Advanced Strength of Materials
Prof. S. K. Maiti
Department of Mechanical Engineering
Indian Institution of Technology, Bombay

Lecture – 10

In this lecture, we would like to consider equilibrium equations in 2-D polar coordinates. In the last lecture, we have derived equilibrium equations in the rectangular polar coordinates. So, in this we will consider the derivation for the 2-D polar coordinates. These are going to be useful for geometries, which are circular.

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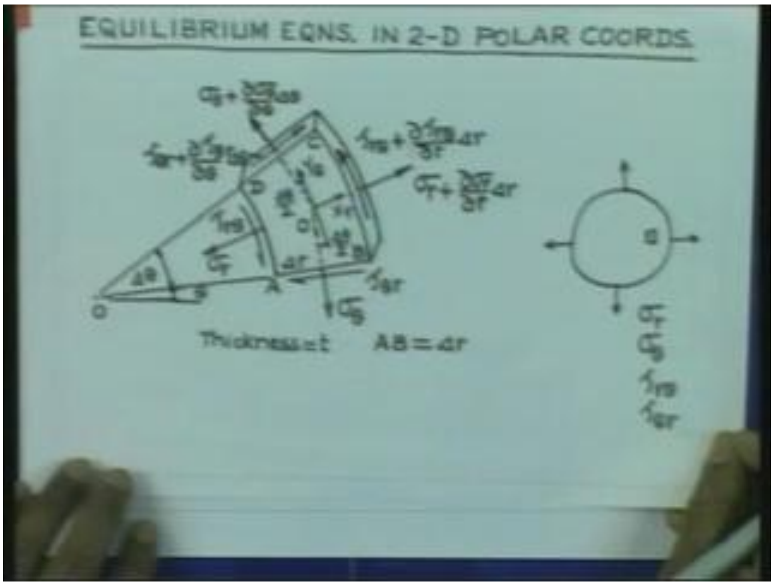

So, under the action of the loading into our disk, it is undergoing between undergo the deformation. And the loading is going to maintain the equilibrium of the whole body. And therefore, at any point there is also going to be we will consider the small element. That is going to be also in equilibrium. Therefore, we will try to consider the small element, here round the point. And try to see, what are the equilibrium conditions needs? So, this element is drawn to a larger scale would like to write to a larger scale.

Let us consider the element to be this one. This is the element. Let us, take the reference theta determinant to be this. So, therefore, let us say that this angle is equal to theta. Then center is located there. At the other end of the element, this is dispose with respect to the earlier age by an angle delta theta. So, this is the center. Now, you would like to consider that this distance that the element be A, B, C, D.

Let us, again assume that the thickness is equal to t, which is uniform all over. And this distance A, B is very small distance represented by delta r. The stresses that are going to be present in this case are sigma r sigma theta and tau r theta tau theta r. The phase here, A D here is that is outer normal is directed in the minus r direction, because r is increasing in this direction that is positive direction. So, outer normal is heretic minus r direction. Therefore, the stress which is going to be the normal stress which is going to be positive is like this that is sigma r.

And the shear stress on this phase will be represented as tau r theta. So, this will be tau r theta. And since it is in negative phase. The positive shear stress will act in the theta negative direction. Theta increases in this direction; we will consider increasing in the anti clockwise direction. So therefore, this is the negative theta direction theta. So, therefore, this tau r theta is positive. Similarly this ((Refer Time: 05:56)) phase, it is outer normal is going to be this one, which is negative theta direction. So, therefore, the stress on this phase which will be considered to be positive, it must be directed in the negative theta direction.

So, it will be indicated in sigma theta. Shear stress theta which is act on this phase, it is acting on theta phase. Therefore, it will be indicated by tau theta r. And since it is negative phase, the positive stress will be directed like this. So, these are the stresses which are going to be coming on the two phases A B and A D. If we consider that the stresses are the continuous functions of r and theta within the body.

Then, we can write the stress on the phase B C starting from the stress acting from the phase A B. We can for example, we can write the normal stress acting here, which is nothing but this is the radial direction r. Therefore, the stress which is going to be the normal to the phase, which is nothing but sigma r. Now, I can write this stress here with respect to this one by the tailor expansion. And hence the stress will be equal to sigma r delta sigma r delta r into delta r. So therefore, this is the stress on the phase B C in the normal direction.

Similarly, the shear stress which is going to be positive on the stress, it must act like this. And it is value can be retained from tau r theta by the tailor free expansion as tau r theta delta tau r theta delta r multiplied by delta r. Now, the stresses on the stress this is theta positive phase, it will be one component is normal stress. And it must be positive in this direction theta increasing direction.

And the shear stress which is theta r, it must be acting like this. And therefore, these two components written from the sigma theta and tau theta r and you can write this to be sigma theta plus delta sigma theta delta theta delta theta plus tau theta r plus tau theta r delta theta multiplied by delta theta. So, that is the stress tau theta r plus delta tau theta r delta theta delta theta.

Out of all these, we also have body forces and this body forces will like to represent by the components in the radial direction by the X r and the tangential direction by y theta. And let us say that, the center of this element is o. A B is delta. So, let us show it here. Now, we would like to consider the equilibrium equations in the radial direction and tangential direction. And at the same time, we would like to consider the equilibrium moment equilibrium at the point o.

Now, let us do this construction. If you consider this direction to be local tangential direction and this angle A D making and the center is delta theta. Then, it is obvious that this angle is going to be delta theta by 2. So, also this angle is going to be delta theta by 2. So, these two angles are same.

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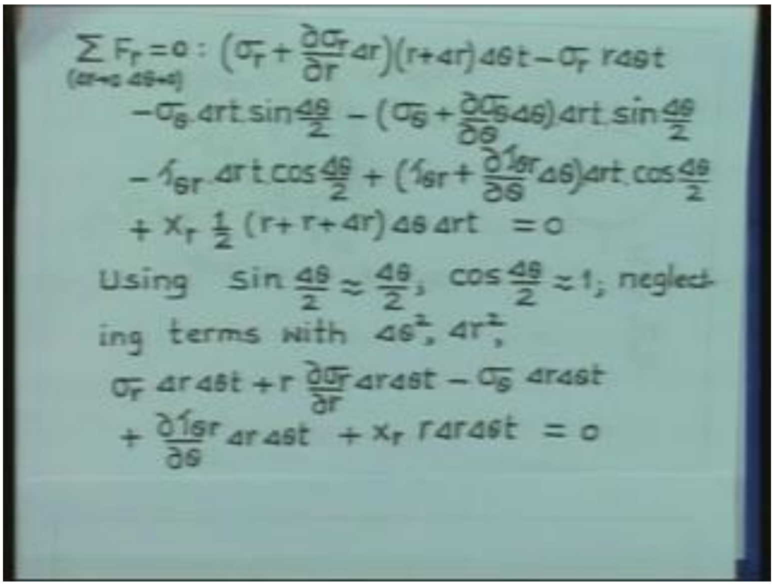

Now, let us consider the equilibrium equation in the direction r. So let us consider the equations. Summations of the forces, in the radial axis are 0, and this is of course in the understanding this delta r is tending to 0 delta theta is also tending to 0. Now, if you consider the forces that are going to come out here.

We will have the radial direction or the continuation in the radial direction coming up from sigma r plus delta sigma r delta r delta r and this sigma r. And the area of the case is nothing but r plus delta r multiplied by delta theta to thickness. And this area here is nothing but r delta theta into thickness. So therefore these two forces, we can write as sigma r delta sigma r delta r into delta r multiplies by r plus delta r delta theta into t minus sigma r into r into delta theta into t.

Now, you are going to get also this sigma theta acting in this directions is also going to give the component in the radial direction. So therefore, that will be sigma theta acting on this area which is delta r into t. And it is sine component sine delta theta by two components in the radial direction. Similarly this stress acting on this area delta r into t it is sine component will be in direction. It is sine component will give us the component in the radial direction.

So therefore, we can write now the two components to be like this. Sigma theta acting on the area delta r into t sine delta theta by 2 and theta is acting on the negative r direction. Similarly, the other one sigma theta plus increment to sigma theta multiplied by delta r into t into sine delta theta by 2 that is also acting on the negative r direction.

Now, we can consider the contributions of tau theta r, tau theta r acting on the area of vector into t. And each component in the radial direction will be cosine delta this angle is delta theta by 2. Therefore, cosine component cosine delta theta by 2 will be then component in the radial direction. And it will be negative r direction and this component is similar. It is acting on the area delta r into t.

And its cosine component will be acting on the radial direction. So therefore, we can now write these two components as follows tau theta r tau theta r into delta r into cos delta theta by 2 plus tau theta r plus increment of tau theta r acting on the area delta r into t cosine delta theta by 2. So therefore, these are the contributions from the boundary stresses.

Now out of all these, we are going to also have the body force. And this body force magnitude, you can find out from the total volume, that total volume we can take to be this distance is r delta theta this distance is r plus delta r into delta theta. So therefore, average of that B C and A D, we can take. We can multiply that delta r into t plus the volume multiplied by this x r that will give us the force. And therefore, if you write that it comes to be X r half r plus delta r delta r into delta theta into delta r t equal to 0.

So, this give the total contribution and if you now try to make use of the fact that sine delta theta by 2 equal to approximately delta theta by 2 for small angle delta theta. Similarly, cos delta theta by 2 equal to 1 and neglecting terms which are going to have involvement of delta theta square delta r square, we find that the there is a great simplification possible.

You can see here that are going to get the contributions like this. Here, in some calculation will be there. So, we have sigma r into delta r into t plus r delta sigma r delta r delta theta t minus sigma theta delta r delta theta t plus delta tau theta r delta r delta theta t plus X r, r delta r delta theta t equal to 0. And this can be further simplified by dividing both sides by r delta r delta theta into t.

We get from this term delta sigma delta r. And from this term, we get one by delta tau theta r delta theta plus sigma r minus sigma theta plus X r equal to 0. So, this is the equilibrium equation in the radial direction. And it is the analog of the equilibrium equation in the X direction that you have the right for the rectangular coordinates.

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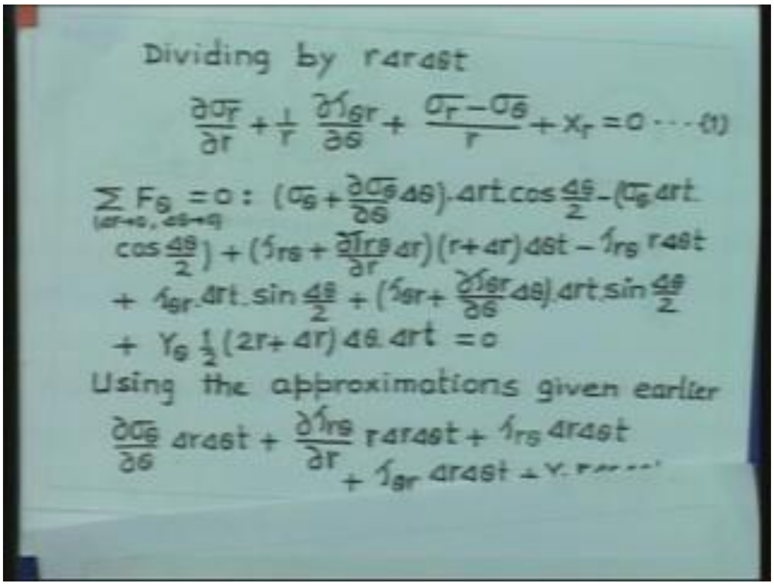

Now, we will consider the equilibrium equations in the Tangential direction. So again, we write some of the forces in the theta direction are 0. And assuming that delta r is very small tending to 0. So also, delta theta is very small. So, delta r is tending to 0. So therefore, if you consider the argument similar to, what we have done earlier.

First of all, we will get the component in the sigma theta direction to be due to this component. This component and it is cosine component that will give us the component in the tangential directions. So therefore, it is acting on the area of delta r into t and it is cosine delta theta by two components will give us the component in tangential direction.

So, therefore, sigma theta delta sigma theta delta theta acting on area delta r into t theta cosine delta theta by 2 that gives us the component in the tangential direction and it is in the positive tangential direction. And this one acting on the area of delta r into t again and its cosine component will give us the component in the negative theta direction. So therefore, we have now the component here sigma r did not come sigma theta into delta r into t cosine delta theta by 2.

Similarly, if you consider the contributions due to this component end this component. They will come out to be this acting on the area of into t. And it is in the angle of delta theta by 2. And therefore, if we take the sine component of that it will give us the component in the positive theta direction. So, we have written here tau r theta plus delta tau r theta into delta r. I am trying to talk about this component.

So, if you try to take this component it is tau r theta delta tau r theta delta r into r. That is acting over the area of r plus delta r into delta theta into t. And this component which is nothing but tau r into theta acting on an area r into delta theta into t. Similarly, if you try to consider this component which I have already told you earlier that this going to give us tau theta r acting on an area delta r into t.

And it is sine component will give us the component in the positive theta direction. So, sine delta theta by 2 will be the component in the positive theta direction. Similarly, this one acting on an area delta r into t. It is component will give us the component in the positive theta direction. So therefore, since this vector is acting like that.

So, it will have component in the direction. Similarly this vector will have component in the positive theta direction. So, they are added. And then finally, we will have the contribution due to the body force. And that body force y theta multiplied by the volume here calculated earlier. And therefore, this going to be given by y theta half 2 r plus delta r into delta theta delta r into t equal to 0.

Now, dividing again using the approximations of the type that you have considered earlier you find that. This relationship gets simplified to this form delta sigma theta delta theta delta r into delta theta into t, plus delta tau r theta delta r delta r into delta theta t plus tau r theta into delta r delta theta into t, plus tau theta r delta r delta theta into t, plus y theta r delta r delta theta into t equal to 0.

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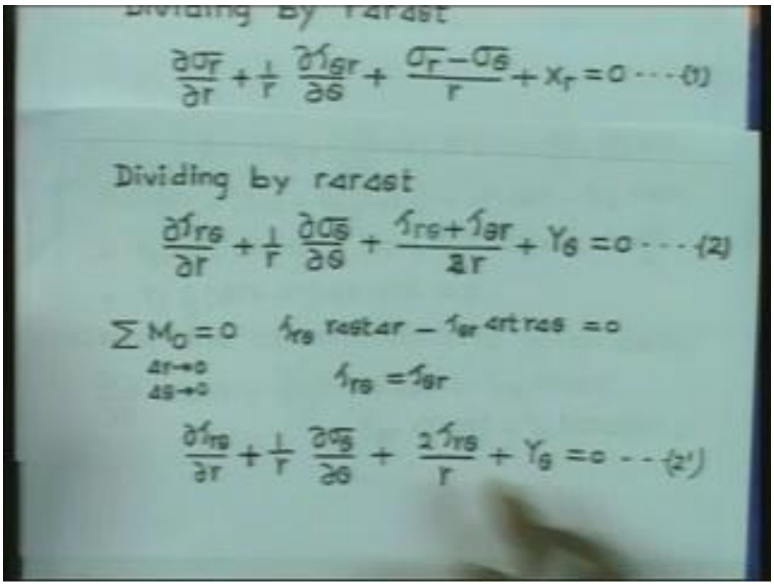

Again you can make use of the division of both sides by r delta r into delta theta into t and that gives us delta tau r theta delta r plus 1 by r delta sigma theta delta theta plus tau r theta plus tau theta r by 2 plus y t equal to 0. So, this is the equilibrium equations equilibrium equations in the second dimensions. So therefore, we have now two equilibrium equations for polar coordinates.

Now, we consider the moment equilibrium equation. So for that, we again get back to our configuration. We take the moment about the point O equal to 0. And we have the picture that delta r is very small. So also, delta theta is very small.

You say again the distribution of sigma r plus delta sigma r delta r delta r such that the resultant acting on the phase at point O. So, as the resultant to sigma will pass through this point. Similarly, resultant of sigma theta and this component will pass through this point and the forces due to x r and y theta are also going to pass through this point. So therefore, the momentum is really going to be due to this shear component acting on the two radial face and these two shear stresses acting on the tangential phase.

Now, we are talking of a case, where in the delta are and delta theta is very small. We can have simple calculation that since delta r and delta theta are almost 0, we can forget about this component. So therefore, we have stresses like tau r theta acting on the phases whose area to be taken to be the minimum whose area can be taken by this average spread of the t k r which is nothing but you can take to the r. We can take to the r into delta theta into t. So therefore, this force is acting on the area of r delta theta into t that is the area on which it is acting. And then, the distance between the two phases is equal to delta r.

So therefore, the moment that is going to come up due to this r theta component is r theta acting on this area r delta theta into t multiplied by delta r. That is acting on the anti clockwise direction. Similarly, if you now consider that the stresses acting on this two phases are tau theta r only neglecting this part of the increment. And if I consider the distance between the two to be divided by r into delta theta then we have tau theta r acting on the area of delta r into t.

And the distance between the two phases is approximated at r into delta theta. And this movement will be acting in the clockwise direction. So therefore, we have now resultant to be tau r theta into r delta theta into delta r minus tau theta r into delta r into t into delta theta equal to 0 that is the moment equilibrium equation. And this comes out to be in simplified form that tau r theta is equal to tau theta r.

So, what you find finally, that the equilibrium equations of the moment gives us the condition that the shear stresses acting on the two orthogonal phase they are going to be equal. So, the shear stress acting on this phase on this phase they are going to be equal. So to sum up, the equilibrium equations here in rectangular polar coordinates we have the first equation in the radial direction given by this one. And tangential direction, this is the equation and moment equilibrium equation gives us tau r theta equal to tau theta r.

Now, you can see that since tau r theta is equal to tau theta r. It will be also proper to write delta tau r theta delta r plus 1 by r delta sigma theta delta theta plus twice tau r theta. I think I have made some mistake here, this should be r. This equation should be r. This should not be two this should be r. So therefore, this is twice tau theta r by r plus y theta equal to 0. So, that is the form of the second equation. So, we have now this is the first equation this is the second equation in equivalent form moment equation this one.

Now, we would like to consider certain stresses in 2-Dimensions which can be specified uniformly by three non-zero component of stresses sigma x sigma y and tau x y. These stresses are termed as plane stress and plane strain conditions.

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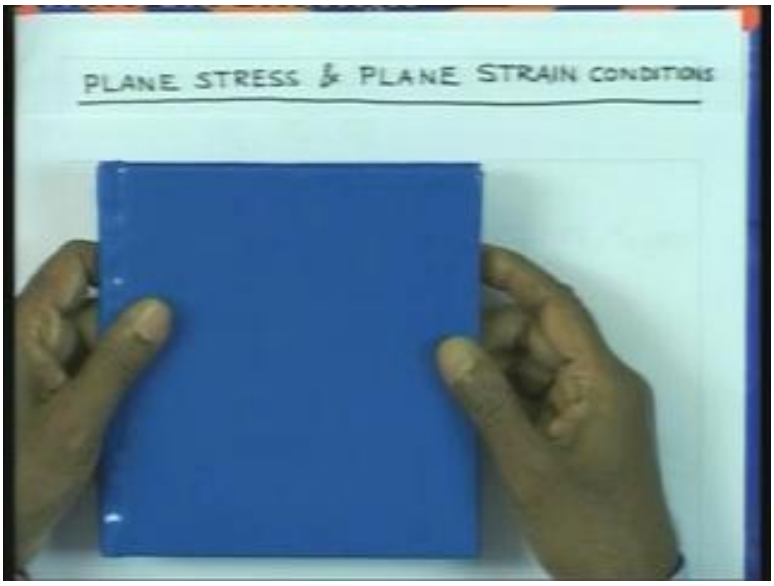

If you consider the thin object like this and if you have the loading of the object at time around its boundary by forces, which are at it is boundary. Therefore, it is loaded in its own plane. It could be loaded like this or it could be loaded like this or may be in combination. So, the loading is in the plane of the body. If you consider this is x direction this is y direction.

And the vertical directions is the z direction that z phase is free of any stresses. Therefore, sigma z is 0 tau z x and tau z y is also 0. That is state prevails in the face here top face. So also, the bottom face the same condition is going to prevail. And now, we can consider that at any intermediate point. Since the thickness is very small the same state will prevail.

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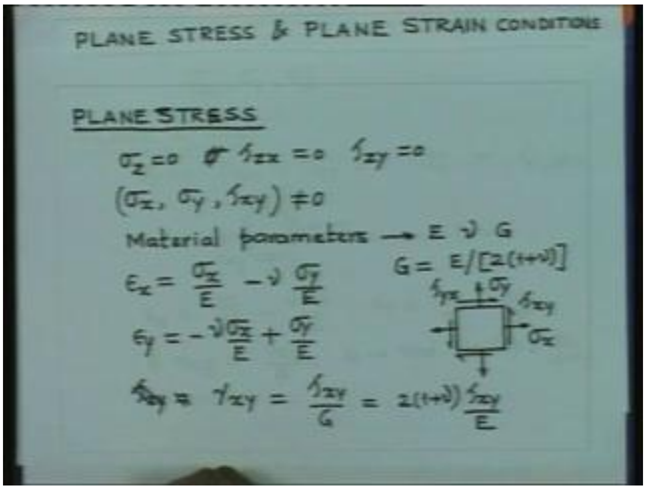

Therefore, the stresses which are going to be 0 are nothing but sigma z equal to 0 and sigma z x equal to 0 and z y equal to 0. These three components are 0. The non-zero components are going to be sigma x sigma y and tau x y. So, these are the three non-zero components. And this particular state of stress is known as plane stress. So, this state of stress is known as plane state of stress. Let us find out the relationship between the stress and strain in this case.

Let us introduce the material parameters E as the modulus of elasticity, nu as the Poisson's stress and G as the modulus of rigidity. And this G modulus of rigidity is related to E and nu by this relationship G equal to E by 2 into 1 plus nu. Whenever the component is loaded with a direction that is going to strain in the direction and that is going to be coupled with the strain in the other two orthogonal directions. And that in fact, is optimal to the Poisson’s ratio.

So, if you now consider the element of the thin plate which is in the state of plane stress that element is going to be subjected to the stresses like sigma x in the x direction sigma y in the y direction and tau x y and y x as the shear stresses. If I know consider the effect of these stresses, we just want to calculate the, if you want to calculate the same amount of the stresses.

First let us like to see, the strain in the x direction. We consider the material to be isotropic. So, that there is no shear there is no shear stress produced by the normal stresses. And therefore, epsilon x is going to be due to the only stress sigma x and sigma y. Due to sigma x, we have the stress in the x direction, sigma x by E and this is going to be causing the strain in the y direction which is nothing but minus nu times the strain in the x direction. So therefore, epsilon l y due to this stress is going to be nu times sigma x by E.

Similarly, if I consider the sigma y stress, it is going to produce the strain in the y direction sigma y by E. So, it is going to be sigma y by E. And this is going to produce a strain in the x direction which is nothing but minus nu times sigma by E. So therefore, this strain is going to be sigma y by E along the x direction. So therefore, along the x direction simultaneous action of sigma x and sigma y, we are going to have strain in the s direction epsilon x sigma s by E nu times sigma y by e and epsilon y equal to nu times sigma x by e plus sigma y by E.

And the shear strain, it is going to produce only by the shear stress. So therefore, the shear strain will be represented by gamma x y, it is nothing but tau x y by G. And this is nothing but 2 into 1 plus nu multiplied by tau x y by E. So, these are the three strain components in the case of planes stress. And if you involve this relation, you try to calculate the stresses in terms of strains using these three relationships. You get a relationship which is of this form sigma x sigma y tau x y is equal to E by 1 minus nu square this matrix multiplied by epsilon s epsilon y gamma x y. So, this gives you the relationship between the stress and strain in the case of plane stress.

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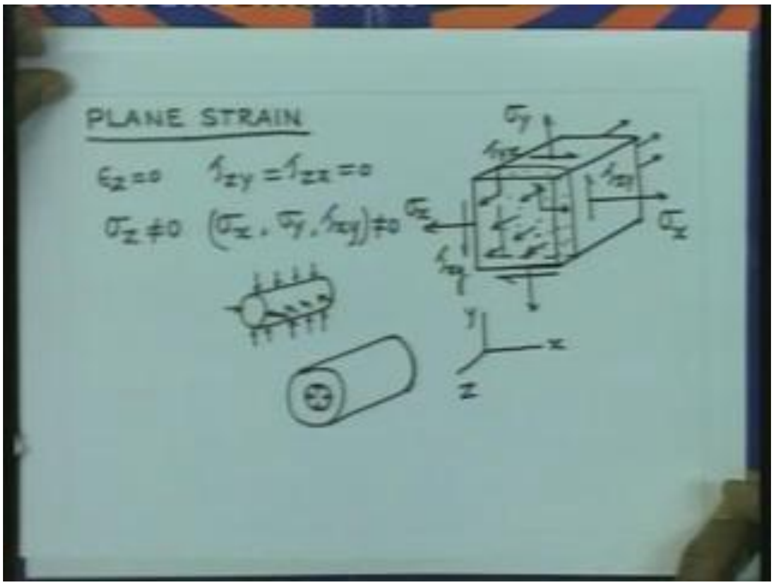

We will consider the other condition which is plane strain condition. If you consider the thick plate and it is if you now have the load of this component by loading like this. We will have sigma x sigma y. And we can also have the shear stresses. And the bottom phase, we can also have the stresses, of course this side we have sigma x and this is tau x y. So, these are the stresses on the phases.

Now, another action of these stresses, if you consider our coordinate are original by x y and z. Under the action of stresses of the four faces, we might have the collision of contraction in the z direction. And if we prevent this contraction by a plane forces in the z direction, so putting the load in the z direction to prevent this contraction.

Then in that case, what you are going to observe is that, all the points which is on the plane of x y will remain in the same plane and if we simply move in the x and y directions. So therefore, if we consider a plane, if we consider a typical plane, this plane which is parallel to x y axis. All the points in the plane will just move in the x and y direction, only and this state of stressing termed at plane strain condition.

Now, in this case you can; obviously, see that, we have epsilon z equal to 0. And we have no shear stress acting on the face. Therefore, we have tau z y and tau z x equal to 0. And we have sigma z non-zero. And therefore, the stresses again are also non zero are sigma x sigma y and tau x y. So, these are the other stresses which are non zero. So in this case, it is the state which is characterized by epsilon z equal to 0 and sigma x sigma y and tau x y are 0 not zero.

So again, it can be describes by sigma x sigma y and tau x y. You can also have a case wherein you can allow a certain degree of contraction in z direction. So therefore, that will also be a case of plane strain. So, which in the two cases possible, where epsilon z equal to 0 and epsilon z equal to constant. These are both termed as plane strain condition. We would like to discuss this case with epsilon z equal to 0.

We will show you the stress sigma z although it is non zero. It is going to dependent on sigma x sigma y at t point. And therefore, this is the case of 2-Dimensional case, where in sigma x sigma y and tau x y are unknown. This sort of situation is going to come up when you consider a bearing roller bearing.

In the case of roller bearing peak of the roller which is loaded in the radial directions it is loaded all along x curved surface by radial forces if you all along the length the radial forces acting. When it is loaded like this, then each plane of this roller will undergo plane strain condition.

Similarly, if you think of a hollow pipe line, which is subjected to internal pressure will find that. Each cross section of that pipe line will undergo deformation which is more or less conforming to plane strain condition. We would like to find out the relation between the stress and strain in such situations.

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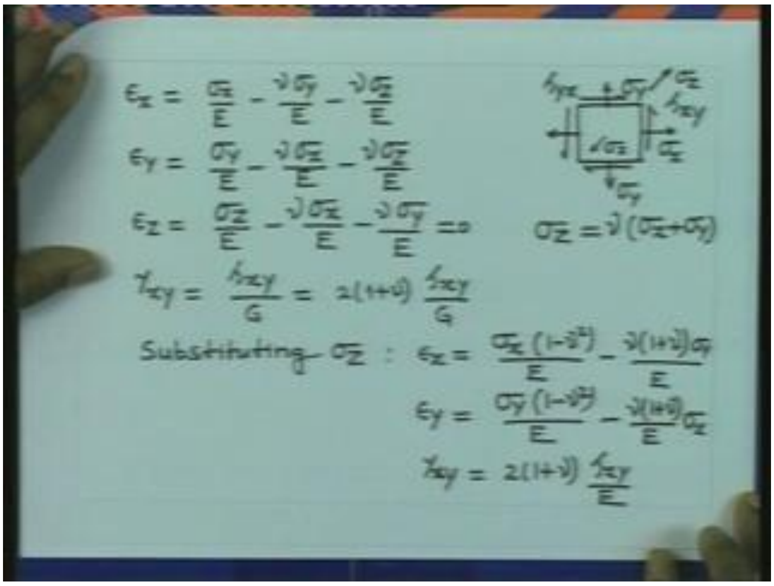

So, a typical element here is going to be subjected to stress like sigma x sigma y. And then, we have shear stress tau x y and tau y x. And also, we have the stress acting in the z direction which is sigma z. The normal stresses will produce normal strain and the shear stresses will produce the shear strain. That is going to valid, if we assume the material to be isotropic.

And therefore, now epsilon x is going to be produced due to sigma x. And it will also receive contributions sigma y sigma z due to Poisson's ratio effect. We can write now epsilon x to be equal to sigma x by E minus nu sigma y by e minus nu sigma z by E. Epsilon y is going to be equal to the sigma y by E minus nu times sigma x by E minus nu times sigma z by E, where nu and E are Poisson's ratio and modulus of elasticity respectively.

Similarly epsilon z is equal to sigma z by E minus nu sigma x by E minus nu times sigma y by E. And since, we have epsilon z equal to 0 for plane strain condition. This gives us sigma z is equal to nu times sigma x plus sigma y. So, in a plane strain state sigma z stress is related to sigma x and sigma y. Therefore, there are only two normal stresses are unknown in this particular case. And this shear strain gamma x y is going to be given by tau x y by G. And since, G is related to e we can write this as two into 1 plus nu tau x y by G. So, these are the relationship in the k sub plane strength. We can replace sigma z by these two equations and that will give us.

So substituting, sigma z we have epsilon x is equal to sigma x into 1 minus nu square by E minus nu into 1 plus nu sigma y by E. Similarly, epsilon y is equal to sigma y into 1 minus nu square by E minus nu into 1 plus nu by E into sigma x. And we have gamma x y as 2 into 1 plus nu tau x y by E. These are the relations in the case of plane strain and if we try to find out the stress in terms of strains. The three components of stresses in the in terms of strain will be as follows. And the plane strain conditions sigma x sigma y and tau x y are nothing but E by 1 plus nu into 1 minus 2 nu into this matrix multiplied by epsilon x epsilon y and gamma x y.

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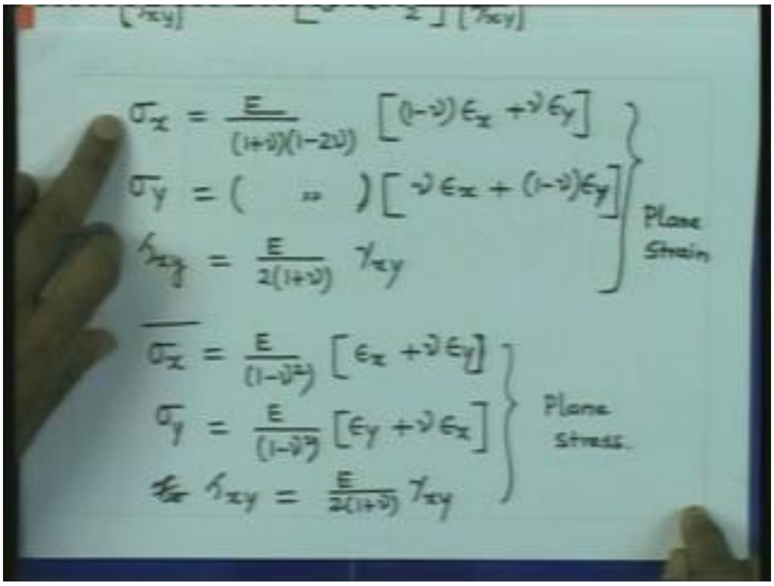

Now, if you try to write these relations in the expanded form you are going to get sigma x is equal to E by 1 plus nu 1 minus twice nu into 1 minus nu epsilon x plus nu times epsilon y. Similarly, sigma y is equal to the same constant will. So therefore, we will just write this constant. Due to and this is nu times epsilon x plus 1 minus gamma into epsilon y.

And shear stress tau x y is equal to E by 2 into 1 plus nu multiplied by gamma x y. So, these are the relations for plane strain. And if you look into the relationship that we had in the case of plane stress, you can again write the same expression for plane stress. It is going to be sigma x is equal to E by 1 minus nu square into epsilon x plus nu times epsilon y.

And sigma y is equal to E by 1 minus nu square epsilon y plus nu times epsilon x. And shear strain gamma x y at it become shear stress tau x y is equal to E by 2 into 1 plus nu gamma x y. So therefore, these are the relations for the plane stress. So just to summarize, what we have talked here, that in the case of plane stress. We have only three non-zero components of stresses sigma x sigma y and tau x y.

Similarly in the case of plane, strain too we are going to have three non-zero components stresses. This is going to be seen in situations like, where we have thin plate like objects loaded in the own plane. And this is going to be plane strain condition is going to be seen in situations, where object is very long. And the loading is having some continuity and symmetry in the axial direction.

Then in that case, in each plane there will be plane strain movement only all the planes are going to move in the same plane. And therefore, the situation is again characterized by the three stresses. And the strains in the case of plane stress are in z direction that is going to be non zero, where as in case of plane strain it is going to be 0 in the z direction.