Convective Heat and Mass Transfer

Prof. A. W. Date

Department of Mechanical Engineering

Indian Institute of Technology, Bombay

Module No. # 01

Lecture No. # 08

Similarity Soln to Temp BL -1

In the last two lectures, we dealt with similarity method for the velocity boundary layer. In lecture 6, we developed the similarity equation and in lecture 7 we obtained solutions to the similarity equations.

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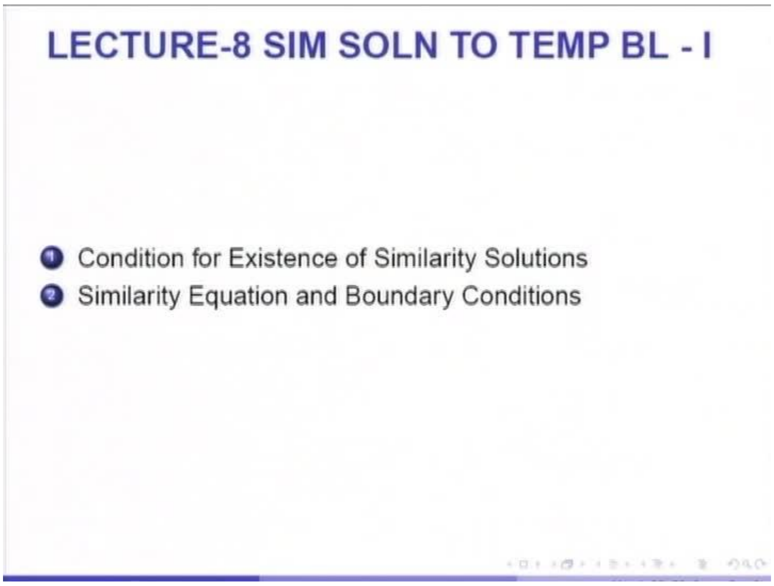

Today, we will deal with similarity method for the temperature boundary layer or the energy equation and like before we shall first establish the condition for existence of similarity solutions to the temperature boundary layer equation and then derive the boundary conditions.

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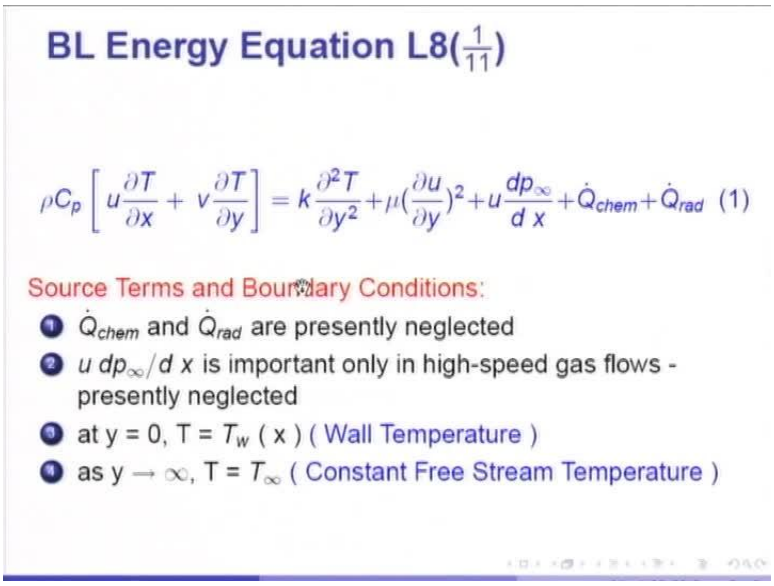

As you will recall, the boundary layer form of the energy equation has convective terms on the left hand side, the diffusion term in the transverse direction, the viscous dissipation term, the pressure work term and two heat generation terms - one due to chemical reactions; the other due to radiation. For our purposes, presently we shall ignore Q dot chem and Q dot rad. As we said earlier, u dp infinity by dx is important only in high speed gas flows of compressible or compressible gas flows. Therefore, the last three terms are neglected because we are largely dealing with uniform property in compressible flows.

The boundary conditions to this equation would be at the wall - y equal to 0- T will equal T wall and that may or may not be a function of x; at y equal to infinity, T will be equal to T infinity and is assumed to be constant in the free stream. It does not vary with x.

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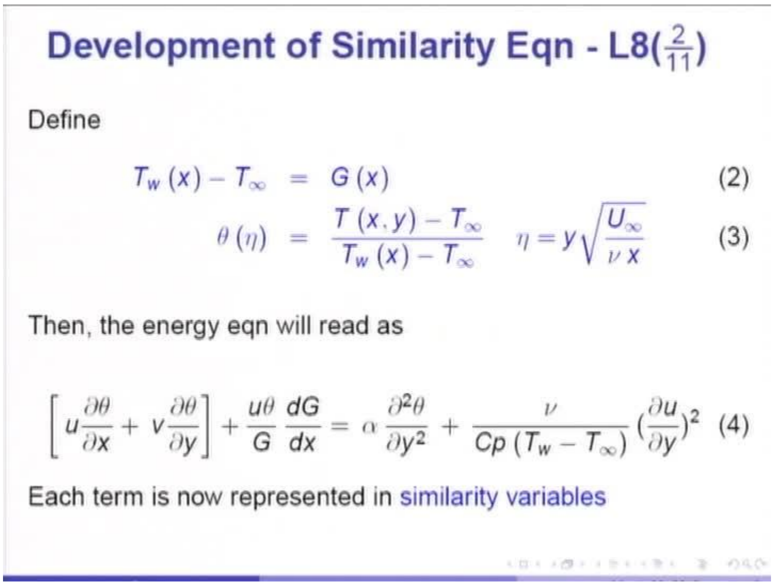

If you want to develop similarity equation, then first of all, we define T w x minus T infinity - that is the reference between wall and infinity states - as sum function G x. We define theta eta, where eta is the similarity variable introduced in the velocity boundary layer equation equal to T at any point in the boundary layer minus T infinity divided by T wall x minus T infinity.

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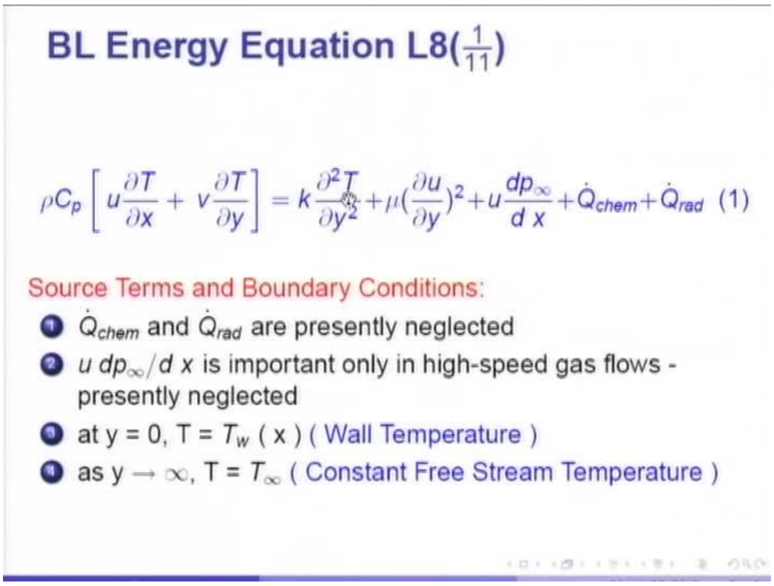

And eta as you will recall equals y into under root U infinity by nu x. If I make the substitutions of these two quantities in the differential equation, you will notice that I am going to replace T in these three terms by theta.

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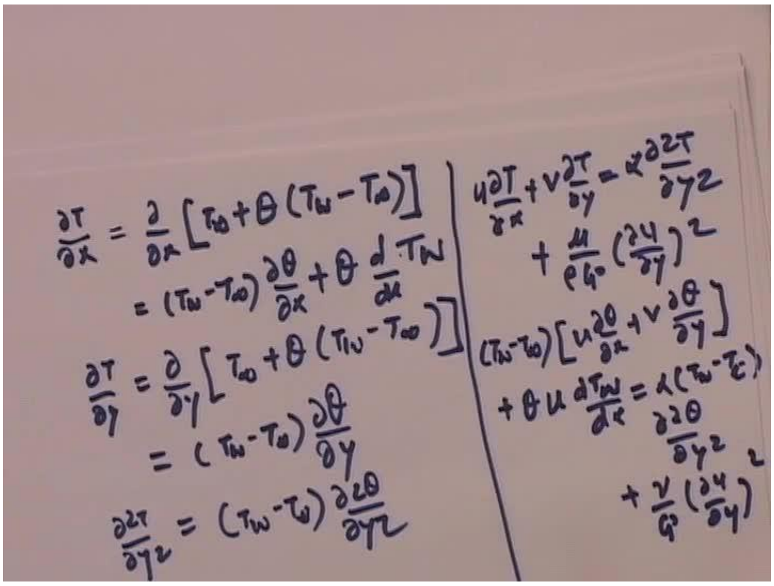

You will notice that dT dx will equal d by dx of T infinity plus theta into T w minus T infinity. Since T infinity is constant, it does not contribute to derivative. Therefore, I would get T w minus T infinity d theta by dx plus theta into d by dx of T w, because T w is a function of x. dT dy; likewise, d by dy of T infinity plus theta into T w minus T infinity and that was simply reduced to T w minus T infinity d theta by dy. Likewise, d2T dy square will simply reduce to T w minus T infinity d2 theta by dy square.

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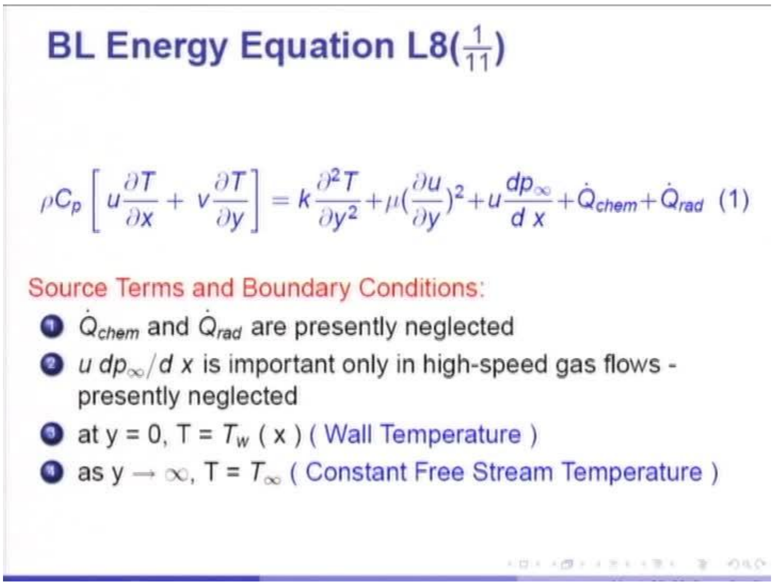

Therefore, upon substitution, you will see the equation would become dividing through by rho C p, I will get u dt by dx plus v dT by dy equal to alpha d2T dy square plus mu by rho C p du by dy square. If I substitute these three derivatives here, I will get T w minus T infinity into u d theta by dx plus v d theta by dy plus theta into u dT w by dx equal to alpha into T w minus T infinity into d2 theta by dy square plus nu by C p du by dy whole square. That is what I will get.

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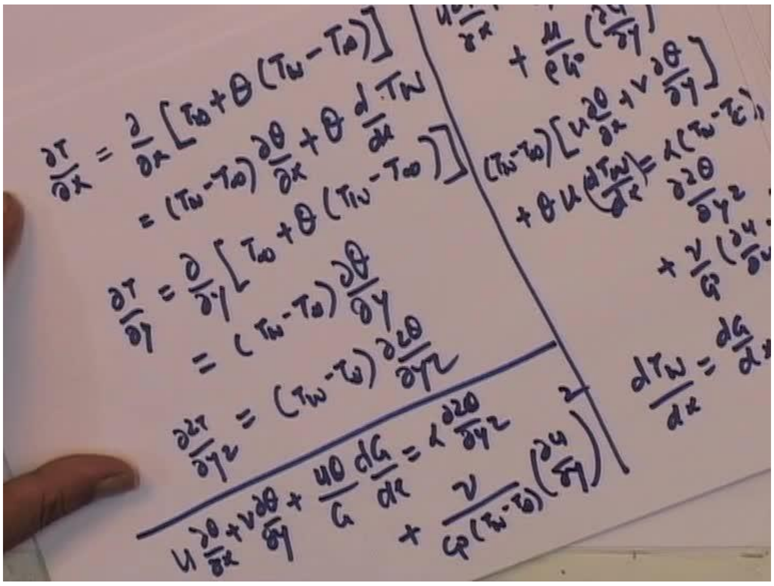

Notice dT w by dx is nothing but dG by dx. T w minus T infinity is nothing but G. Therefore, if I divide through by G, I will get u d theta by dx plus v d theta by dy plus u theta by G dG by dx equal to alpha times d2 theta by dy square plus nu by Cp T w minus T infinity du by dy whole square.

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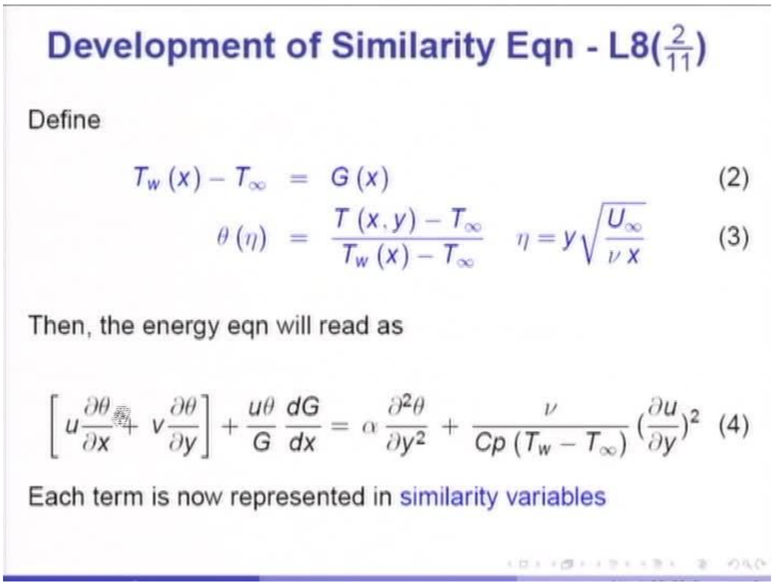

This is the last equation that you see here. We now wish to represent each term in terms of similarity variables.

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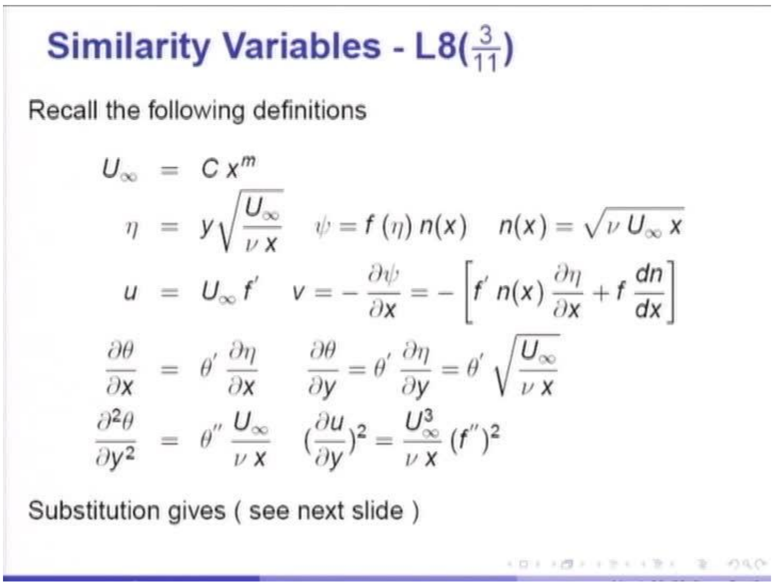

Recall that for the velocity boundary layer, the similarity variables are U infinity equal to C x raise to m, eta as we saw before is - y into square root of U infinity divided by nu x psi - is f eta into n x a function of x. n(x) itself is under root nu U infinity x. We define u over U infinity equal to f dash. Therefore, u will become equal to U infinity f dash. v which is equal to minus d psi by dx will become minus into f dash into n x into d eta by dx plus f into dn by dx - follows from this definition of psi.

d theta by dx would be simply theta dash into d eta by dx, where theta dash is d theta by d eta. d theta by dy would be equal to theta dash d eta by dy and d eta by dy would be simply under root nu U infinity x. d2 theta by dy square would be d theta double prime U infinity nu x into du by dy whole square equal to f double prime square nu x.

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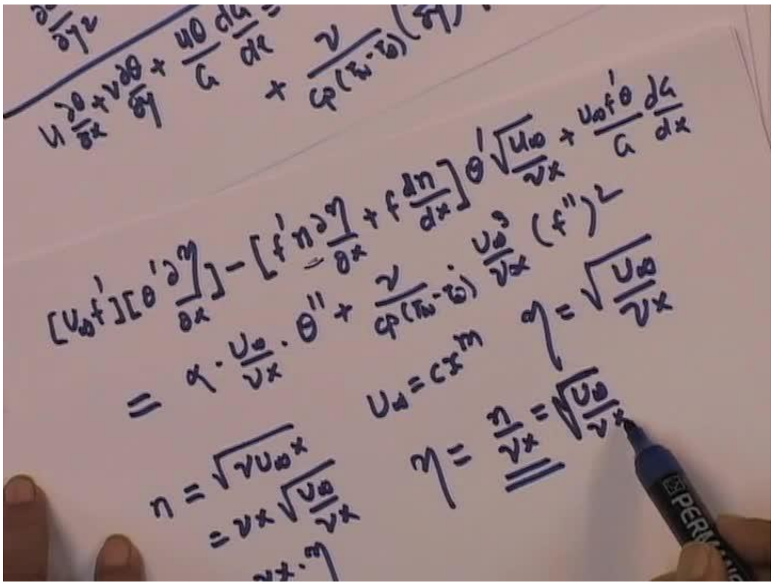

So, you will see that if I now substitute for u d theta by dx and so on and so forth in this equation, then I would get U infinity into f dash into d theta by dx which is d theta - this is theta dash - d eta by dx minus f dash n d eta by dx plus f dn by dx into d theta by dy which is theta dash into under root U infinity by nu x equal to alpha times -I must include - u theta - u is equal to U infinity f dash theta divided by G dG by dx equal to alpha times U infinity by nu x into theta double prime plus nu by Cp T wall minus T infinity into U infinity cube by nu x into f double prime whole square.

It is very clear then. Now, notice that dn by dx, since n is equal to under root nu U infinity x, U infinity is equal cx raise to m and eta is equal to under root U infinity by nu x. I can show that eta is equal to n by nu x which is equal to under root U infinity under n divided by nu x which will be under root U infinity nu x.

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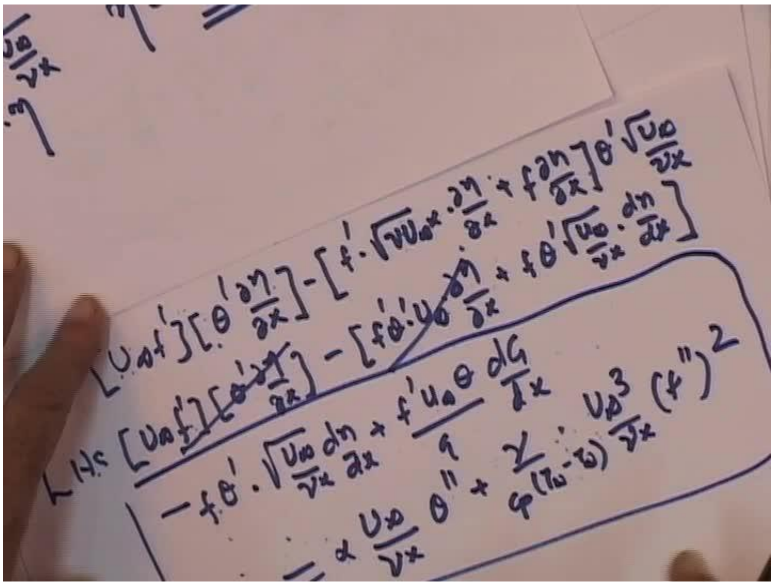

Remember, if I were to replace this n by n nu x U infinity by nu x, then you will see that I get U infinity f dash into theta dash d eta by dx minus f dash. n is equal to under root nu U infinity x d eta by dx plus f dn by dx into theta prime under root U infinity by nu x. Therefore, you will see this becomes U infinity f dash into theta dash d eta by dx minus f dash into theta dash U infinity d eta by dx plus f theta prime under root U infinity by nu x dn by dx.

Therefore, you will see this term gets canceled with this term. I have now minus f theta dash - this is the left hand side- f theta dash under root U infinity by nu x dn by dx plus f dash U infinity theta by G dG by dx equal to alpha U infinity by nu x theta double prime plus nu by Cp T w minus T infinity U infinity cube by nu x into f double prime whole square. This would be the equation.

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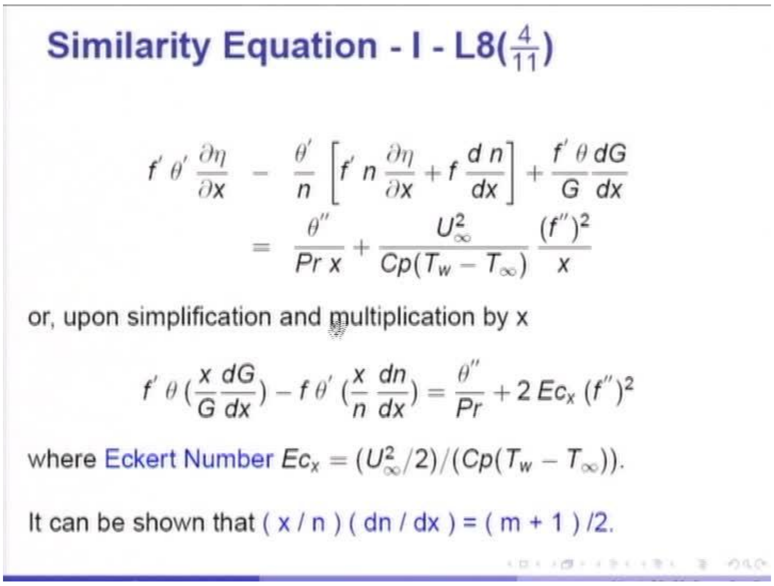

That is what I have shown on this figure. You will see these two terms now get canceled. U infinity by nu x, as you can see is simply n over nu x, so, if I divide through now by U infinity by x in this equation, then you will get minus f theta prime into under root nu x by U infinity dn by dx plus f dash. The equation I have written here is quite correct.

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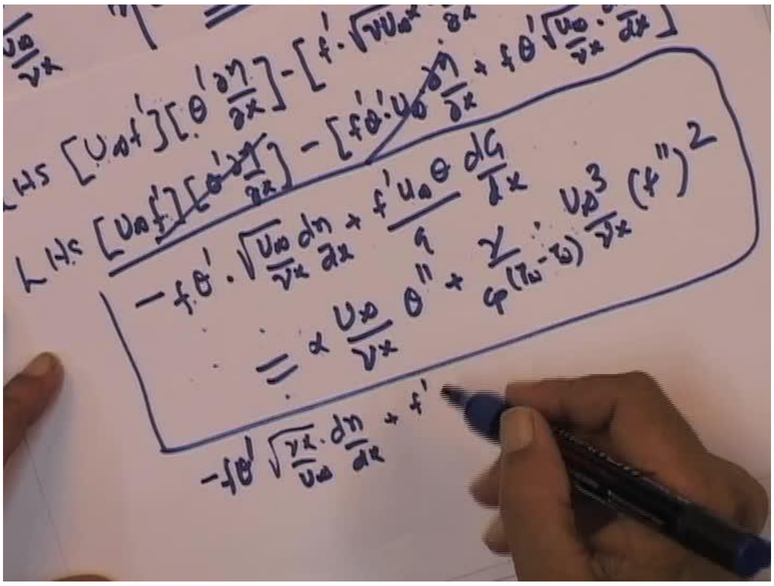

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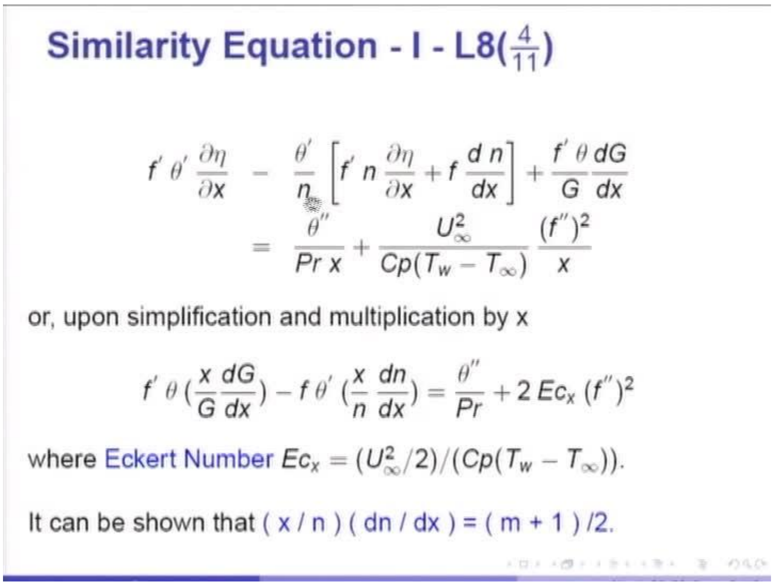

I hope you will check the algebra. n and n will get canceled and this term gets canceled with that one and minus theta dash n f dn by dx is equal to all this. f dash theta and this will become U infinity squared because I have divided through by this. So, this will go and this will go. I will get U infinity by a nu x is n over nu x, so that is why u get n here and divided by n and that is how this and this term gets canceled. You then have this term, f dash theta d divided by Gx and multiplying through by x, I get f dash theta x dG by dx minus f theta dash x over n dn by dx equal to - that is this term - theta double prime divided by Prandtl plus 2Ec x f double prime square where Eckert number Ec x is defined as U infinity square divided by 2 divided by Cp into T w minus T infinity.

Now, to show this x over n dn by dx equal to m plus 1 by 2, that is a straight forward thing, in the sense that - remember n is equal to under root nu U infinity into x, but U infinity is equal to cx raise to m. Therefore, this becomes nothing but c times nu x raise to m plus 1 and that is equal to under root c nu x raise to m plus 1 by 2 -that is –n. dn by dx therefore, will be under root c nu into m plus 1 by 2 into x raise to m minus 1 by 2. Therefore, x divided by n dn by dx will be equal to x divided by under root c nu x raised to m plus 1 by 2 into under root c nu into m plus 1 by 2 into x raised to m minus 1 by 2.

So, you get this gets canceled with that; this is x raise to 1, so, this becomes x raise to minus 1 by 2. So, this this and this gets canceled and you simply get m plus 1 by 2

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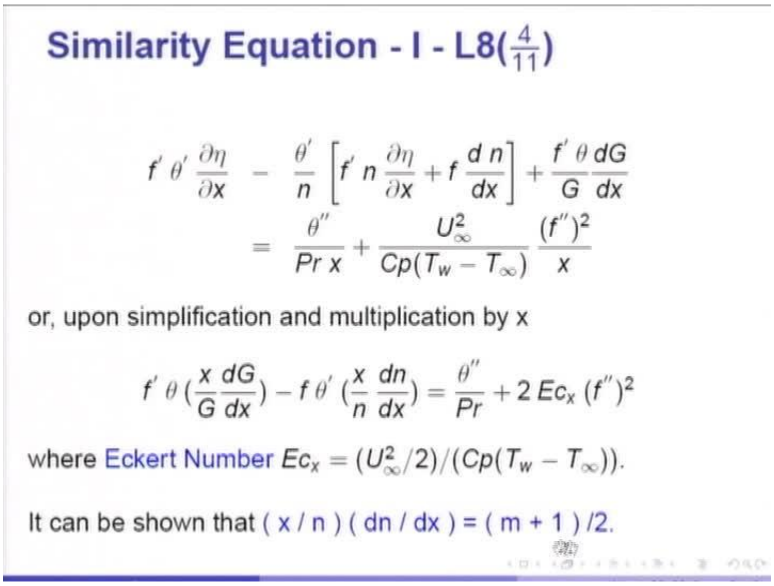

That is what is shown here. This is the pressure gradient parameter m and therefore, our equation would become like this - I have transferred all terms on the right hand side and multiplied through by Prandtl number, then you will see our equation becomes theta double prime plus Prandtl into m plus 1 by 2 f theta dash minus f dash theta x by G dG by dx, which is the wall temperature variation of a function plus 2 Ec x f double prime square equal to 0.

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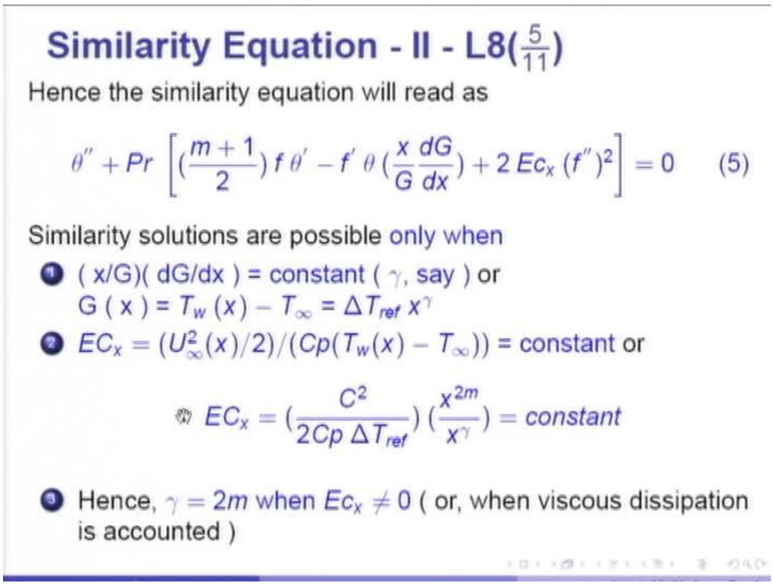

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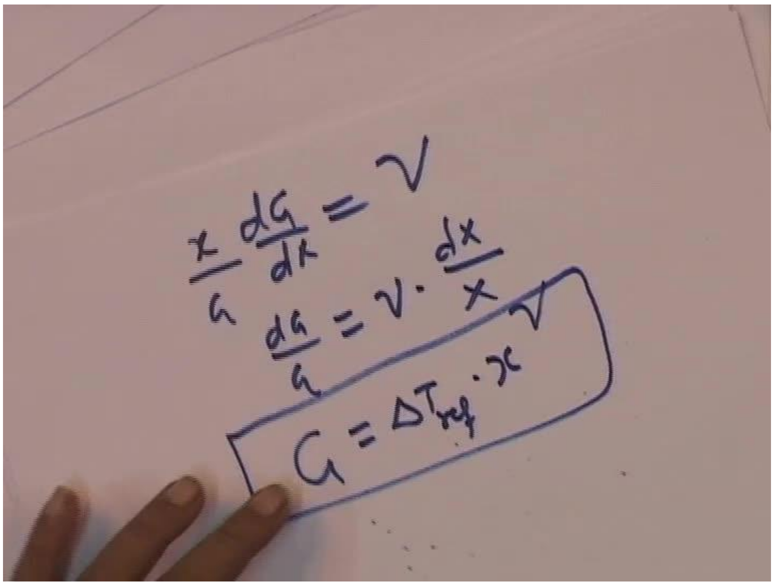

As I said before, this equation will be a perfectly ordinary differential equation; if this quantity x divided by G dG by dx is a constant. Let us say, it is gamma. Then, the solution to G will be simply sum delta T ref x by G dG by dx equal to gamma gives me dG by G is equal to gamma times dx by x. The solution therefore, is G times some constant delta T ref into x raise to gamma.

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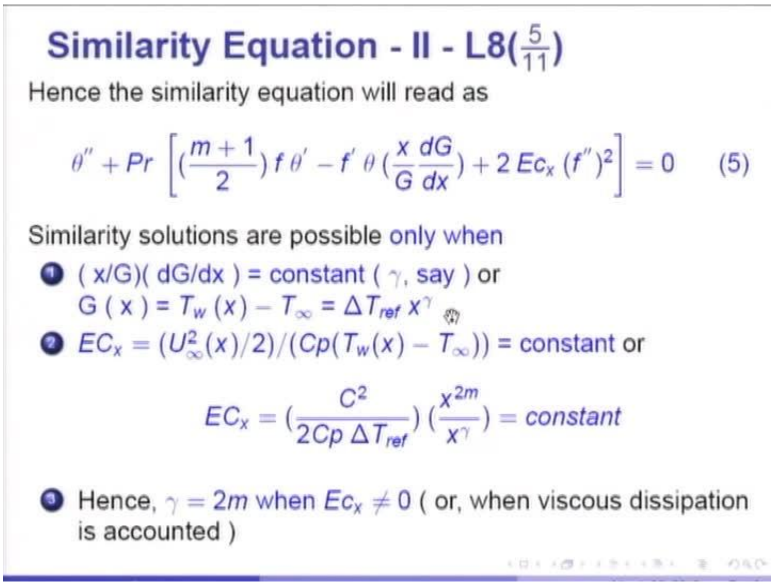

Unless the wall temperature variation is of this form, gamma is an arbitrary constant. Then, similarity solutions will be possible. Likewise, Ec x should also be a constant. But what is Ec x?

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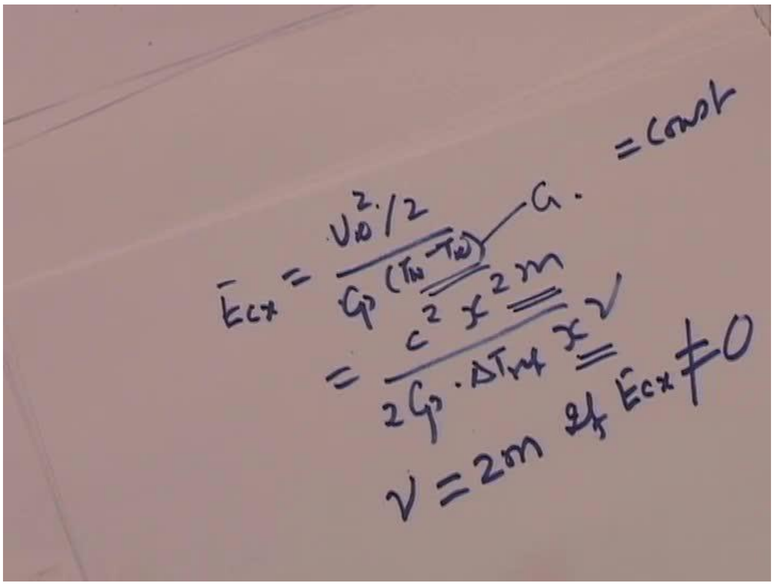

Ec x is U infinity squared by 2 divided by C p into T w minus T infinity and this is equal to c squared x raise to 2m divided by 2 times C p - and this is - delta T ref x raised to gamma. G is T w minus T f. Therefore, Ec x will be constant, if this is constant then this and this must cancel which means gamma must equal 2m, if Ec x is not equal to 0.

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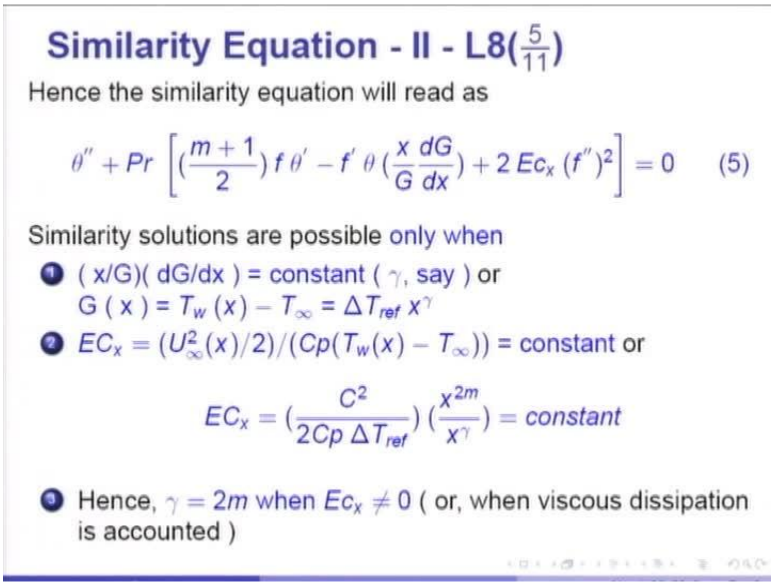

Remember similarity solutions to the temperature boundary layer are possible when Ec is ignored - that is viscous dissipation is ignored. Then, gamma can take absolutely arbitrary values. But, when viscous dissipation is included, gamma can take only restrictive values - gamma equal to 2m.

This is because there is a connection between the viscous dissipation term and the velocity solution. Such odd constraints are put in similarity method. So, these are conditions for existence of similarity solution: the wall temperature variation must follow this law, must be proportional to x raised to gamma and EC x if EC x is included, then gamma must equal 2m.

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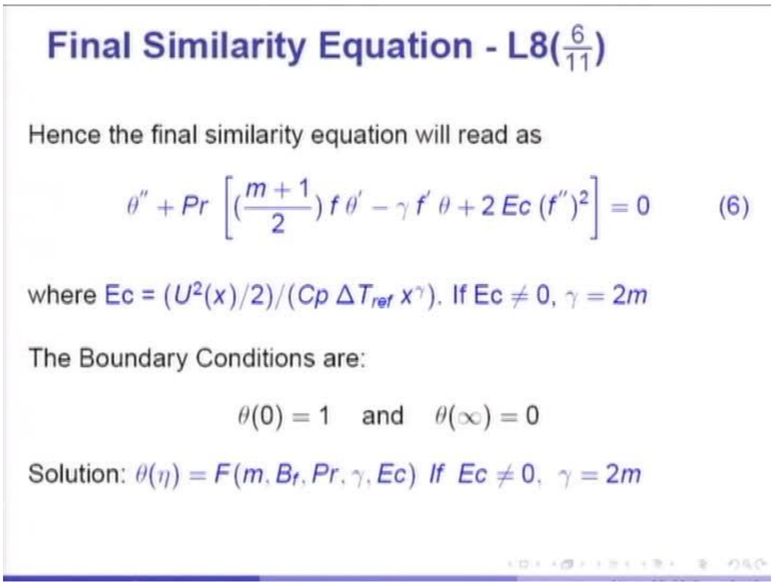

So, then our final energy equation is of this type, theta double prime plus Prandtl m plus 1 by 2 f theta dash minus gamma f dash theta. This represents the wall temperature variation, this is the viscous dissipation and this is the pressure gradient and this is the Prandtl number with a reminder that where Ec is equal to this and if Ec is not equal to 0 then gamma is equal to 2m, otherwise, gamma can take absolutely arbitrary values.

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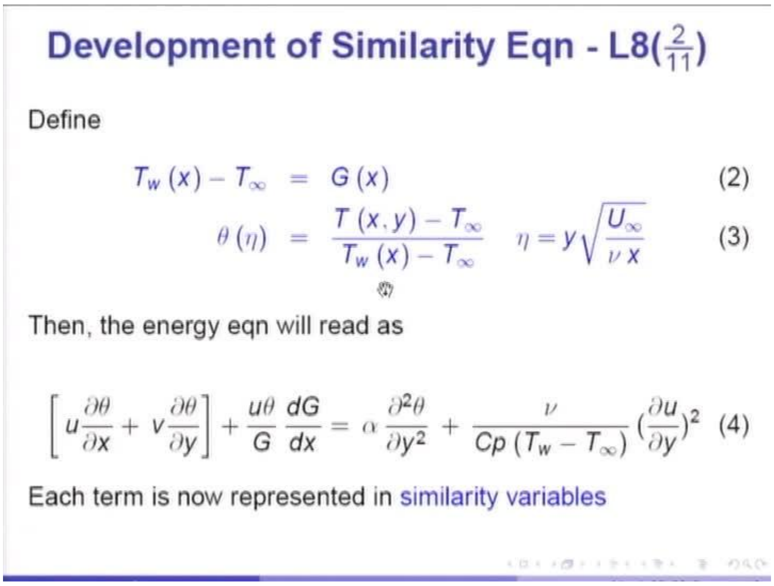

What would be the boundary conditions? That is straight forward to see, if you recall the temperature. So, at the wall, eta equal to 0 - theta 0 will be 1 and at infinity state, the numerator will be 0, so, theta infinity will be 0. That is what you see here.

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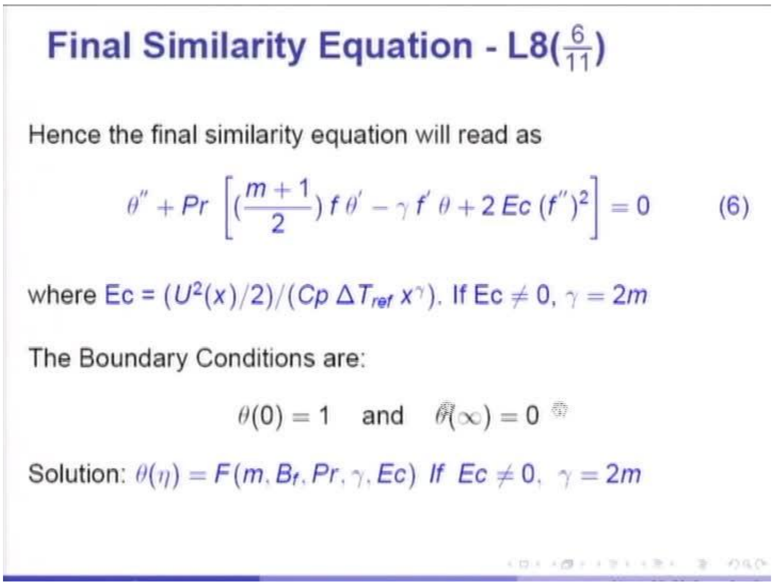

The boundary conditions are this and what would the solutions look like, theta eta will be functions first of all m and B f, because f f dash and f double prime are functions of m and B f and in addition you have Prandtl number, the wall temperature variation parameter and the viscous dissipation parameter, Ec. This is what we expect.

How do we solve this second order differential equation- like we did in the in the case of velocity boundary layer, where we had a third order equation.

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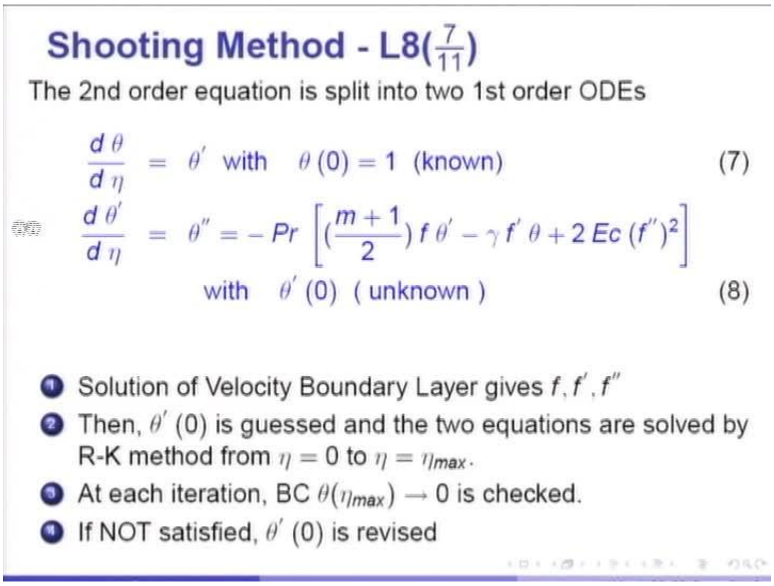

We split this up into 2 first order differential equations. The first equation is simply d theta by d eta equal to theta dash with theta 0 equal to 1 which is known; wall temperature is known. Then, d theta dash by d eta equal to theta prime which is equal to minus Prandtl all this. But, we do not know the theta dash 0.

Like we did in the in the velocity boundary layer equations, first of all, we obtain for a given m and B f gives f f dash f double prime. We then guess theta prime 0 and solve these two equations by Runge-Kutta method from eta equal to 0 to eta equal to eta max. At each of iteration, we check the boundary condition at the outer edge whether theta eta max has tended to 0 or not, if not, we revise theta dash 0 and continue the solutions again.

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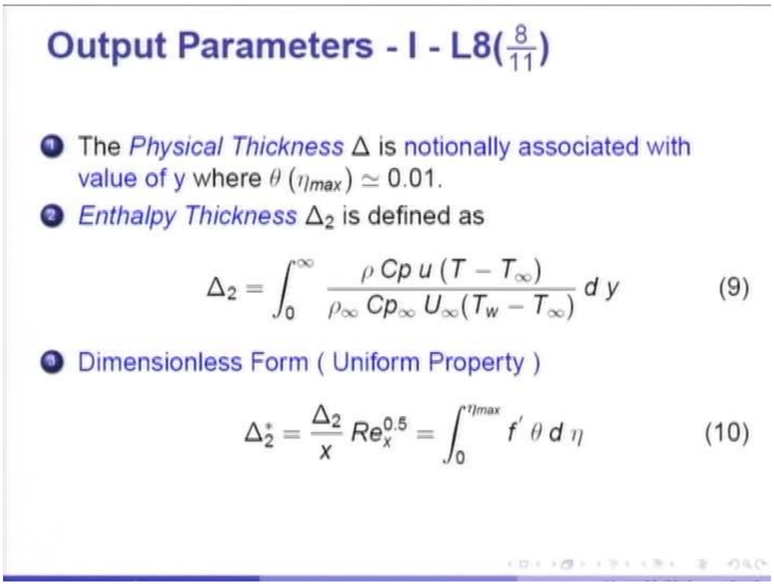

When the solution is obtained, we define as we did in case of velocity boundary layer, the thicknesses of the temperature boundary layer.

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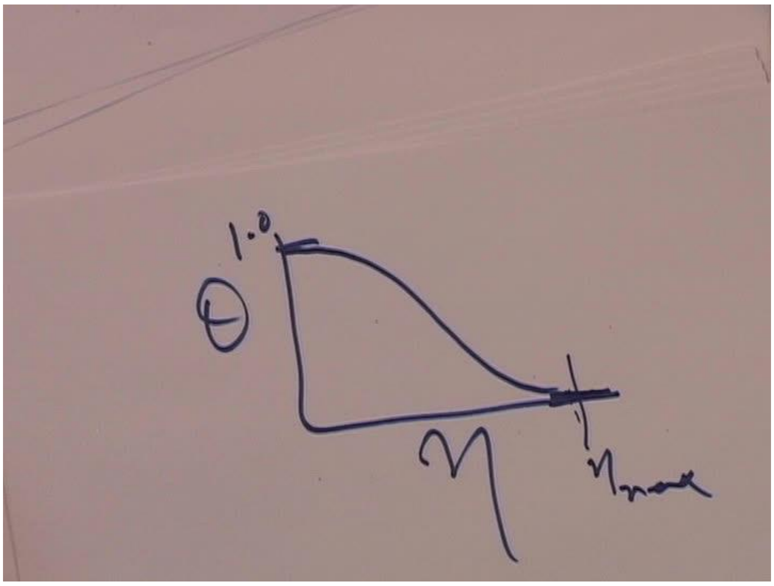

The physical thickness, delta is - remember our solution will look like this - theta verses eta, at wall it will be 1 and we expect the solution to go something like this with this 0 boundary condition at eta equal to some eta max. So, how do we choose the value of the thermal boundary layer thickness?

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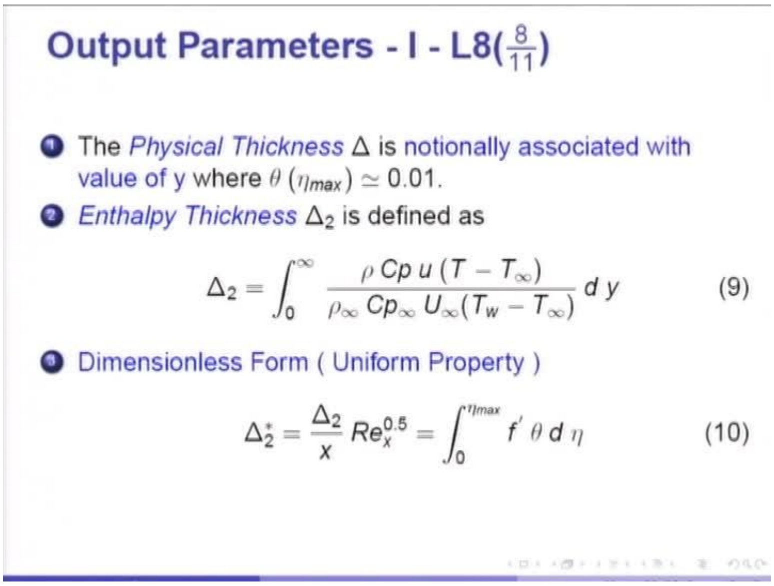

As we choose velocity boundary layer thickness and say that when f dash eta tends to about 0.99, we say that is velocity boundary layer thickness.

Likewise, in temperature boundary layer, we say when theta tends to about 0.01 - that is somewhere here - then that would represent. But there is a notional thickness. There is no exactness. We can impart the exactness by defining another thickness called the enthalpy thickness. It is defined as 0 to infinity rho Cp u T minus T infinity, which is the actual enthalpy within the boundary layer when integrated to dy and this is the enthalpy that would be carried in the layer, if there was no boundary layer.

So, for uniform property flow, rho Cp gets canceled with this; u over U infinity will be simply f dash and T minus T infinity over T wall minus T is our theta. Therefore, this relationship - if I change y to eta then delta 2 star like delta 2 by x into Reynolds x to the 0.5 equal to 0 to eta max f dash theta d eta - this is the dimensionless form for a uniform property enthalpy thickness.

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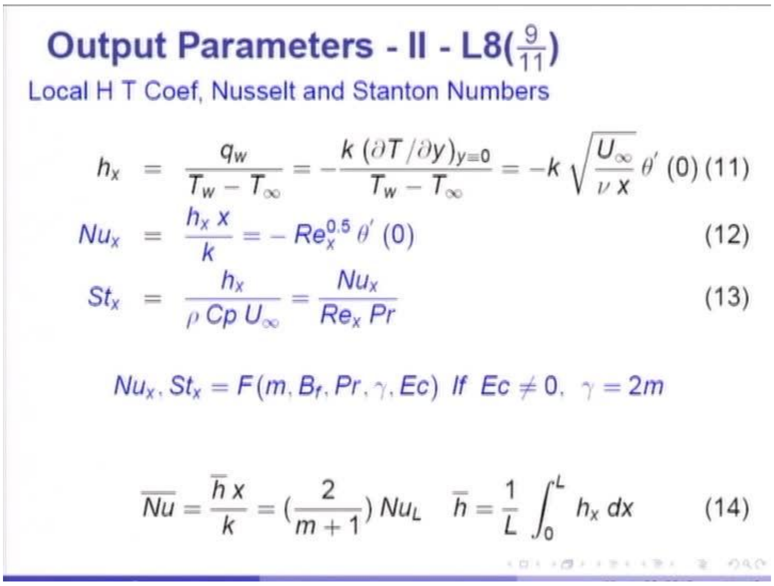

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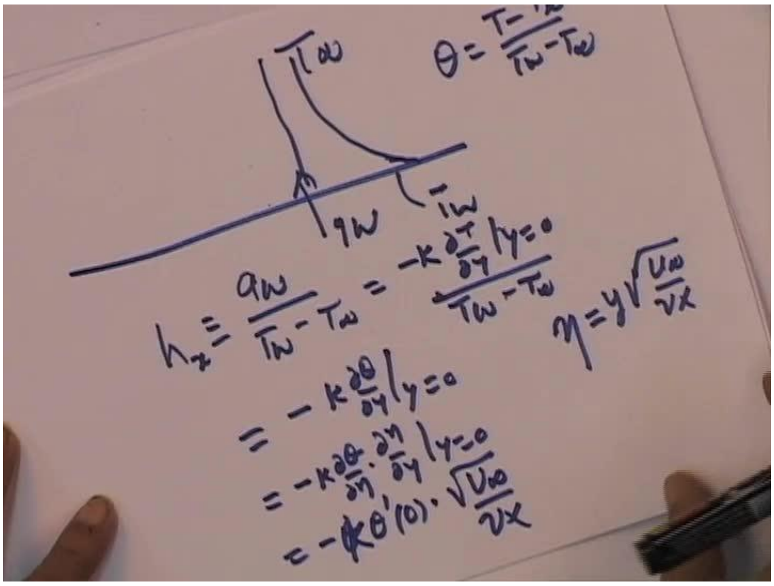

Remember we obtained solutions by guessing the value of theta prime 0.How is it related to the heat transfer coefficient, h x? The local heat transfer coefficient will be q wall divided by T wall minus T infinity - remember at any x, if this is q wall - T w and T infinity – then, h x is simply q w divided by T wall minus T infinity. This is the definition, you will recall. That would equal, if this was the temperature profile - the gradient of temperature will be minus k dT by dy at y equal to 0 divided by T w minus T infinity.

Since theta is equal to T minus T infinity over T w minus T infinity, I can straight away say, this is equal to minus k d theta by dy at y equal to 0; this is dT divided by T w minus T infinity. If I now change, eta is equal to y times U infinity by nu x. Then, you will see this can also be written as - minus k d theta by dy d eta into d eta by dy at y equal to 0. That is equal to 0 minus k minus k theta dash 0 into under root U infinity by nu x.

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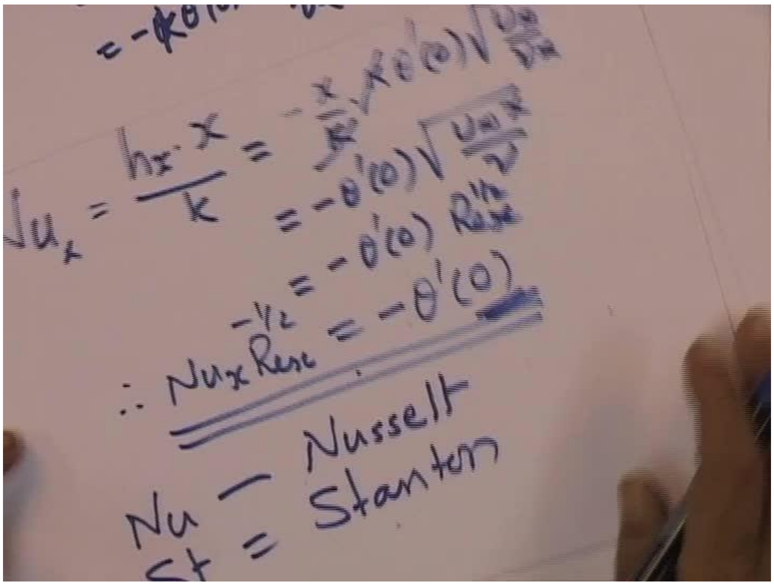

I can develop further as - h x into x by k equals the Nusselt number n u x. Then you will see this becomes equal to minus x by K into K theta prime 0 into U infinity by nu x and you will see K and K gets canceled and you will get this as - minus theta prime 0 into under root U infinity x by nu. That is nothing but - minus theta prime 0 Re x to the half. Therefore, I can say that - Nu x Re x to the half minus half is equal to minus theta prime 0. Our R-K solution gives us the value of theta prime 0 which in turn give us the value of Nusselt number.

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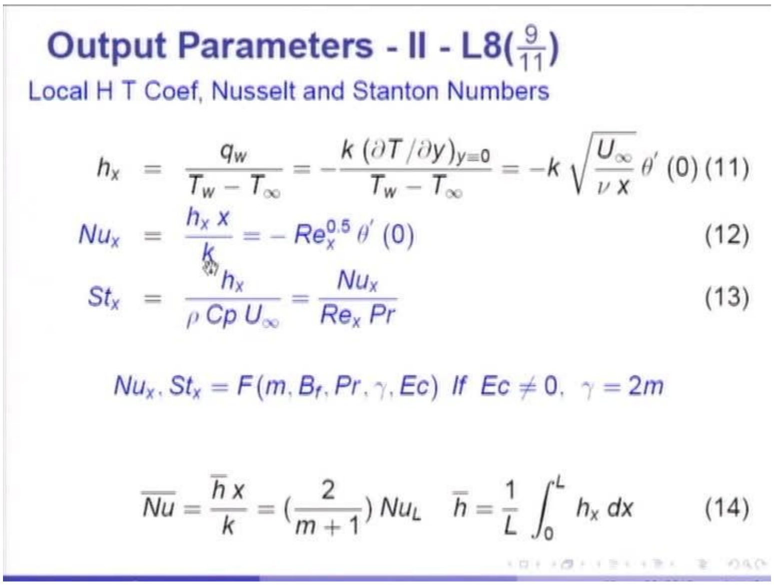

That is what I have derived here; Nu x is equal to that. Sometimes, in boundary layer theory, we define, Stanton x Nu is Nusselt named after the scientist Nusselt. Similarly, St is Stanton named after a scientist Stanton and that is defined as - h x divided by rho Cp U infinity. Remember there is no length dimension in the definition of Stanton number and this is found sometimes quite useful in defining a dimensionless heat transfer coefficient.

It is simply - Nusselt x divided by Reynolds x into Prandtl. This is simple algebra to show that if I divide this quantity by Reynolds x and Prandtl, you would get Stanton. Then, the solution is essentially Nusselt x or Stanton x equal to is a function of m B f Prandtl gamma Ec and with the remainder. If Ec is not equal to 0, gamma is equal to 2m. Knowing local value of h x, you can always calculate average value of heat transfer coefficient and average value of Nusselt number.

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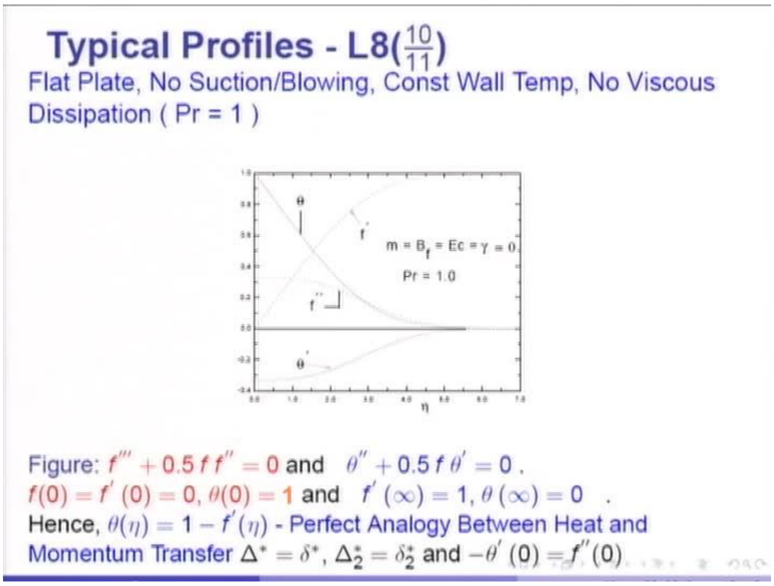

So here is the first solution obtained to the thermal boundary layer equation taking the case of a flat plate.

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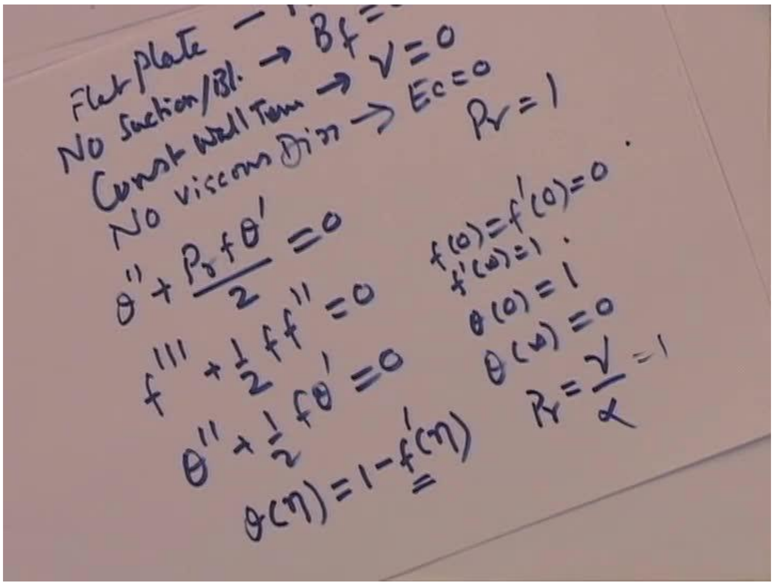

Flat plate means - m equal to 0; no suction blowing means - B f is equal to 0; constant wall temperatures means gamma is equal to 0; no viscous dissipation - means Ec is equal to 0.

We have a very simple equation. Our thermal equation will simply look like - theta double prime plus Prandtl into f theta dash divided by 2 equal to 0. The velocity equation would look like - f triple prime plus half f f double prime equal to 0. So, if I now say that Prandtl number is equal to 1, then, theta double prime will equal 1 by 2 f theta prime equal to 0.

The boundary conditions here are - f 0 equal to f dash zero equal to 0 and f dash infinity equal to 1. Whereas, the boundary conditions here are - theta 0 equal to 1 and theta infinity equal to 0. You will see therefore, that there is a remarkable similarity between these two equations. If I were to say that theta eta equals 1 minus f times eta, then, I do not have to solve the temperature equation; simply the value of velocity boundary layer equation itself will show me that.

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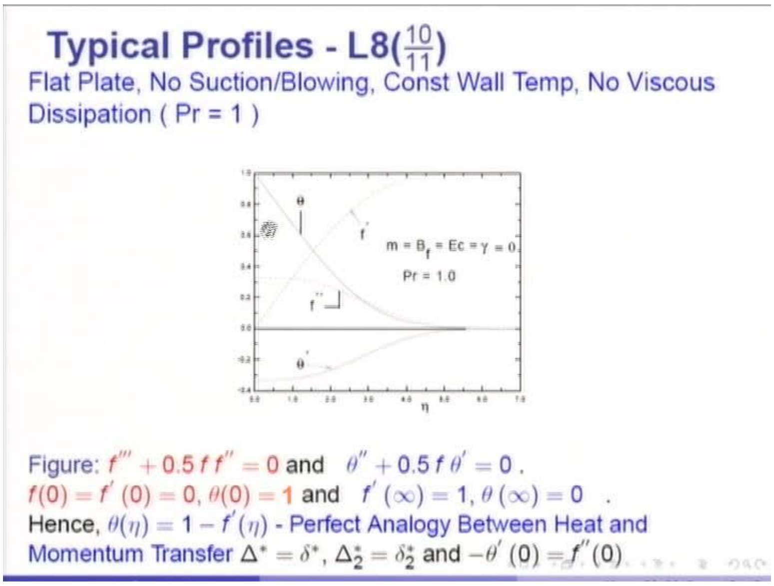

So, you will see here the earlier solution, f for f dash - this is a velocity boundary layer equation for Prandtl equal to 1 and the boundary layer thickness for the velocity was about 5. Likewise, the temperature profile theta should be 1 minus f dash eta. So, theta is nothing but - 1 minus f dash eta. This is the f double prime 0 which is going. Here, it is the theta double prime theta double prime or theta prime at x equal to 0.

This shows that perfect analogy between heat and momentum transfer exists for Prandtl equal to 1. Remember Prandtl is simply - nu by alpha. If this is equal to 1, the rate of diffusion of momentum and heat are equal. In such a case, there is a perfect analogy between heat and momentum transfer and all thicknesses would be equal. So, thermal boundary layer thickness delta star would equal a velocity boundary layer thickness delta star.

Enthalpy thickness, delta 2 star would equal momentum thickness delta 2 star, minus theta prime 0 would simply equal f double prime 0 as seen in the figure here with a negative sign as you see - this is theta dash 0 and this is f double prime 0. That is what you see here. This reference case is of great value to us because we can now go on comparing the effects of m, B f, gamma and Ec and Prandtl number on solutions.

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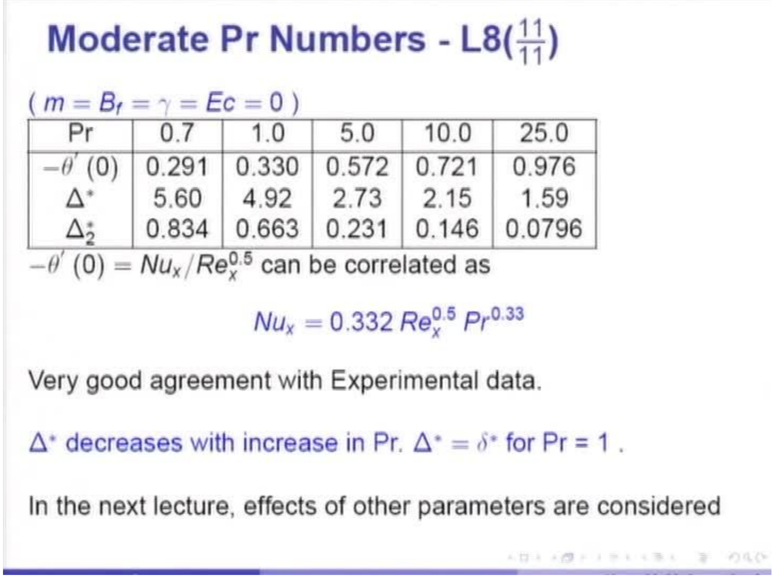

Moderate Pr Numbers - L8($\frac{11}{11}$)

$(m = B_f = \gamma = Ec = 0)$

Pr	0.7	1.0	5.0	10.0	25.0
$-\theta'(0)$	0.291	0.330	0.572	0.721	0.976
Δ^*	5.60	4.92	2.73	2.15	1.59
Δ_2^*	0.834	0.663	0.231	0.146	0.0796

$-\theta'(0) = Nu_x / Re_x^{0.5}$ can be correlated as

$$Nu_x = 0.332\, Re_x^{0.5}\, Pr^{0.33}$$

Very good agreement with Experimental data.

Δ^* decreases with increase in Pr. $\Delta^* = \delta^*$ for Pr = 1 .

In the next lecture, effects of other parameters are considered

Here are the solutions generated by Runge-Kutta computer program. What I am now done is allowed for the variation of Prandtl number by keeping m B f gamma Ec all equal to 0. So, we have case of flow reflat plate at constant wall temperature; no suction and blowing and viscous dissipation is ignored; then you see value of minus theta prime 0 - for I have computed from 0.7 to 25 as the Prandtl number. So, you will see at Prandtl number 1, this is the well-known velocity solution where perfect analogy between momentum and heat transfer exists, 0.33, 4.92. But, when the Prandtl number is reduced to 0.7 which is for the gas, you get theta prime 0 is lowered to 0.291, the thermal boundary layer thickness increases and so does the enthalpy thickness increase.

On the other hand, at 5 which is small as water, you get higher heat transfer 0.572, 2.73 and 0.231 still higher 0.721. These are the really the organic liquid range in which this still increases; so, you have minus theta p.rime 0 increases with Prandtl number. So, minus theta prime 0 is Nu x Re x to the half

Now, if I were to correlate these values - theta prime 0 as a function of Prandtl number, I would see that it will become almost 0.33 into Prandtl raised to one third. Therefore, you can develop from the numerical solutions or the similarity solution an expression for Nusselt number as Nu x is equal to 0.332 Reynolds x to the half Prandtl.

This result looks remarkably like an experimental correlation and indeed this result has been found to be an excellent agreement with the experimental data. One point to note, however is that - delta star, the thermal boundary layer thickness increases with decrease in Prandtl number. Another way of saying - thermal boundary layer thickness decreases with increase in Prandtl number and therefore, as we move towards oils and other things where the Prandtl number is very large, you would see that the thermal boundary layer thickness would be much smaller than the velocity boundary layer thickness.

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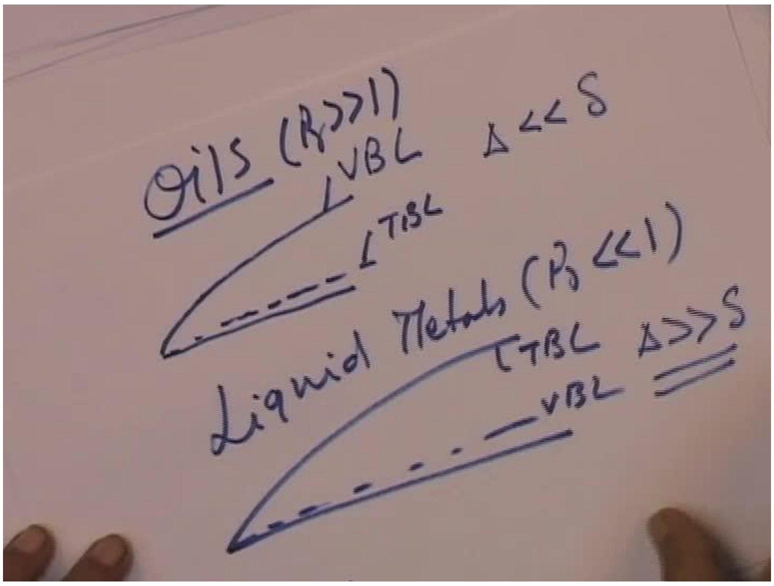

In other words, as we move towards oils - which is Prandtl number very much greater than 1, then, you will get situation like - this will be the velocity boundary layer and the temperature boundary layer would just develop like that.

On the other hand, for liquid metals when Prandtl number is very much less than 1, then, you will get - this would be temperature boundary layer and the velocity boundary layer will be this. So, here delta is much smaller than delta whereas, here delta would be much greater than delta. This important deduction we will make use of it in several further developments of both the similarity method as well as the integral method that we will be discussing much later.

With this, I have given you a simple sample solution for a very special case of m equal to 0, B f equal to 0, gamma equal to 0, Ec equal to 0. In the next lecture, I will explore the influences precisely of these parameters on heat transfer rate.