel with this term and this term is u nu by r square this is negative and this term is also going to be nu u by r square, but this is positive sign.

So, therefore, this terms is going to cancel with this one. So, finally, we are left with here one more term here, 1 by r du dr and then we have 1 by r u by r. So, therefore, u by r square equal to 0. So, therefore this is the governing equation involving the displacement u.

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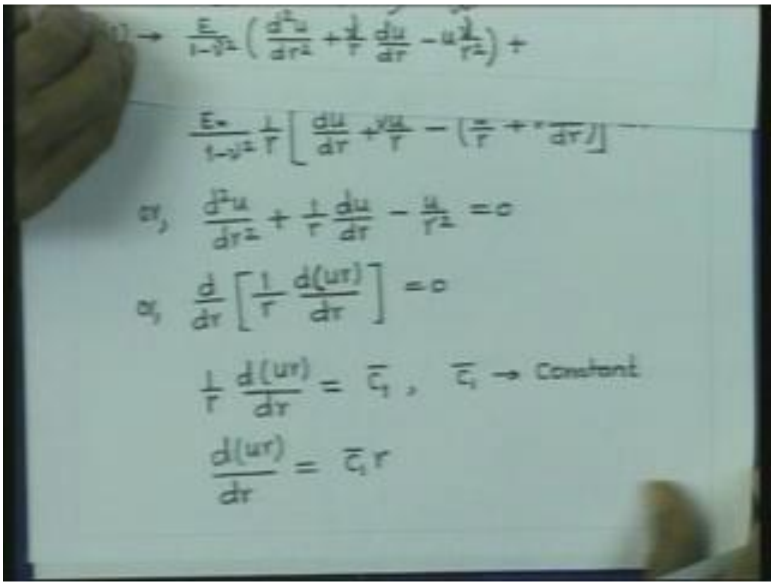

And this equation can be written in this form, this is du dr 1 by r d ur dr equal to 0. So, this equation really gets converted to this form and it is easily integrable. So, therefore, we can now write 1 by r d ur dr is equal to some constant let us say c 1 bar, c 1 bar is a constant. Now, if we continue further, we will have d ur dr is equal to c 1 bar into r.

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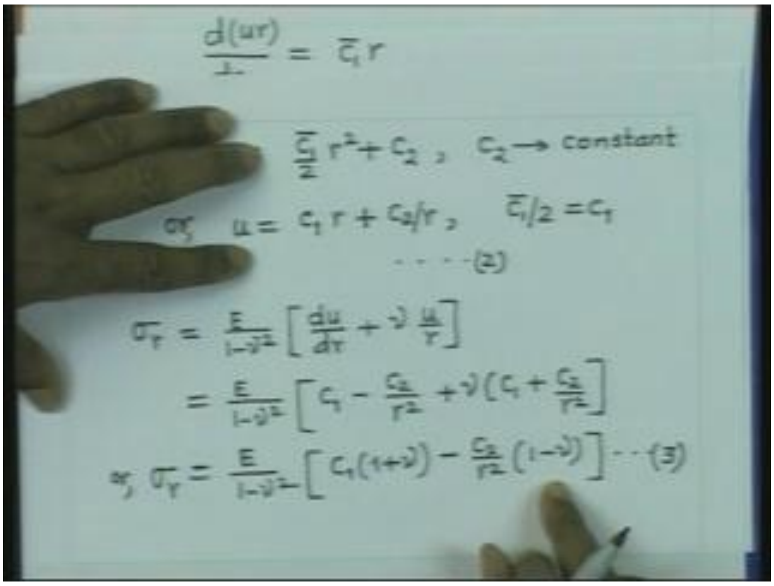

So, this will now should can be further integrated and we get now, ur is equal to c 1 bar by 2 r square plus some constant, let us say c 2, c 2 is another constant of integration. Now, we can write this thing, in the form u equal to let us say c 1 r plus c 2 by r, where we have substituted c 1 bar by 2 equal to c 1. So, this is the displacement variation with radius. So, this displacement variation with radius let us consider that relation to be 2, we do not know the constant c 1 and c 2.

Now, we can go back to the expression for the stresses and substitute the value of u thereby we get now sigma r, sigma r is equal to E by 1 minus nu square du dr plus nu times u by r. So, if we now consider substitution. So, it gives us nu square. So, it is going to be c 1 minus c 2 by r square plus nu times c 1 plus c 2 by r square. So, this can be rearranged to give sigma r is E by 1 minus nu square it is c 1 into 1 plus nu c 2 by r square into 1 minus nu.

So, that is the expression for radial stress, in terms of the radius and the constant c 1 and c 2. So, we will have to determine these two constants of integration from the consideration of loading of the body. So, we will look into the determination of these two constants. A specific situation where we have both internal pressure and external pressure.

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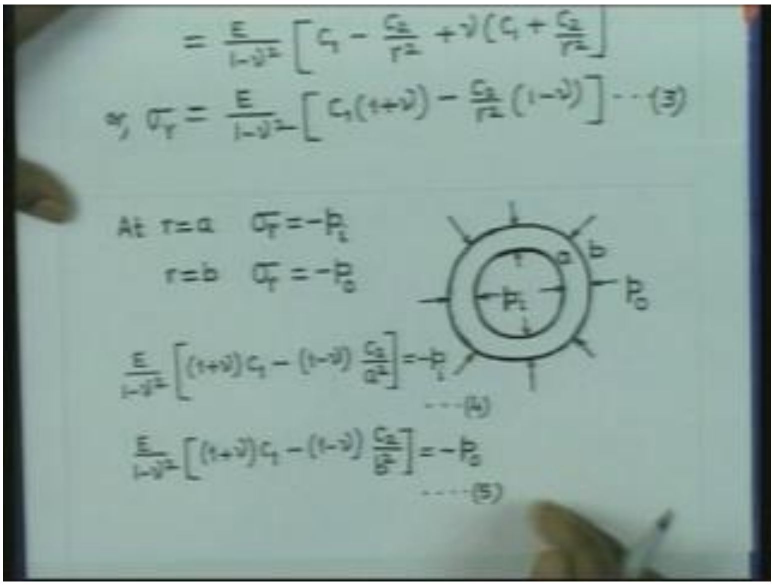

Let us indicate the internal pressure by pi, external pressure by po and the radii are a and b. Now, in this case the stress in the radial direction, if you consider the boundary conditions existing in this problem, what we have at internal radius that is r equal to a, what is the radial stress. The radial stress sigma r is nothing, but minus pi. So, the magnitude of pi will be given to you and therefore, the radial stress is going to be just minus pi.

Similarly, at external radius, r equal to b your radial stress is equal to minus p 0, just I want you to reflect upon that if we consider this surface its outer normal is directed like this. But this pressure is acting in the negative r direction therefore, the stress is negative and it also from the fact, that these stresses are compressing therefore, they will have to be negative. Now, using these two boundary conditions of the problem and using the relationship of sigma r with radius, we can evaluate the two constant.

So, therefore, if we substitute now what we have is E by 1 minus nu square 1 plus nu c 1 minus 1 minus nu c 2 by a square is equal to minus pi. So, let us say that this equation number 4. Similarly, we can write E by 1 minus nu square 1 plus nu c 1, 1 minus nu c 2 by b square is equal to minus po, that is our equation number 5. So, from these two equations, we have two unknowns, c 1 and c 2 and we can determine them. I will not go for the steps I, will give you directly the value of the two constants c 1 and c 2.

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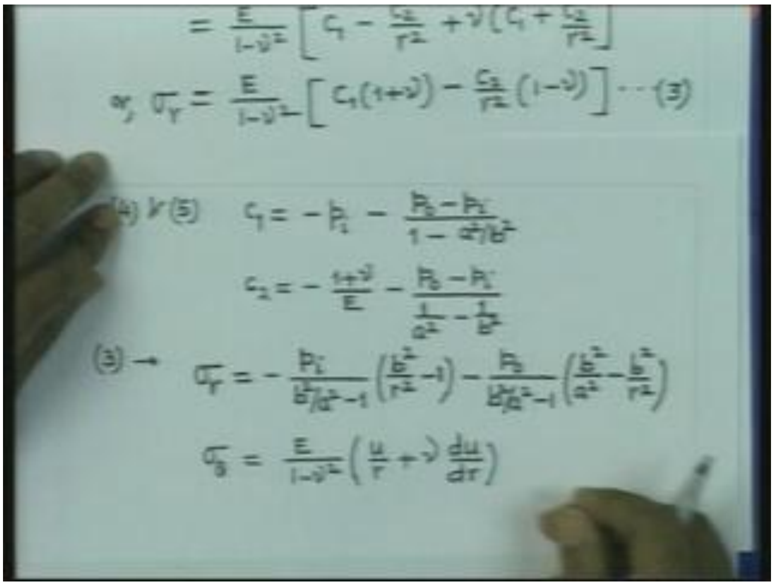

This c 1, so from 4 and 5, we have c 1 is equal to minus pi minus p 0 minus pi 1 minus a square by b square and constant c 2 is equal to minus 1 plus nu by E minus p 0 minus pi 1 by a square minus 1 by b square. So, these are the two constants, once we go back to the equation number 3 and substitute the values of c 1 and c 2, we can get the values of the stress. So, therefore, we had the sigma r stress given here in terms of c 1 and c 2 and r.

So, therefore, if we substitute the value in this expression, so therefore, from 3 we can now obtain sigma r as minus pi divided by b square by a square minus 1, b square r square by r square minus 1, minus p 0 b square by a square minus 1 divided by b square by a square minus 1 multiplied by b square by a square minus b square by r square. So, that is the expression for sigma r. Similarly, we also had sigma theta is E by 1 minus nu square into u by r plus nu times du dr. Now, that we have got u, we can substitute here thereby we will get the value of sigma theta. So, therefore, this sigma theta will look like this.

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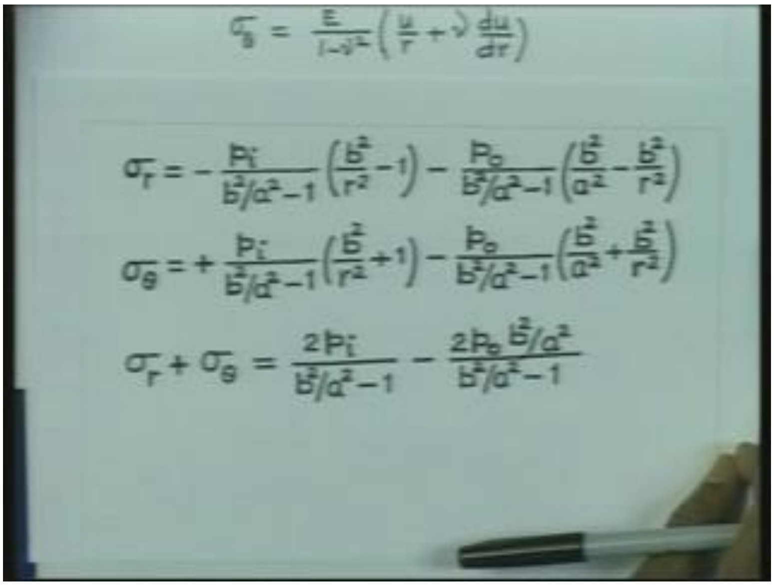

So, your sigma r is this sigma theta is pi by b square by a square minus 1 into b square by r square plus 1 minus p 0 by b square minus a square minus 1 into b square by a square plus b square by r square. You know that this particular stress distribution is such that if you take the sum of the two stresses, that is sigma r plus sigma theta it is equal to the terms involving r cancels. And we have sum of the two stresses is equal to 2 pi divided by b square by a square minus 1 minus 2 p 0 b square by a square divided by b square by a square minus 1.

So, the sum of the two stresses is independent of radius, these are something very important for the thick cylinder that the sum of the two stresses is a constant. Now, let us try to look in to what these variations are in specific cases. So, if we now consider just internal pressure loading. So, if we consider the internal pressure loading only then in that case you.

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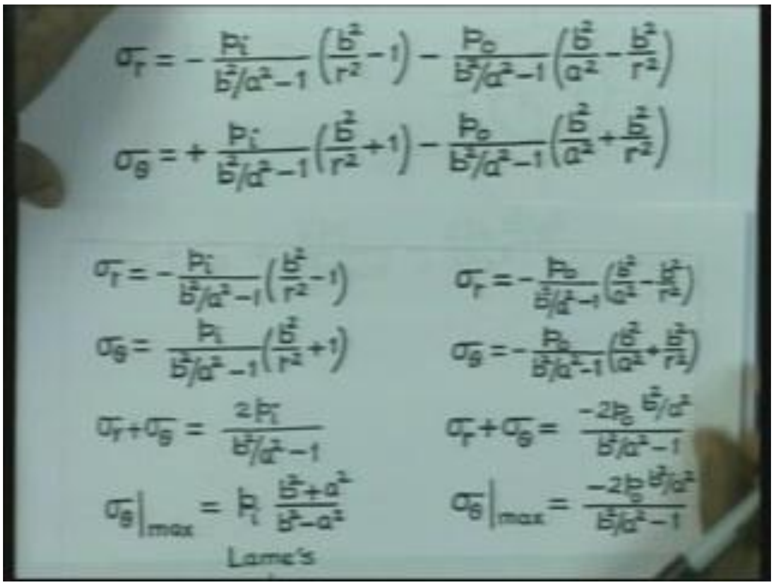

So, what I am saying that if p 0 is absent then your sigma r is given by this expression, minus pi by b square by a square minus 1 into b square by r square minus 1 and sigma theta is equal to pi by b square by a square minus 1 into b square by r square plus 1 and again sum of the two stresses is simply 2 pi by b square by a square minus 1. These stresses are going to be distributed. So, if we try to now see the distribution of the stresses in the wall of this cylinder, it will be like this.

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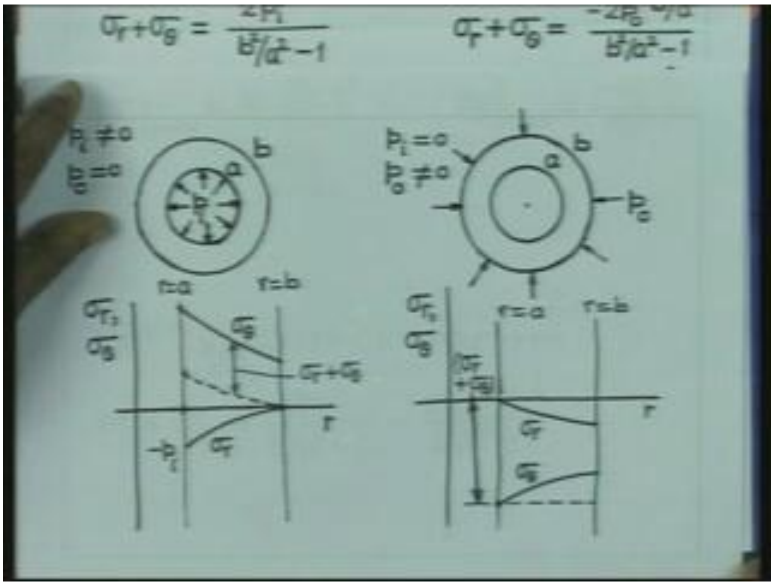

Let us plot the radius in this direction. So, we are considering this is the situation pi is not 0, p 0 is 0 and we have the radial direction. So, radius is plotted in this direction and we would like to plot the stresses sigma r and sigma theta in this direction. Then we have the internal radius is here. So, therefore, this is the internal radius. So, this is r equal to a let us consider that, this is the external radius r equal to b. ((Refer Time: 47:21))

Now, the variation of the stress, this stress you can see that it is as r increases the value of this term within the bracket is going to be reducing. So, therefore, the radial stress gradually decreases at r equal to a, you find that this value of this stress is minus pi and at r equal to b the value of the stress is 0. So, therefore, it is gradually reducing and obviously, the slope d sigma or dr it is going to be a positive decreasing slope. So, therefore, the variation of the stress, radial stress it is going to be reducing in magnitude and it is going to vary like this.

So, it is a positive slope, the value of the stress here is minus pi and this is the variation of sigma r. Similarly, if you consider the variation of the hoop stress, ((Refer Time: 48:23)) when r is minimum you have the maximum value, when r is maximum you have the minimum value and it is gradually reducing in magnitude and the slope d sigma theta dr is going to be negative. So, therefore, because you see that d sigma theta dr is proportional to minus 2 by r cube. So, therefore, it is going to be negative slope.

So, therefore, the variation of the hoop stress is going to look like this. So, this is the variation of sigma theta stress and the maximum stress is going to occur at the internal radius. Now, what is sigma r plus sigma theta. So, this is the stress sigma r, which is always negative. So, if we take the mirror reflection of this curve.

So, this is the mirror reflection and therefore, sum of the two stresses is nothing, but this. So, this is the sum of the two stress, sigma r plus sigma theta which is constant, this is what is the sum of the two stresses. So, you see that maximum stress in the thick cylinder is going to be the hoop stress at the internal radius. And the maximum stress in the radial direction is at the outer radius and at the outer radius we have hoop stress, which is going to be tensile and it has magnitude less compare to the value at the internal radius.

Now, if we consider similar thing for the cylinder subjected to only external pressure loading. So, we consider a case like this, then in that case the term symbol the pi will be 0 and ((Refer Time: 50:28)) this stress sigma r and sigma theta is going to be given by these expressions and sum of the two stresses is going to be given by again by this expression. So, if we now plot the stresses along similar line that we have done in the case here, we find the variation to be as follows.

So, here the radial stress variation, radial stress is ((Refer Time: 51:23)) when r is equal to a this term cancels therefore, sigma r is 0. When r equal to b then we are going to have minus p 0. So, the stresses are always negative and the slope of the curve is negative and it is decreasing slope ((Refer Time: 51:47)) because r it is going to be proportional to d sigma r dr is going to be proportional to minus 1 by r cube, so therefore as r increases, the slope going to decrease. So, therefore, the variation of the stress will be like this from the internal radius to external radius.

So, if you consider that, this is the internal radius. So, let us say that this is r equal to a and this is r equal to b, this is how the variation of sigma r is going to look like. By similar consideration, ((Refer Time: 52:32)) you find now that sigma theta, sigma theta when r is minimum we have the maximum hoop space and when that is when r is equal to a we have the maximum hoop space and when r is equal to b we have the minimum hoop space.

And in this case, the slope of the curve is proportional to 1 by r cube again. So, therefore, it is positive slope and it is gradually decreasing slope. So, therefore, the variation of the stress is going to look like this. So, the stresses are going to vary, now what about the sum of the two stresses sigma r plus sigma theta. So, therefore, this is negative, this stress is also negative. So, therefore, as you add up you get this common value and this is nothing, but sigma r. So, this is the value, which indicates sigma r plus sigma theta which is a constant.

Now, in this case the maximum stress is again going to be the hoop stress and it is occurring at the internal radius. These stresses therefore, are very important from the point of view of design, so therefore, let us look into their values. If, you consider the maximum hoop stress in the case of the internal cylinder, internally pressurized cylinder. So, if you consider the internally pressurized cylinder then the hoop stress that is going to come up at the internal radius is going to be given by ((Refer Time: 54:38)) pi into b square plus a square divided by b square minus a square.

This particular stress is known as Lame's stress, it was first calculated by Lame. Similarly, the maximum stress that we get in the case of the externally pressurized cylinder, it is going to be occurring again at the internal radius and it is value is given by ((Refer Time: 55:09)) minus 2 p 0 into b square by a square divided by b square minus a square minus 1. So, that is the maximum hoop stress that is going to occur in the case of externally pressurized cylinder.

So, we have been able to calculate the stress distribution in the wall of the cylinder, you can see that the stress is not constant, it is continuously varying and it is giving the highest value, in both the case of internally pressurized cylinder, externally specialized cylinder at the inner radius. Now, just I want you to reflect upon, what are the nature of these two stresses sigma r and sigma theta. What I mean is that are these stresses principle stresses. I would like you to come to a conclusion about the two stresses sigma r, sigma theta in relation to this question.