Advanced Strength of Materials
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Lecture – 12

In this lecture, we would like to see relations between elastic constants. We have introduced elastic constants like, E modulus of elasticity, nu Poisson's ratio, G modulus of rigidity. I have already indicated that G is related to E and nu for the isotropic materials. So, therefore, we would like to see what sort of relationship exists between these constants. And at the same time, we can have another elastic constant, which is K the bulk modulus will show that this bulk modulus is also related to modulus of elasticity and Poisson's ratio in the case of isotropic materials.

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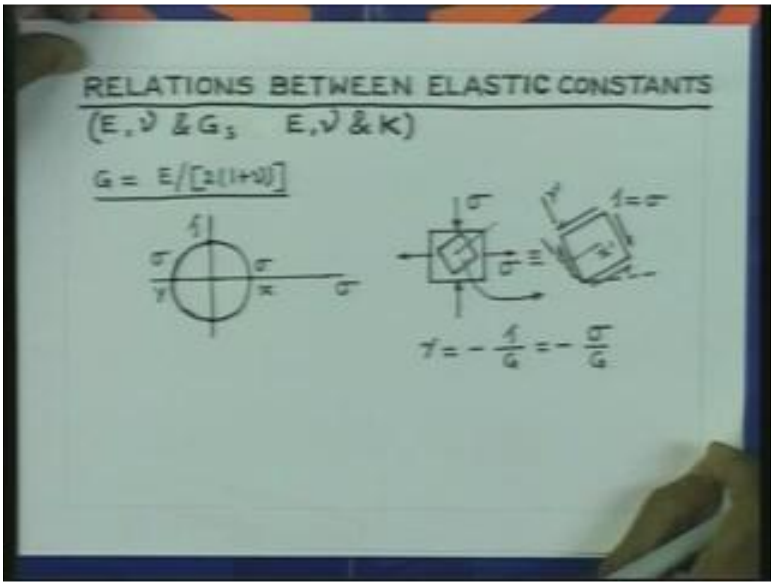

So, first we will try to derive the relationship between G on one hand, E and nu on the other. So, this relationship is of this form, so we would like to derive this relationship. To do this, let us consider a state of stress at a point in a material which is isotropic. This material point is subjected to stresses. Let us say sigma in the x direction and sigma in the y direction, but this is compressive, there is no shear stress. So, therefore, these are principle directions.

Now, if you try to draw the Mohr circle for this case, in two dimensions will have normal stress in the x axis and the shear stress in the y axis. So, now if I consider this given stress, so therefore, we have let us say sigma and sigma. So, you can draw now the Mohr circle this is x axis, this is y axis. So, therefore, the Mohr circle going to be something of this sort in this case. So, therefore, this is the location of x axis, this is the location of y axis.

Now, you see the on a plane which is at an angle of 45 degree with the positive x direction. So, therefore, if you consider a direction like this at an angle of 45 degree. So, on the plane perpendicular to it, we are going to get, no normal stress. And then the shear stress, which is going to be of magnitude equal to this radius which is nothing but sigma in this case.

So, if we now try to draw an element, which is oriented like this, this element. So, we draw this element separately here, this is the element and it is subjected to stresses, this is tau acting there. So, tau acting on this space is nothing but it is in this direction. Similarly, the ((Refer Time: 04:43)) stress here on the normal plane is this and so also this stress here is this. So, let this tau is of magnitude equal to sigma, because this is sigma, this is also sigma, this distance and this distance both are sigma.

Therefore, what we find is that under the action of the loading here, this is equivalent to; this is really equivalent to this loading, which is purely shear loading. And therefore, this shear strain that is going to be coming up. You can see that under the action of this type of loading we are going to get the increase in this angle here. So, this angle is going to increase.

So, if you consider this is the direction let us say x dash and this is the direction y dash. So, the angle between the two directions is going to increase. So, therefore, the shear strain is negative and hence this shear strain is nothing but minus tau by G, which is nothing but, minus sigma by G, because tau is equal to sigma. Now, I can calculate the strain in the direction x and y.

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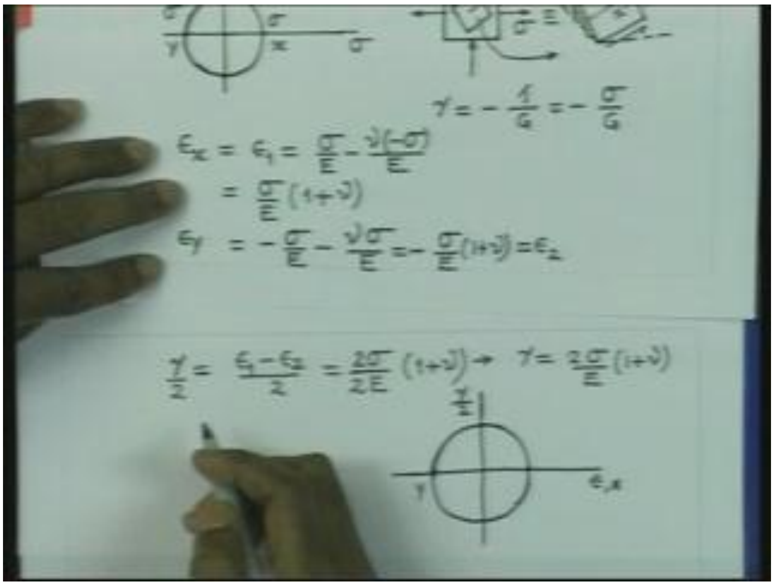

So, epsilon x and in this case, this is the maximum principle stress, this is the minimum principle stress. So, therefore this is epsilon 1 which is nothing but sigma by E or sigma x stress is sigma, sigma by E minus nu times sigma y by E and that minus sigma y is nothing but, minus sigma. Therefore, this divided by E and hence this is equal to sigma by E multiplied by 1 plus nu.

Similarly, if I calculate the strain in the y direction, which is nothing but the minimum principle stress direction. So, therefore, that is going to be given by sigma y is minus sigma. So, minus sigma by E minus nu times, sigma exceeds sigma. Therefore this is nothing but, minus sigma by E into 1 plus nu. So, therefore, these are the two strains. Now, if I calculate the shear strain and I think you should have no difficulty in appreciating that this epsilon y is nothing but the minimum principle strain.

So, therefore, now if we consider the shear strain, maximum shear strain. So, that again gamma is equal to, this is going to be gamma by 2 is equal to epsilon 1 minus epsilon 2 by 2. And this is sigma by E, 2 times sigma by E into 1 plus nu, because this sigma epsilon 1 is this, epsilon 2 is this, therefore this gives us this value. And therefore, again gamma is equal to 2 times, there is one half here, so therefore, this divided by 2. So, therefore gamma is nothing but 2 sigma by E into1 plus nu.

The maximum shear strain, if you see them again Mohr circle for strain, if you draw you will find like this, the Mohr circle would be here in with this normal strain in this direction we plot shear strain by 2. Now, in this case the strain in the, one strain epsilon x is nothing but this, epsilon y is this and therefore, we can draw now the Mohr circle. So, now again the Mohr circle have the similar shape and therefore, this shear strain it is really going to be given by the radius of the Mohr circle and therefore, gamma is again going to be given by this.

Now, I must mention that under the action of this sort of strain, we are going to get an increase in the angle of the, angle between that two principle directions. So, therefore, here this angle you can see that this angle ((Refer Time: 10:18)) between x dash and y dash direction is going to be actually increasing, so this again at an angle of 45 degree. So, therefore, this strain is going to be occurring at an angle of, let me make it clear that this maximum shear strain is going to occur at an angle of 45 degree. Since, this angle is 90 degree it is going to occur at an angle of 45 degree with the x direction.

So, therefore, again you find that the maximum shear strain is going to occur in this direction ((Refer Time: 10:56)) or in this coordinates and again this angle is going to increase. Therefore this shear strain that we have got, it is really a negative shear strain. In fact, this gamma maximum is nothing but minus gamma and that is equal to minus 2 times sigma E, sigma 2 times sigma by E into 1 plus nu .

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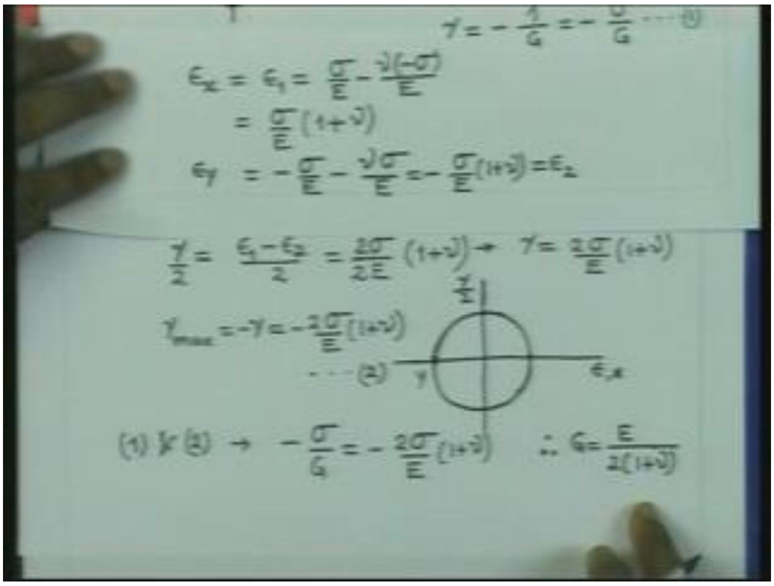

So, here again the strain that we have got is nothing but this. So, therefore, if you like to consider that, this is relation 1 and this is relation 2. So, coupling this relation 1 and 2 what do you get is this, that minus sigma by G is equal to minus 2 times sigma by E into 1 plus nu therefore, G is equal to E by 2 into 1 plus nu. So, this is the relationship that exists between the elastic constants in the case of isotropic material.

Now, this you might be having some doubt about why this has become negative. Now, I must mention that the strain, this strain is going to be on the Mohr circle, Mohr circle whatever is the positive strain or it is shown in an upper half, in physical situation it is going to be negative strain or the angle will increase. So, therefore, thus the region we have taken is negative, that is another logic to take this negative sign. So, that is how we have established that the two or rather three elastic constants, in the case of isotropic material E, nu and G are interrelated. Now, we would like to consider the relationship between K E and nu.

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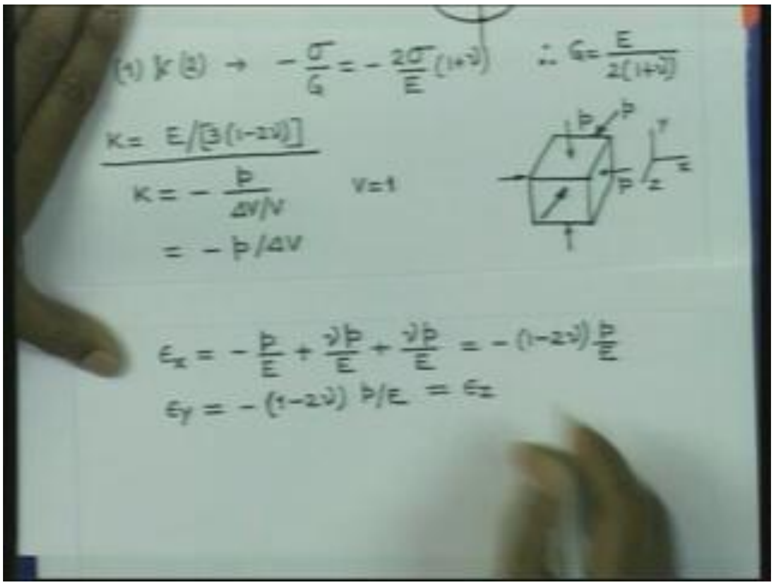

So, therefore, we will now like to establish that, this K is going to be given by E by 3 into 1 minus 2 nu, so we would like to derive this relationship, this K is bulk modulus. So, let us consider a cube, let us consider a cube again our coordinates would be as usual x y and z. Now, let us consider that this unit cube is subjected to hydrostatic pressure and therefore, we have stress let us say in the x direction it is p. So, also in the y direction it is p. So, z direction also we have the same pressure.

So, this is the state of stress on this unit cube. So, now the bulk modulus is defined by minus pressure divided by the volumetric strain. So, therefore, it is nothing but delta v by

v. So, in our case since v is equal to unity, we have taken 1 unit cube. So, therefore, this is nothing but p divided by delta v. Now, if we consider the strain in the x direction, epsilon x it is going to be sigma x by E minus nu times sigma y by E minus nu times sigma z by E.

So, in this case it is going to be sigma x is p. So, therefore, it is p by E minus nu times sigma y by E. So, therefore, it is since sigma y is negative. So, therefore, it becomes nu p by E. So, also for the z direction we will have nu p by E. So, therefore, it is nothing but 1 minus 2 nu p by E that is the strain in the x direction. Now, epsilon y without much further elaboration we can write that we will have the same state of strain in the y direction. So, therefore, it is going to be 1 minus 2 nu into p by E, so also we are going to get epsilon z.

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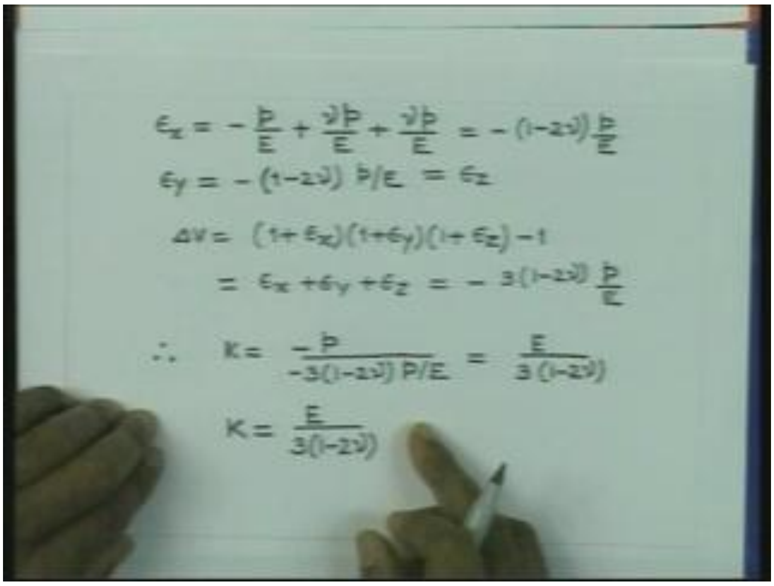

Now, the increase in volume, so if I consider the change in volume it is nothing but new volume is going to be 1 plus epsilon x into 1 plus epsilon y into 1 plus epsilon z minus initial volume is 1. So, this is, since this strains are of small order of magnitude we can write this thing as epsilon x plus epsilon y plus epsilon z and this is nothing but 3 times 1 minus 2 nu into p by E.

So, therefore, that is the change in the volume therefore, we have k, (Refer Time: 18:25) k given by p by delta v. So, therefore, this is nothing but k is equal to minus p by minus 3, 1 minus 2 nu p by E and therefore, this is equal to E by 3 into 1 minus 2 nu. So, the modulus of bulk modulus of elasticity of an isotropic material is also related to the modulus of elasticity and Poisson's ratio by this relationship k is equal to E by 3 into 1 minus 2 nu.

So, therefore, you see in the case of isotropic material, we have only 2 independent elastic constants E and nu, all other constants are derivable from these 2 constants. And in the whole of stress strain relationship in three dimensions, we have only 2 elastic constant to relate the stresses to strains. On the other hand, if it is orthotropic material in three dimensions we are going to have 9 constants relating stresses to strains. And if it is fully anisotropic material then we are going to have 21 constants relating stresses to strains. Now, we would like to consider, calculation of strain energy of deformations. So, we will begin with very simple case.

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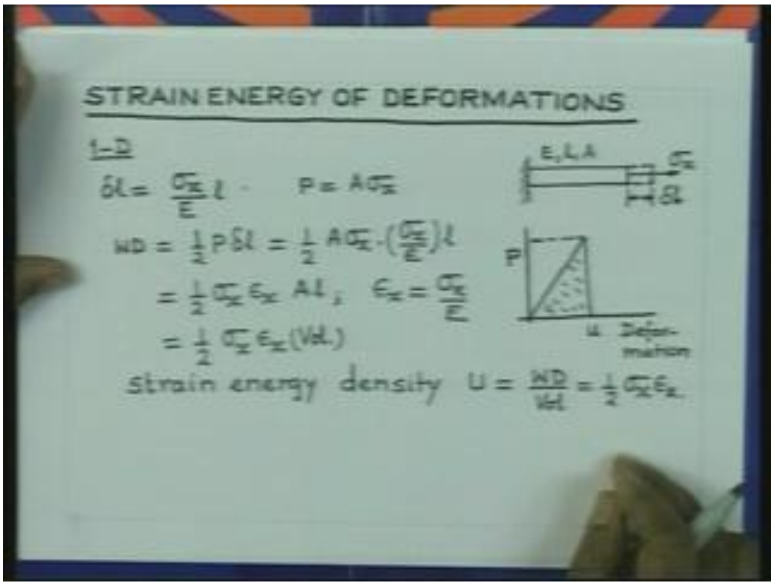

Let us consider, just one dimension. Consider a rectilinear component of modulus of elasticity E, length is l, cross sectional area is A. Now, it is subjected to some stress in the axial direction that is sigma x. The force therefore, acting at the end of this body is nothing but sigma x into A under the action of the load it is going to expand. That means point of application of the force is going to shift by a distance. Let us say that is equal to delta l that is the moment of the point of application of the load and this delta l is equal to it is nothing but sigma x by E is the strain in the component multiplied by l.

So, therefore, this is nothing but strain into the length. Now, this force let us say that force is equal to p which is nothing but A into sigma x. In the elasticity we try to consider that the application of the force is such that it is done quasi statically, thereby indicating that we do not increase the force p suddenly, we go on increasing the load slowly. So, that at every stage there is equilibrium or the system goes through quasi static states of the equilibrium and the load versus the deformation, it varies, so this deformation if you plot it in this direction.

So, this load versus deformation it will varies like this. So, anyhow got to a particular level, the amount of energy that has been put inside the body or the work has been done to the extent given by this triangle. So, therefore, that work done is nothing but if you consider that this deformation is equal to u. So, therefore, the work done is pu by 2. So, that half factor comes up because we do not allow the force suddenly, we apply the force quasi statically.

So, therefore, the work done in this case is nothing but half p into delta l. So, therefore, this is nothing but half p is A into sigma x and delta l you can write delta l to be sigma x by E multiplied by l. And this we can write to be nothing but half sigma x into epsilon x multiplied by A into l, wherein epsilon x is the strain in the x direction and in this case it is nothing but sigma x by E. And this is equal to sigma x into epsilon x into the volume of the body and this work done by the external force gets stored in the body as strain energy.

So, therefore, this strain energy in the body is also equal to the work done by the external force. Now, if we calculate this strain energy per unit volume that will be strain energy density. So, that strain energy density, we represent this thing by symbol U, it is going to be WD, work done by volume and it is nothing but half into sigma x into epsilon x. So, in the one dimension the strain energy density is going to be half sigma x into epsilon x.

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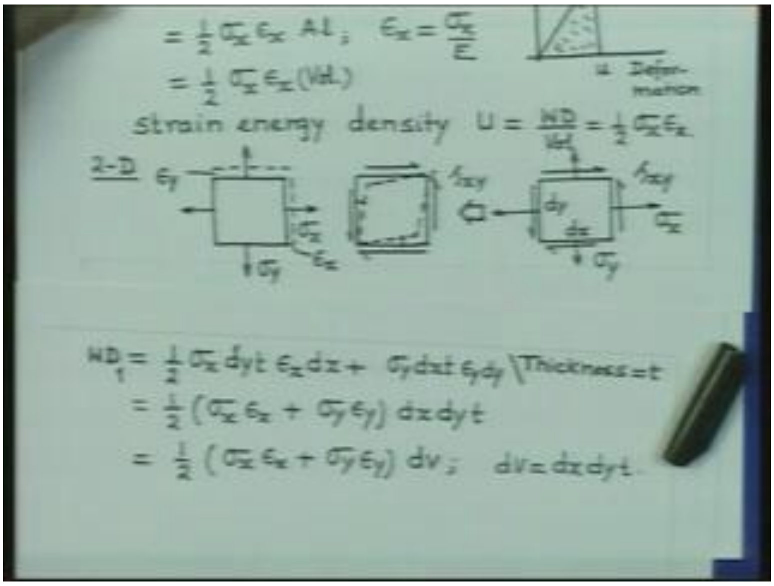

Now, if we consider two dimensions, we will have three components of stresses, sigma x, sigma y then we are going to have shear stresses, which we indicate by tau xy. The two normal stresses, they are going to produce normal strains and this shear stress is going to produce shear strains. Note that this shear stress is not going to produce any normal strain. So, therefore, the work done by the normal stresses and the shear stresses can be calculated separately.

The deformation that is going to come up, if we would like to show that you are going to have under the action of the normal stresses, let us say we will have simply normal stresses here, sigma x, sigma y. And then we will consider the shear strain, rather shear stress here acting. So, we are trying to just consider the loading to be, so on like this under the action of the two normal stresses we are going to have moment. So, also we would like to calculate the energy now.

Let us consider that this dimension is equal to dx, this is dy and thickness let us say that thickness is uniform for this element thickness is equal to t. If we consider that this strains here is epsilon x, the strain here is epsilon y then you can understand that the work done ((Refer Time: 28:53)) by the sigma x stress is going to be the magnitude of the load is sigma x multiplied by dy into t that is the force in this direction and it has moved by a distance of epsilon x multiplied by dx.

So, strain is epsilon x this distance is delta dx. So, therefore, total moment is epsilon x into dx. So, this is the work done and since it is quasi static, we will have this much as the work done. Similarly, we can write the force acting on this phase is nothing but sigma y into dx into t that is the force and the distance by which the moment has taken place, it is epsilon y into this distance dy, so epsilon y into dy. So, therefore, that is the work done by the y force.

So, therefore, let us consider this is WD 1, so therefore by simplifying what we get is that half sigma x epsilon x plus sigma y epsilon y into dx dy into t and this is nothing but half sigma x epsilon x plus sigma y epsilon y into the volume let us say this is dv, wherein dv is equal to dx dy into t. Now, we would like to consider the work done by the shear part or shear stress part, so this is tau xy

Obviously under the action of this shear stress, you will assign that this component is going to distort like this, let us approximately show that this is what is going to be distortion. ((Refer Time: 31:26)) This is what is going to be distorted form of the component. And consider this angle to be its total shear strain is gamma xy, then this is going to be gamma xy, this is going to be ((Refer Time: 31:55)) this is also going to be gamma xy by 2

Now, we can turn this component, whether we can rotate this distorted form in the clockwise direction, then we are going to get that distorted set to be like this, it is going to be oriented like this.

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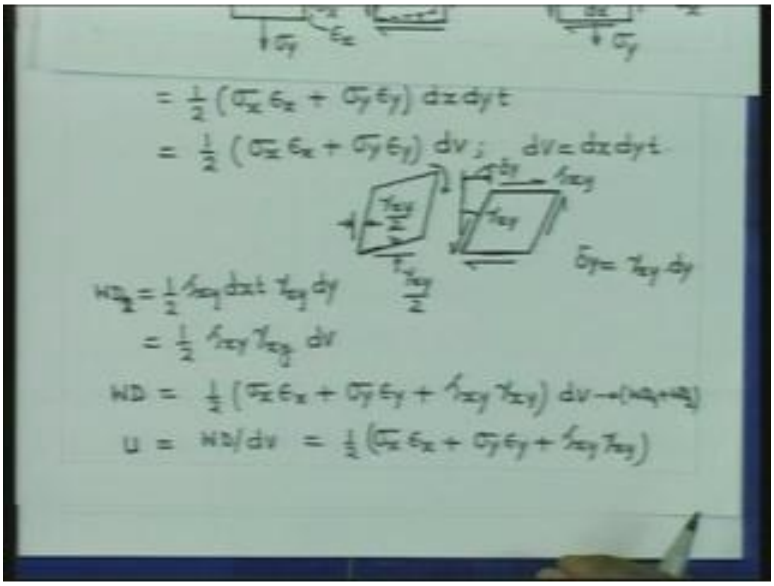

So, if we draw this same component I think, I would like to show that one separately. So, we have this. So, this angle this angle, this is gamma xy by 2, these angle is also gamma xy by 2. Now, by turning this in this direction what we have achieved is that this angle is equal to gamma xy. So, therefore, if we now consider the stresses acting here, this is how the stresses are acting on this element.

Now, we will find that this force which is acting on this stress tau xy, which is acting on this phase which is dxy, the area is dx into t that is shifted by a distance of this much magnitude. So, therefore, this is the shift of the quantum application of this load, which is tau xy into dx into t and if this angle is gamma and this height is equal to dy then this moment is nothing but this moment delta y delta y is nothing but it is gamma xy multiplied by dy.

So, we can write, now this work done due to the shear component of loading is nothing but we have the force which is tau xy acting on an area of dx into t and the displacement is gamma xy multiplied by dy. So, that is the work done, but again it is quasi static. So, half factor will come and if you rearrange this is going to be half tau xy into gamma xy into dv, this would be let us consider WD2.

So, therefore, we have two components of the energy. So, total energy that is going to be their strain energy I mean this work done, total work done is going to be now half sigma x epsilon x sigma y epsilon y plus tau xy gamma xy into dv. So, therefore, the total energy is sum of these two, so which is obtained from WD 1 plus WD 2. And the strain energy density, which is energy per unit volume is WD by dv that comes out to be half sigma x into epsilon x sigma y into epsilon y plus tau xy into gamma xy.

So, that is the form of the strain energy density in two dimensions. I would like you to reflect upon, if the problem is plane strain problem then in that case we are going to have some stress acting in the z direction as well. Then we consider that there is no strain in the z direction therefore, there is no displacement in the direction of z and hence that is going to be no contribution from the stress sigma z and hence the energy calculation in the case of plane strain as well will be given by this formula.

That is why this formula is really true for both plane stress and plane strain, which are both two dimension. Now, you can surely well consider the extension of this consideration to two dimensions, I leave it to you for calculating the energy, strain energy density in three dimensions.

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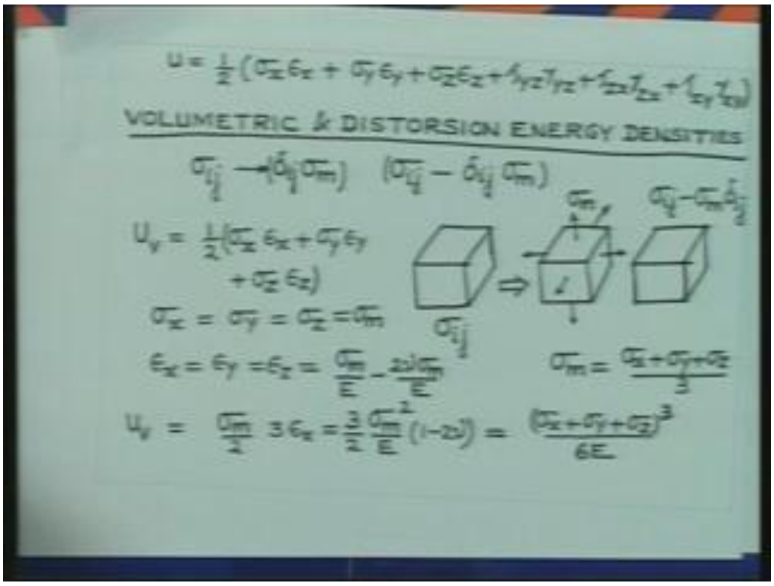

So, in three dimensions you will find that this strain energy density U is going to involve all the 6 components of stresses and strains and it will be like this sigma half into sigma x into epsilon x sigma y into epsilon y plus sigma z epsilon z plus tau yz gamma yz tau zx gamma zx plus tau xy into gamma xy. So, that is the form of the energy in three dimensions. Now, we would like to consider that this total strain energy density can be split into two parts. One corresponds to change of volume and the other corresponds to deformation or distortion of the element.

So, they are volumetric and distortion energy densities. In fact, this corresponds to hydrostatic component or the volumetric component or spherical component of this stress tensor and this distortion energy density corresponds to the deviatory part of the stress tensor. So, we will have the stress at a point, given by sigma ij and this can be split into two parts, one is sigma m the mean stress and the other part is nothing but sigma ij minus delta ij into sigma m.

So, this is the volumetric part and in fact, if you would like to show it as a tensor this will be nothing but delta ij into sigma m. So, that is the volumetric part. So, an unit cube subjected to the stresses. So, let us say that this is subjected to stress sigma ij and this is now we can show equivalently by the stress sigma m, sigma m acting in all the three directions and here in the stress is sigma ij minus sigma m into delta ij. And I am sure you recollect, that this sigma m is nothing but the sum of the three normal stresses divided by 3.

So, therefore, it is sigma x plus sigma y plus sigma z by 3. So, if we want to calculate the strain energy density due to the volumetric part then we will have this Uv. If we make use of the formula for three dimensions, it is going to be half sigma x epsilon x plus sigma y epsilon y plus sigma z epsilon z. In this case, sigma x sigma y equal to sigma z equal to sigma m. So, therefore, this volumetric part we have this and epsilon x is equal to epsilon y equal to epsilon z.

If we consider the material to be isotropic, it is going to be equal and this is nothing but sigma m by E minus nu times sigma m by E minus nu times sigma m by E. So, therefore, it is 2 nu sigma m by E. So, this includes the effect in the other two directions. So, therefore, if we now calculate Uv. So, that is going to be sigma m by 2 into 3 times epsilon x and therefore, it is nothing but sigma m square we can have this constants 3 by 2 sigma m square by E into 1 minus 2 nu.

So, that is the volumetric strain energy density and sigma m is since given by these. We, can write this thing as sigma x plus sigma y plus sigma z by whole cube by it is going to be 6 E. So, that is the volumetric strain energy density. Now, if you are working in terms of the principle stress at a point.

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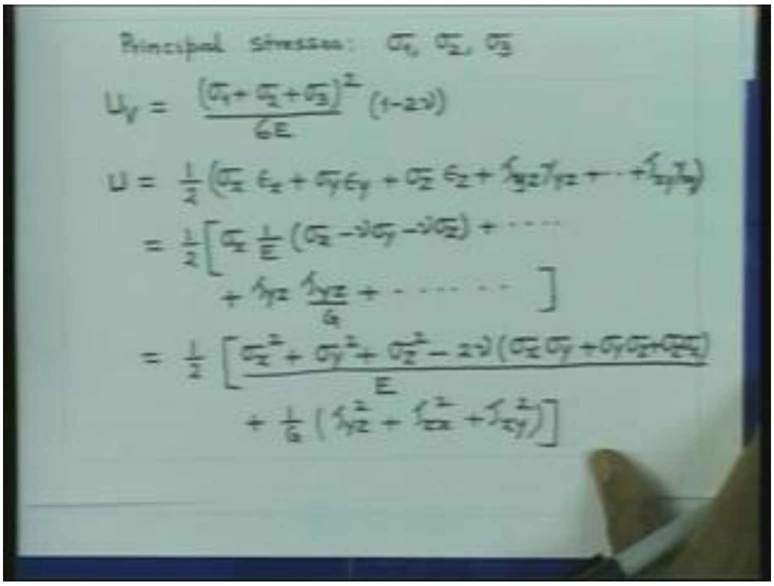

So, if we are working in terms of the principle stresses sigma 1, sigma 2 and sigma 3 then this Uv is going to be nothing but sigma 1 plus sigma 2 plus sigma 3 whole square by 6 times E. Of course, we will have the constants 1 minus 2 nu that multiplier will always remain. Now, the total strain energy density. So, total strain energy density U is due to the stresses sigma ij that we can write, half sigma x epsilon x plus sigma y epsilon y sigma z epsilon z plus the product of the shear stress and shear strains. So, this will be gamma xy.

Now, if I am trying to work in terms of the Cartesian components of these strains then we will have this given by, sigma x multiplied by it is 1 by E sigma x minus nu times sigma y minus nu times z. So, that is the strain in the x direction. So, we will have terms involving all the three directions and the shear stress multiplied by the shear strain is nothing but tau yz by G and we will have also other two components of shear stress and strain.

So, finally, what we find is that this is going to be half it is sigma x square plus sigma y square plus sigma z square 2 times nu sigma x into sigma y plus sigma y sigma z sigma z into sigma x divided by E that is the first part and then we are going to get 1 by G tau yz square plus tau zx square plus tau xy square. So, this is the strain energy, total strain energy density and now if we try to make use of the principle stresses.

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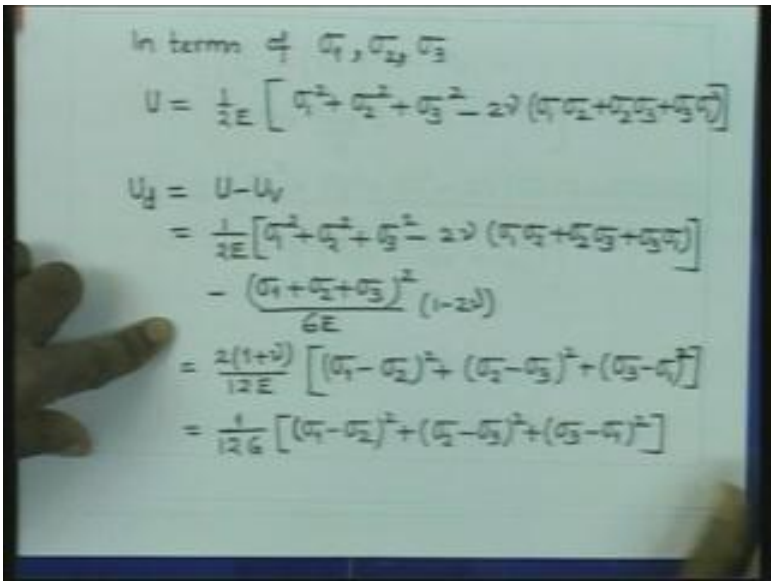

In that case, in terms of the principle stresses sigma 1, sigma 2, sigma 3, the principle stresses at a point we will have these 0. Then in that case your total strain energy density is going to be 1 by 2 E sigma 1 square plus sigma 2 square plus sigma 3 square 2 nu sigma 1 sigma 2 sigma 2 sigma 3 plus sigma 3 sigma 1. So, this is the total strain energy density. Now, if I look into the volumetric strain energy density that was given by Uv. So, therefore if, we now calculate the difference between the two, that will give us the energy density due to the distortion part.

So, that is nothing but U minus Uv and this is going to be 1 by 2 E sigma 1 square plus sigma 2 square plus sigma 3 square minus 2 times sigma 1 sigma 2 plus sigma 2 sigma 3 plus sigma 3 sigma 1 minus sigma 1 plus sigma 2 plus sigma 3 whole square by 6 E 1 minus 2 nu. And if you simplify this, this comes out to be 2 into 1 plus nu by 12 E into sigma 1 minus sigma 2 whole square sigma 2 minus sigma 3 whole square sigma 3 minus sigma 1 whole square.

And since E by 2 into 1 plus nu is G we can write this thing as 1 by 12 G sigma 1 minus sigma 2 whole square sigma 2 minus sigma 3 whole square plus sigma 3 minus sigma 1 whole square. So, this is very important please note this, that we have finally, got the distortion energy density given by this expression and the volumetric strain energy density is given by this relationship.((Refer Time: 52:31)) So, volumetric energy density is given by this expression and this distortion energy density is given by this expression.

You can also see that sigma 1 minus sigma 2 is nothing but related to maximum shear stress. So, this is the maximum shear stress multiplied by 2 in the plane 1 and 2. Similarly, this is the maximum shear stress in the plane 2 3 multiplied by 2, this is the maximum shear stress in the plane 3 1 multiplied by 2 and therefore, this distortion energy density is really due to the shear deformation. Now, I would like you to consider some points for thinking, first thing is that the Mohr circle diagram that we have considered earlier, what will happen to the Mohr circle diagram if the stresses are 0.

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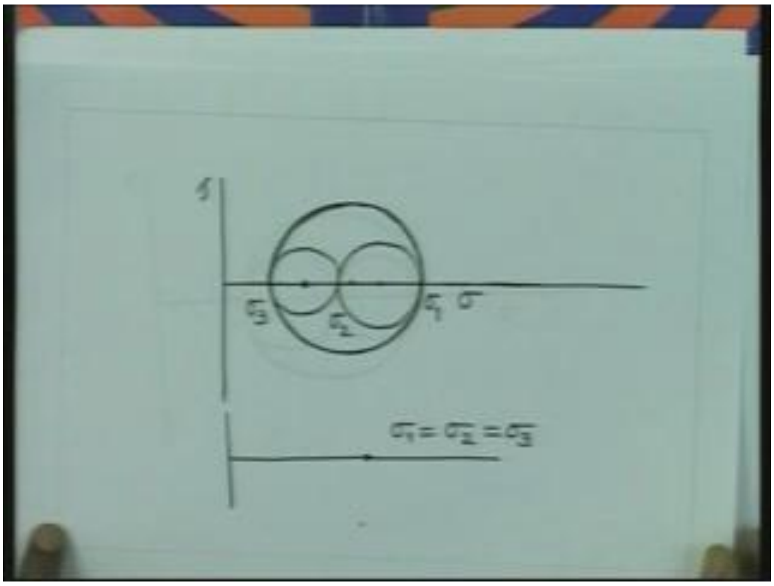

So, let us say that we have the Mohr circle diagram like this in three dimensions. And we have indicated that this is the sigma axis, this is the tau axis and now the stresses are sigma 1 sigma 2 and sigma 3. So, these are the three principle stresses. If the three principle stresses are equal then Mohr circle is going to be one point, so the case, wherein sigma 1 equal to sigma 2 equal to sigma 3 that becomes a point.

And therefore, any three directions can be taken as principle directions and it is good to consider three orthogonal directions, mutually orthogonal directions as the principle directions. Now, other important point that I would like you to also note that on this Mohr circle diagram, the two directions the direction sigma 1 is given by this radial direction, sigma 2 is given by this radial direction and they are 180 degree apart. As in this case of two dimensions, you find that the two principle directions are 180 degree apart.

Similarly, here the direction sigma 2 is given by this direction, direction sigma 3 is given by this direction they are 180 degree apart. It is again that the angle, the physical angle gets doubled up in the Mohr circle diagram. Similarly, if you consider that this is the centre of sigma 1 and sigma 3 this centre, this is your direction of sigma 1 this is direction of sigma 3 again they are 180 degree apart. This is one point, please reflect upon then other point I would like you to also consider, that the change in the volume, change in the volume due to the deviatory part of the stress is 0.

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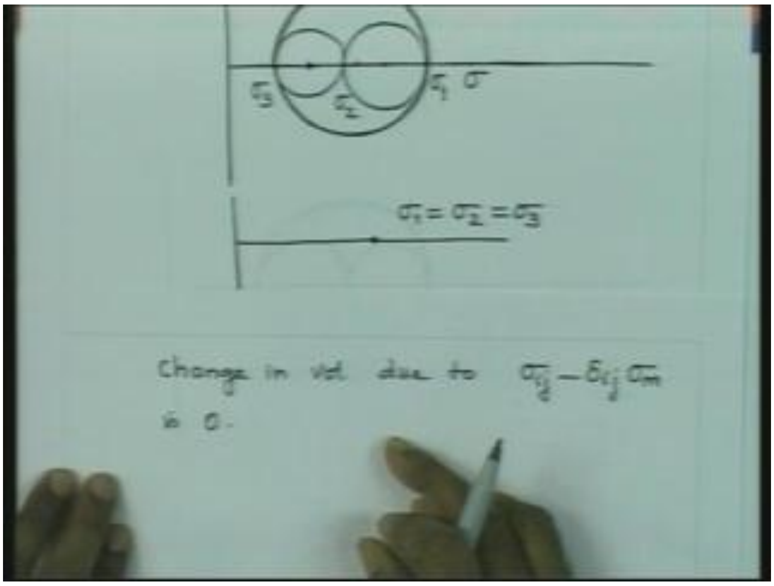

So, therefore, the change in volume, in volume due to deviatory part that is sigma ij minus delta ij sigma m is 0. So, I would like you to derive this relationship. It is very simple that change in volume is due to the normal stresses and in this case three normal stresses are going to be sigma x minus sigma m sigma y minus sigma m sigma z minus sigma m.

And therefore, you calculate the change in the or rather strain in the three Cartesian direction add them up, and we will find them this change in the strains or rather that the change in volume which is nothing but change the sum of the three strains are going to be 0. So, please I want to repeat that change in the volume is nothing but some of the three strains in the Cartesian direction and that is going to add up to 0. So, change in the volume due to the deviatory part of the stress component is 0.