Convective Heat and Mass Transfer

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Module No. # 01
Lecture No. # 06
Similarity Method

In the last lecture, we derived the Boundary layer equations of a two-dimensional boundary layer and I ended by saying that there are three methods of solving these equations. The first one is the similarity method, the second is the integral method and the third is the finite difference or the finite element method.

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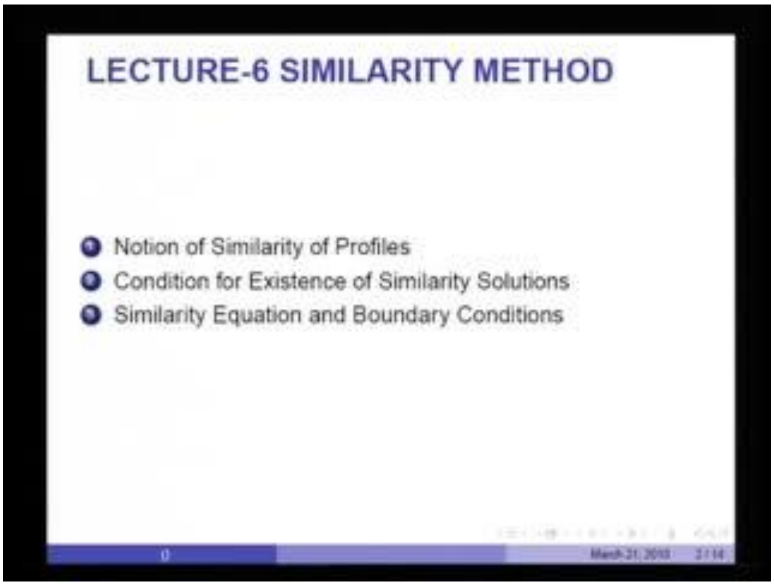

Today, I am going to develop the similarity method. The topics then are to introduce the notion of similarity of profiles. I will establish what are the conditions for existence of similarity solutions and then, I will develop the similarity equation with appropriate boundary conditions.

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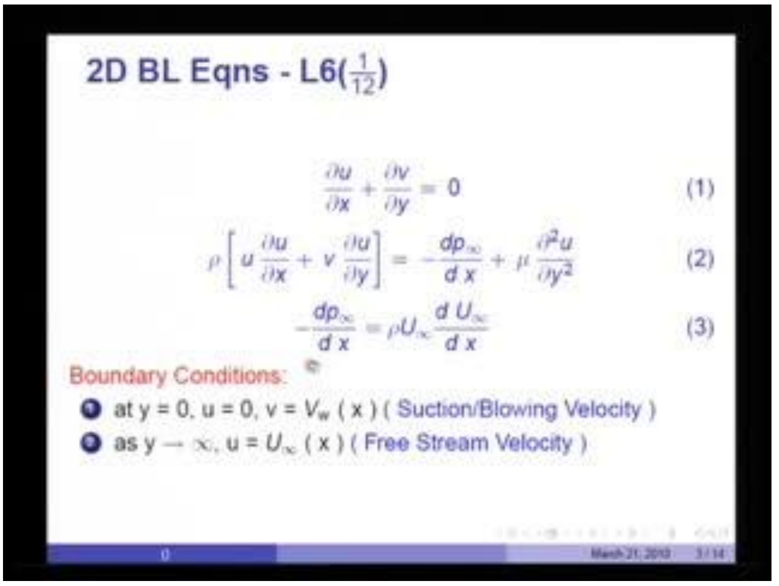

Let us recall the equations. The equation is the so-called mass conservation or the continuity equation for a constant property steady flow, du by dx plus dv by dy equal to 0. These are the convection terms in the momentum equations – the pressure gradient and the viscosity effected term, mu d 2 u dy square. We said that if this equation was written in the infinity state, then the pressure gradient could actually be written as rho U infinity d U infinity by d x.

I shall be making this substitution here (Refer Slide Time: 01:41) in all subsequent developments. The boundary conditions are obvious at y equal to 0; that is, at the wall, the stream-wise velocity u will be 0, but there may be suction or blowing at the wall. So, we shall say v equal to V w, which may be function of x. As y tends to infinity, u will tend to U infinity at x, which is the Free Stream Velocity. So, we have two variables here: V w as a function of x and U infinity as a function of x. U infinity as a function of x represents the pressure gradient according to equation 3 here.

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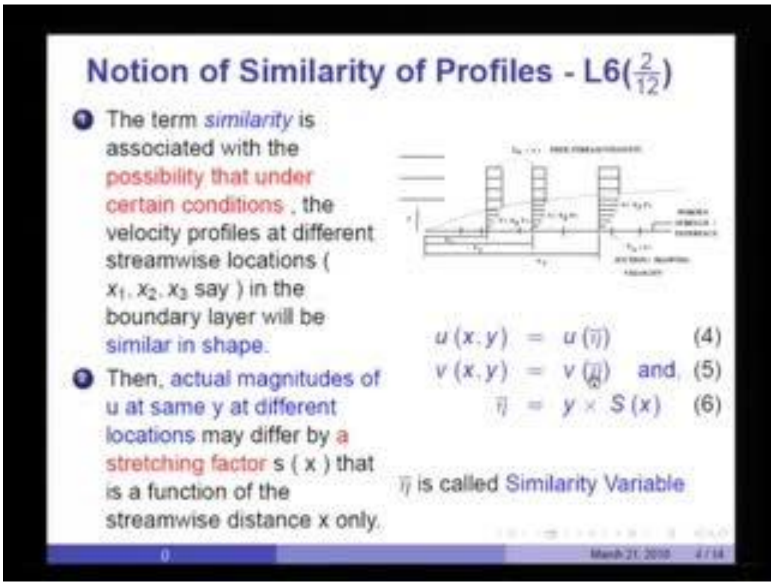

What is the notion of similarity profiles? The term similarity is associated with the possibility that under certain conditions the velocity profiles at different stream-wise locations, x 1, x 2, x 3, say in the boundary layer will be similar in shape.

Here (Refer Slide Time: 02:48) is a surface over which the flow is flowing and the boundary layer development is taking place. The velocity profile at x 1 is as shown here, at x 2, it is as shown here, and at x 3, it is as shown here. In terms of boundary conditions, they are u equal to 0, u equal to U infinity here, u equal to 0, u equal to infinity, and u equal to 0 and u equal to infinity. The question we ask is - they look similar? Yes, but can they be collapsed on to a single curve? Now, this idea of a collapse is very important and that is really what underlies the similarity.

The profiles at different positions, x 1, x 2, x 3 will collapse on each other if their shapes at each position were equal. What does that mean? It means that the actual magnitudes of u at some y at different locations would differ simply by a stretching factor, s. That is a function of stream-wise distance, x. In other words, if the value of y here; the physical value of y here (Refer Slide Time: 04:08) were stretched by s as a function of x. Then, it is quite possible that the velocity profiles would collapse on each other. Basically, then what we are saying is – u, which was a function of x and y can now be written as u as a function of eta bar, which is a new variable. It is called the Similarity Variable. v, which was a function of x and y is a function of v eta bar. eta bar is equal to y, the transverse distance multiplied by some function of x.

What we are aiming at is – equations, which are functions of two variables x and y going to be made functions of one variable. Therefore, a partial differential equation would now be converted to an ordinary differential equation in eta bar. Let us say - how this is done on the next slide.

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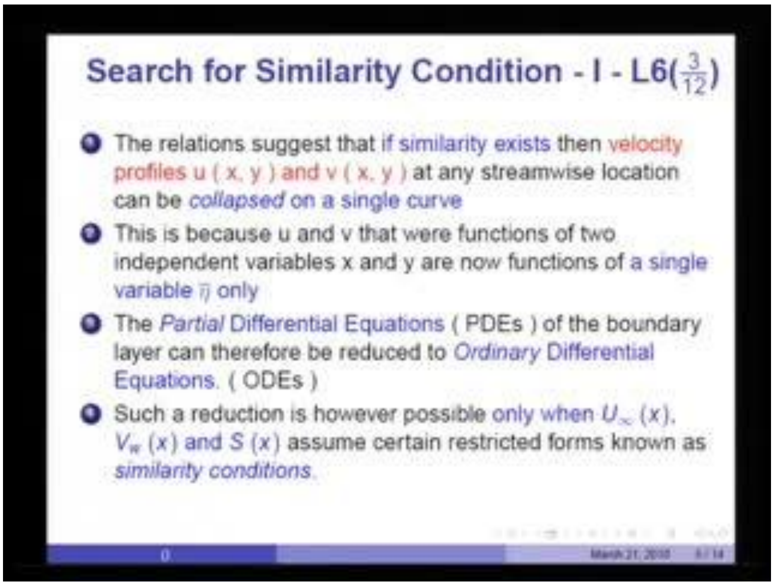

The relations suggest that if similarity exists, we want to search for conditions for similarity to exist. So, the relations that I showed on the last slide suggest that if similarity exists, then there is velocity profiles u and v at any stream-wise location can be collapsed on a single curve. This is because, u and v that are functions of two independent variables x and y are now functions of a single variable eta bar only. The Partial Differential Equations therefore, will be reduced to Ordinary Differential Equations. However, such a reduction would be possible only when U infinity x V w as a function of x and the stretching factor S as a function of x assumes certain restricted forms known as similarity conditions. It is these special forms that we wish to discover and they are called the similarity conditions.

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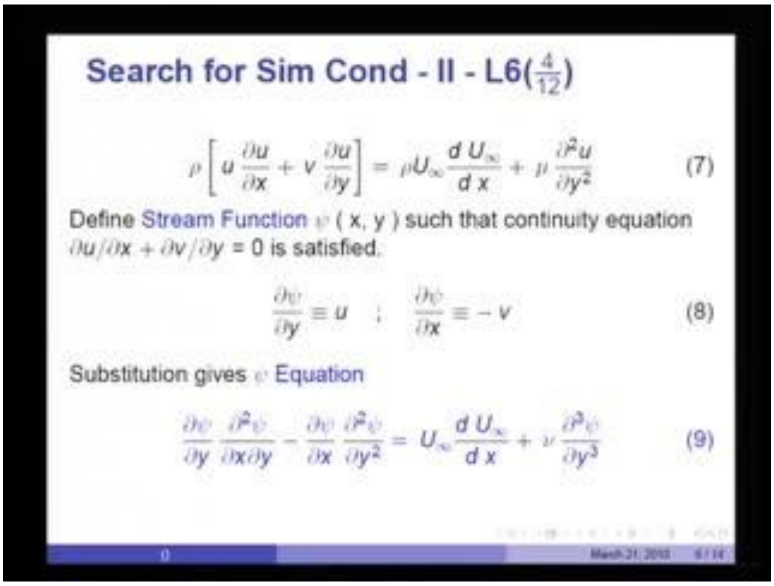

I have written the momentum equation again rho u du by dx plus v du by dy equal to rho U infinity d U infinity by d x plus mu d 2 u by dy square.

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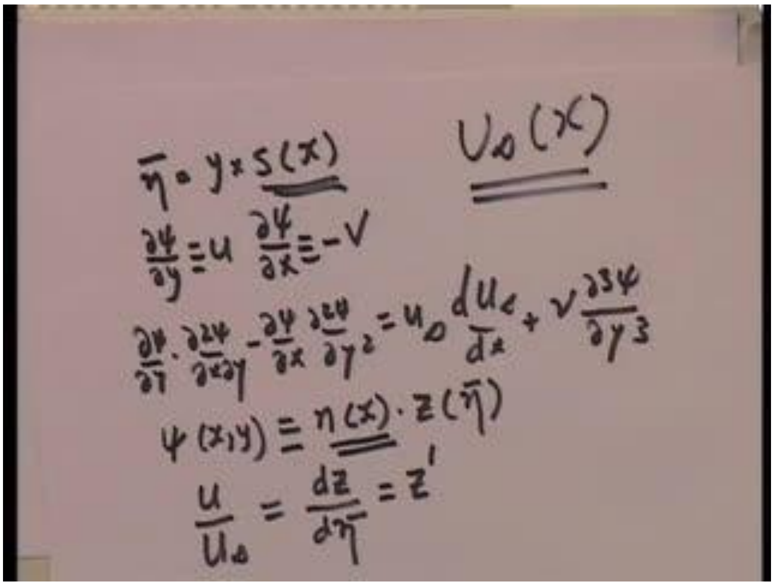

We have already introduced a variable called eta bar as y into s x. We introduce another variable psi x y such that d psi by dy is defined as u and d psi by dx is defined as minus v. Then, you will notice that if I substitute for u (Refer Slide Time: 07:05) in the continuity equation and for v also in the continuity equation, I will get d 2 psi dx dy minus d 2 psi dx dy equal to 0. Therefore, these definitions have not fallen from the skies; it is simply; they satisfy the continuity equation. psi is called the Stream Function.

Now, if I have to substitute for all these quantities (Refer Slide Time: 07:33), then you will see the momentum equation will become… I am already going to divide by rho then the momentum equation will become (Refer Slide Time: 07:43) d psi by dy into du by dx would be d 2 psi by dx dy minus d psi by dx, which is v; d 2 psi by dy square equal to U infinity d U infinity by dx plus mu divided by rho, which is nu times d3 psi by dy cube.

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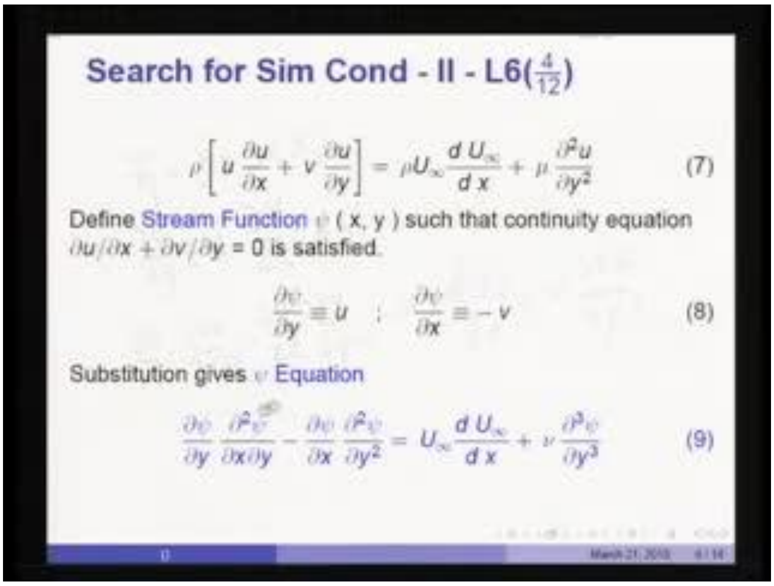

This is the psi equation we would get, where psi already satisfies the continuity equation du by dx plus dv by dy equal to 0. Now, we are going to make some further definition.

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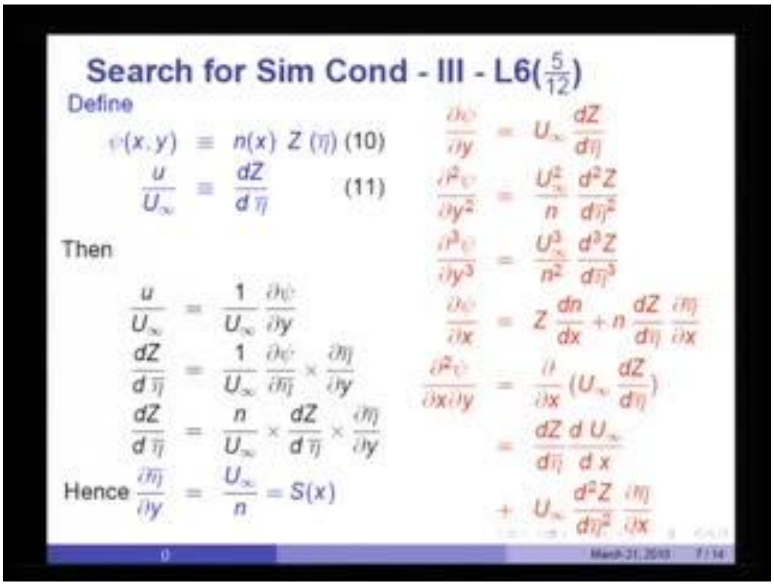

Now, it is obvious. Since u is a function of x and y and v is a function of x and y, psi would also be a function of x and y. That function I am introducing a (Refer Slide Time: 08:49) psi x y equal to some function n x into z times eta bar; z as a function of eta bar. I am also going to define u over U infinity equal to dz by d eta bar. Since z is a function only of eta bar, I can also write this as z dash.

Now, we wish to establish (Refer Slide Time: 09:24) the relationship between eta bar U infinity and the n. So, we have introduced another function n x (Refer Slide Time: 09:33), we have already got one function s x, and we have the third function U infinity x. We want to establish a relationship between these three variables.

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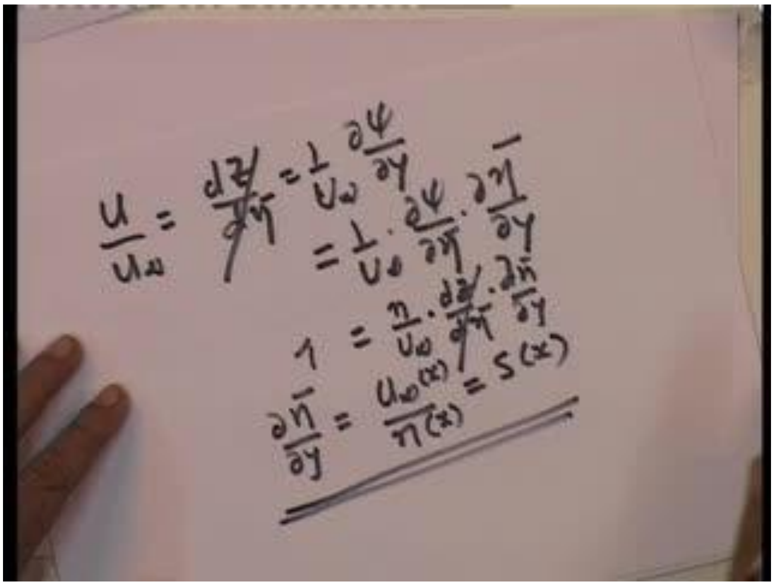

We will recall that u over U infinity as I defined is dz by d eta bar and that is equal to 1 over U infinity into d psi by dy. However, notice that I can write this as 1 over U infinity d psi by d eta bar into d eta bar by dy, which I can write as…

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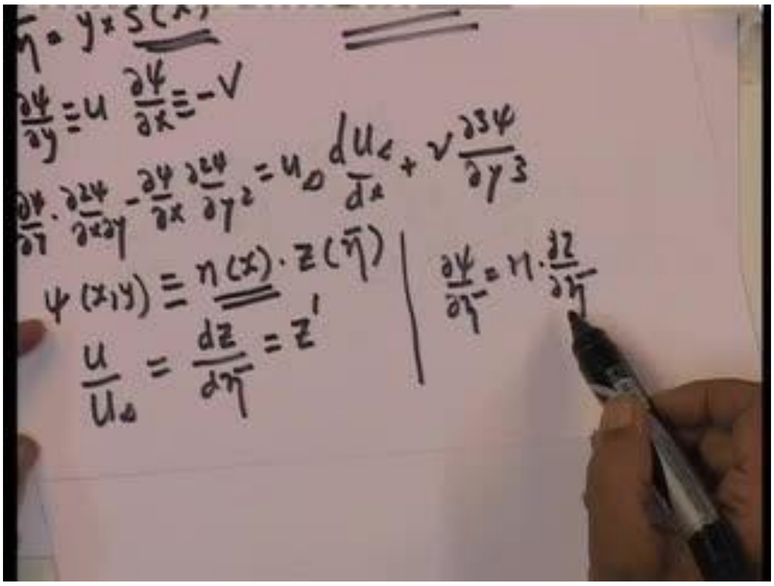

What is d psi by d eta bar? You will see d psi by d eta bar would be simply n times dz by d eta bar simply because n is a function of x and z is a function of eta bar. Therefore, you will see I can replace this as (Refer Slide Time: 10:52) n over U infinity into dz by d eta bar into d eta bar by dy. So, I can cancel dz by d eta bar on both sides and I will have that as 1. Or, d eta bar by dy will be equal to U infinity by n; both are functions of x (Refer Slide Time: 11:21). That would equal s x as per our definition because remember: (Refer Slide Time: 11:30) eta bar is y into s x. So, d eta bar by dy would be simply s x.

This is a very important relation (Refer Slide Time: 11:40) because it establishes relationship between three functions of x that we have introduced.

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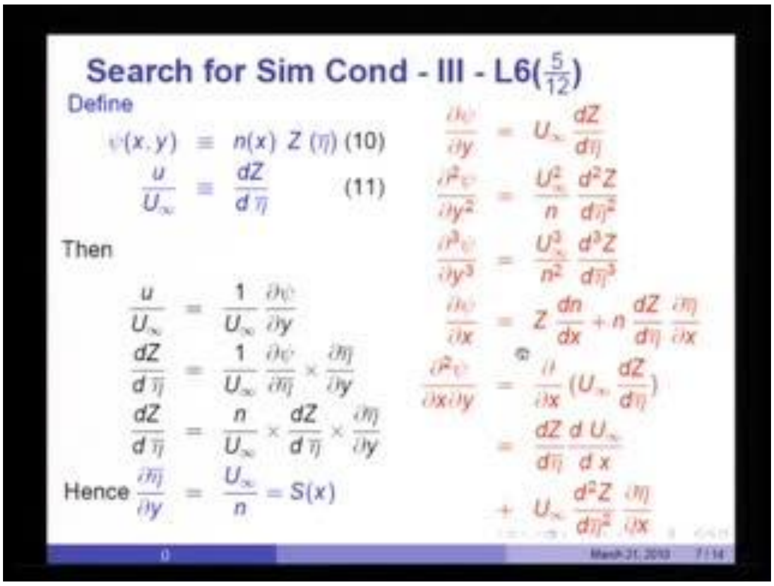

Now, it follows quite obviously; d psi by dy would be equal to U infinity into dZ by d eta bar. From this definition itself u over U infinity is dZ by d eta bar. d 2 psi by dy square would then be U infinity square by n into d 2 Z by d eta bar square. d 3 psi by dy cube will be U infinity cube divide by n square into d 3 Z by d eta cube. d psi by dx; that is where is a little we…

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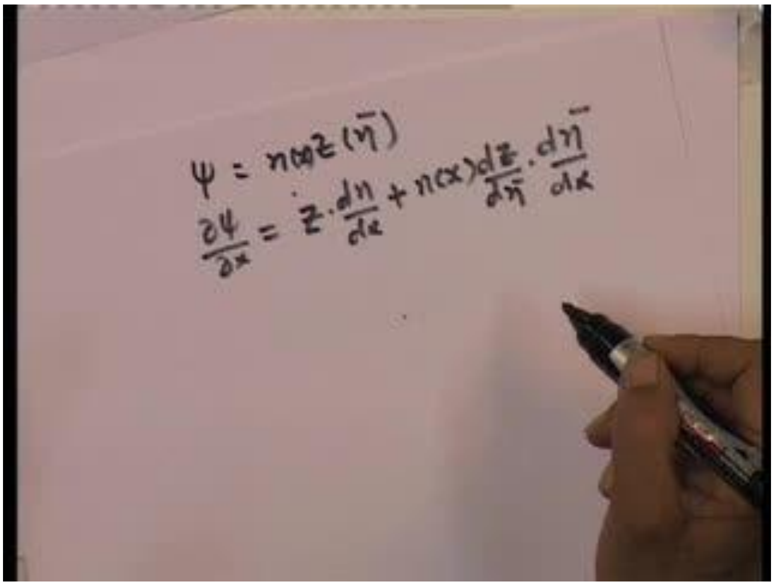

Remember: psi is n times Z and x times Z eta bar.

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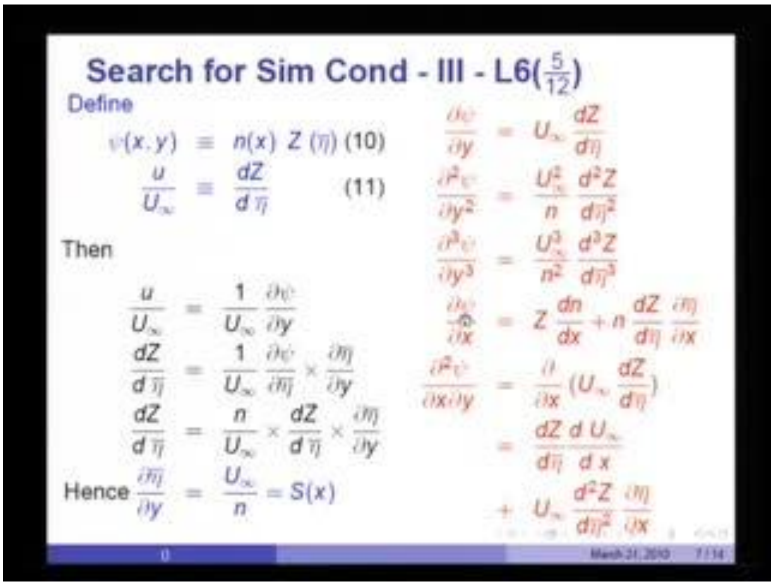

Therefore, d psi by dx would be Z times dn by dx plus n x dZ by d eta bar d eta bar by dx because remember: eta bar is function both of y and x d psi by dx equal to Z dn by dx plus n dZ by d eta bar d eta bar by dx.

We also need d 2 psi by dx dy and that could be d by dx of d psi by dy; d psi by dy is simply U infinity into dZ by d eta bar. Multiplication of the differentiation of a product would give dZ by d eta bar d U infinity by dx plus U infinity d 2 Z by d eta bar d eta bar by dx.

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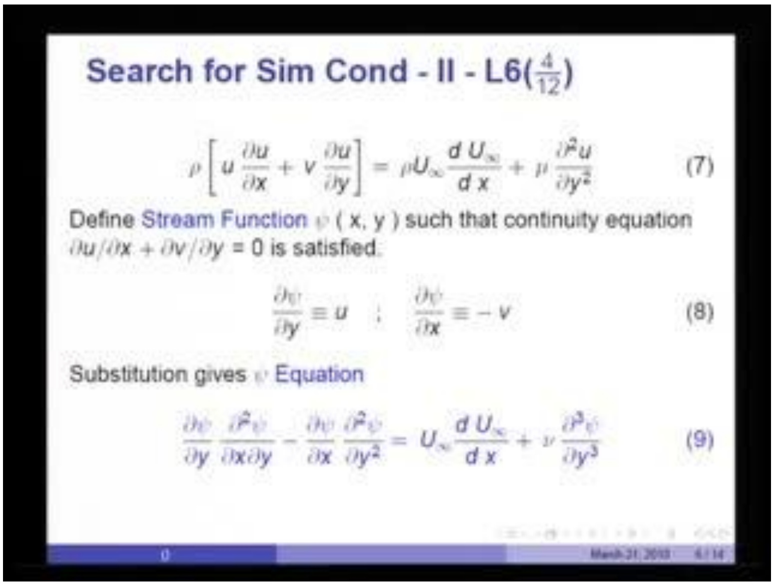

If I were to introduce all these quantities (Refer Slide Time: 13:30) in the last equation on the previous slide, then you will notice that I would get…

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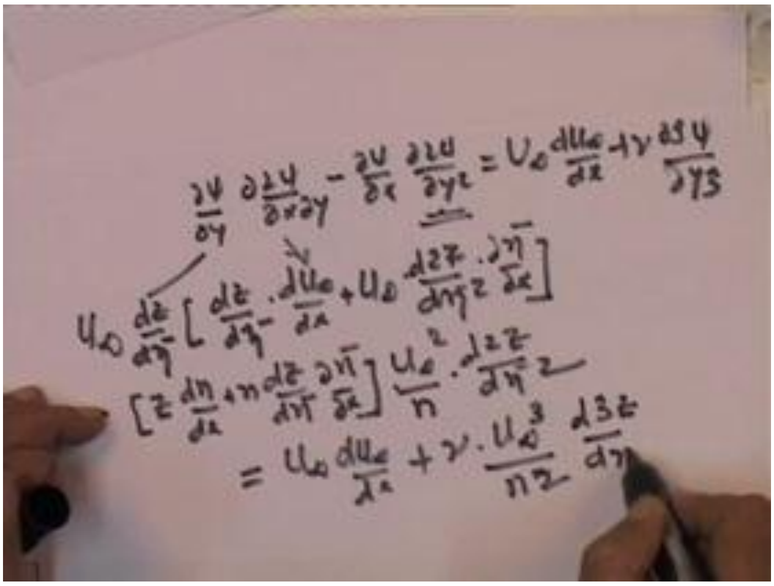

I am repeating the equation d 2 psi by dx dy minus d psi by dx d 2 psi by dy square equal to U infinity d U infinity by dx plus nu d3 psi by dy cube.

And if I go to the next slide again this d psi by dy will be U infinity into dZ by d eta bar into d 2 psi by dx dy, which is dZ by d eta bar into d U infinity by dx plus U infinity into d 2 Z by d eta bar square into d eta bar by dx. This is the first term (Refer Slide Time: 14:46). Minus d psi by dx is Z times dn by dx plus n times dZ by d eta bar d eta bar by dx into d 2 psi by dy square, which is simply U infinity squared by n into d 2 Z by d eta bar square. That would equal U infinity d U infinity by dx plus nu times U infinity cube divided by n square into d 3 Z by d eta cube.

Now, observe that (Refer Slide Time: 15:49) U infinity dZ by d eta, which is really the d psi by dy multiplied by U infinity d 2 dZ by d eta square d eta bar by dx in this first term. Cancels because of the negative sign here with the second term here (Refer Slide Time: 16:08); n dZ by d eta bar d eta bar by dx U infinity square by d 2 Z by d eta bar square because n and n cancels here. So, this term and this term (Refer Slide Time: 16:21) will vanish.

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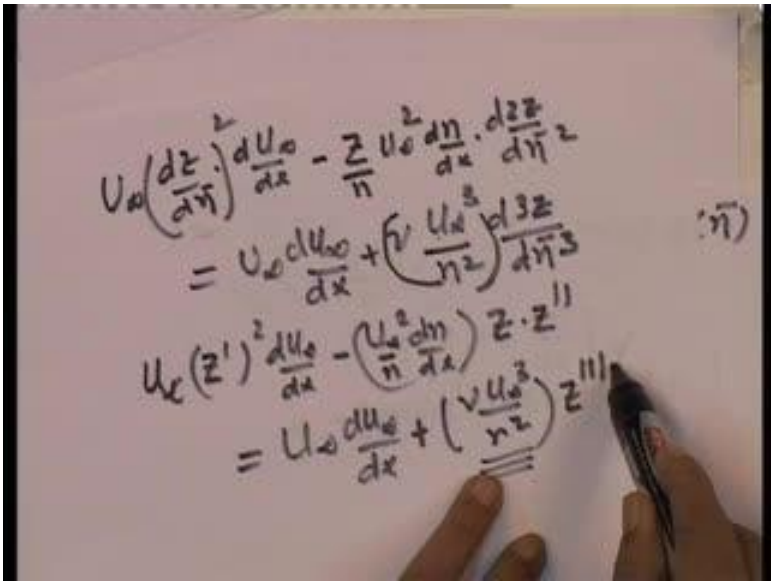

As a result, then I would have the equation, which would look like this; U infinity U infinity into dz by d eta bar whole square (Refer Slide Time: 16:36) into d U infinity by dx would be the first term; that is what I have written here (Refer Slide Time: 16:41).

The second term will be (Refer Slide Time: 16:43) minus Z dn by dx multiplied by U infinity square by n d 2 Z by d eta bar square and that is the term I have written here

(Refer Slide Time: 16:54); would equal U infinity d U infinity by dx plus nu U infinity cube by n square d 3 Z by d eta bar cube. I simply replace these (Refer Slide Time: 17:08) by prime quantity, Z prime square; Z Z prime; Z into Z double prime. This becomes the Z triple prime here (Refer Slide Time: 17:19).

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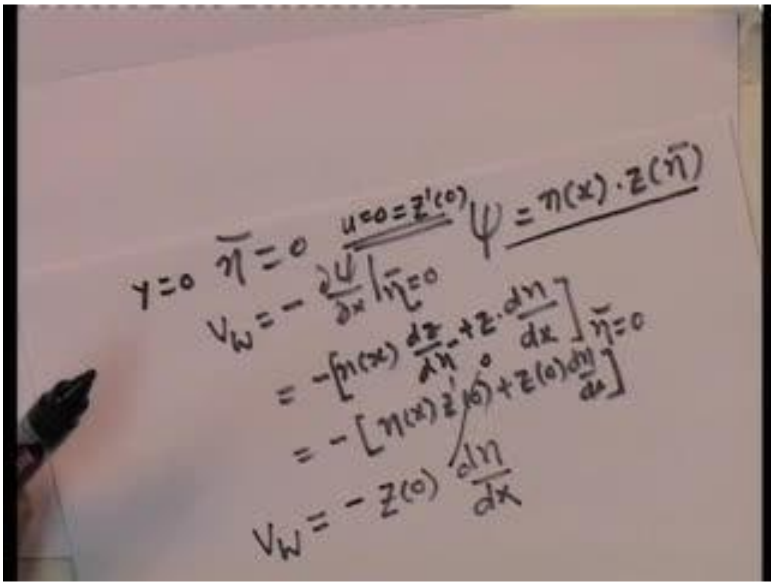

What are the boundary conditions? First of all, at y equal to 0, eta bar equal to 0, I have u equal to 0 or that will be equal to Z prime 0; that is, the no slip condition, which I have mentioned here.

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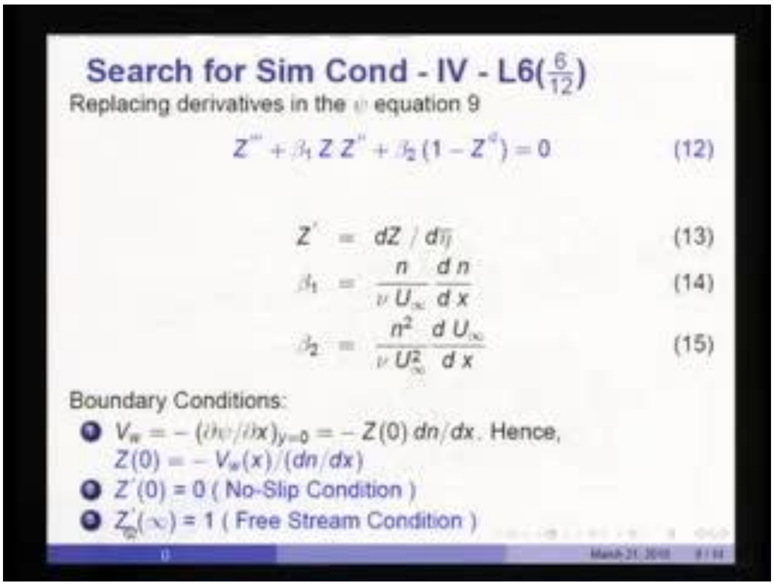

In the infinity state, u equals U infinity. Therefore, Z prime U infinity will equal 1. However, at y equal to 0, there is a suction or blowing velocity, V w. V w will be (Refer Slide Time: 17:53) d psi by dx; n eta bar equal to 0. Since psi is equal to n x Z eta bar, I will have n x dz by d eta bar plus Z dn by dx, all at eta bar equal to 0. If I replace dz by d eta by Z prime 0, this will be 0 (Refer Slide Time: 18:15), but then as we know, this is due to no slip condition that is 0. Therefore, I get V w is equal to minus Z 0 dn by dx. Or, in other words (Refer Slide Time: 18:32), Z 0 equal to minus V w divided by dn by dx as shown. Remember: this is a third order equation and therefore, I need three conditions. I have provided 2 conditions at eta bar equal to 0 and one condition at eta bar equal to infinity. Therefore, the problem statement is now complete.

When would this equation be a perfect ordinary differential equation? It would be a perfect ordinary differential equation only if beta 1 and beta 2 are absolute constants and not functions of x. Remember: beta 1 is entirely a function of x; beta 2 likewise is entirely a function of x, and Z 0, which is V w divided by dn by dx is also a function of x. However, unless these three quantities (Refer Slide Time: 19:25) are absolute constants; beta 1, beta 2 and Z 0 are absolute constants, we do not have an ordinary differential equation with appropriate boundary conditions.

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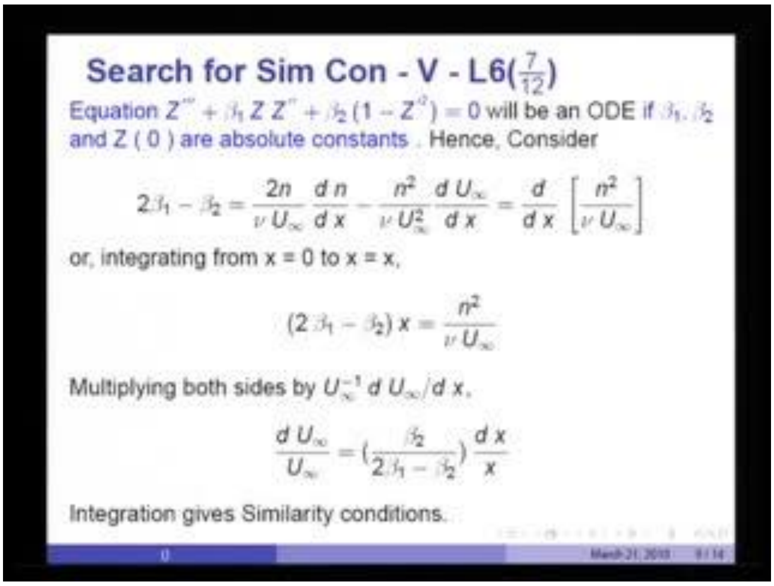

That tells us in a way what the similarity conditions are going to be. So, the equation Z triple prime plus beta 1 Z Z prime plus beta 2 into 1 minus Z prime square equal to 0 will be an ODE if beta 1, beta 2 and Z 0 are absolute constants.

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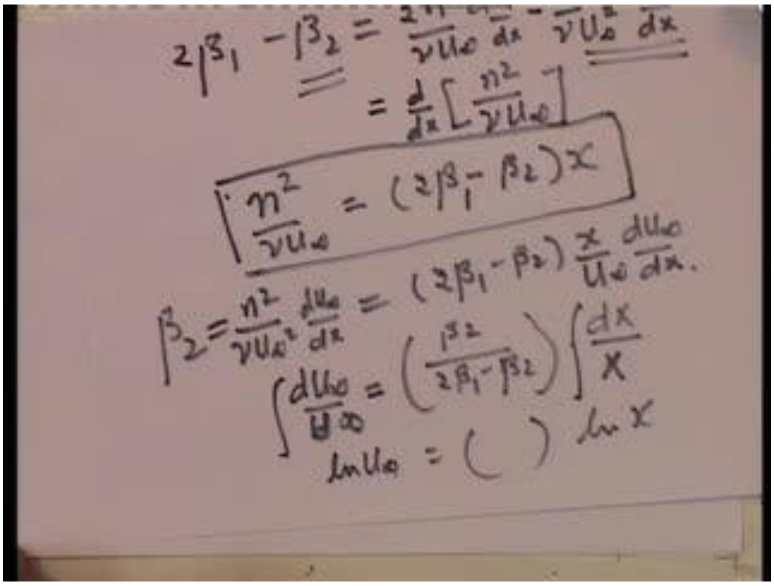

Now, consider 2 beta 1 minus beta 2; we have already written the definitions of beta 1 and beta 2 on the previous slide (Refer Slide Time: 20:05). So, I am writing now 2 beta 1 minus beta 2 equal to 2n divided by nu U infinity dn by dx minus n square by nu U infinity square d U infinity by dx.

If I have to write the right hand side, you will see it can be written as d by dx of n square divided by nu U infinity, where nu is the kinematic viscosity and it is a constant. If I were integrate this equation from x equal to 0 to x equal n, then I will get n square by nu U infinity equal to 2 beta 1 minus beta 2 times x because beta 1 and beta 2 are now taken as constants with respect to x.

Now, I multiply both sides by (Refer Slide Time: 21:26) one over U infinity d U infinity by d x. Then, you will see n square nu U infinity square d U infinity by dx will equal 2 beta 1 minus beta 2 x divided by U infinity d U infinity by dx. However, if you see the left hand side now, it is nothing but beta 2. Therefore, I have a relationship that d U infinity by U infinity will be equal to beta 2 over 2 beta 1 minus beta 2 into dx by X.

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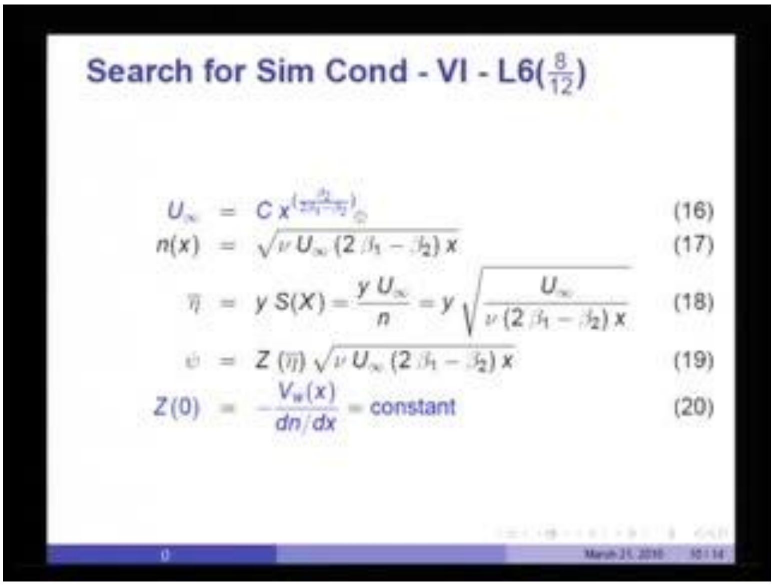

If I integrate this (Refer Slide Time: 22:27), it gives me the first similarity condition that U infinity must vary as C x raise to beta 2 over to beta 1 minus beta 2. If I integrate this from 0 to x, you will see (Refer Slide Time: 22:45) ln U infinity will be constant times ln X and therefore, the relationship follows. So, ln U infinity equal to same constant ln X or U infinity would be constant times C x raise to beta 2 over 2 beta 1 minus beta 2.

What about n x? You will see n square over nu infinity is equal to 2 beta 1 minus beta 2 x. If I multiply through by nu infinity and take a square root, then n will be (Refer Slide Time: 23:34) under root nu infinity 2 beta 1 minus beta 2 x. You recall – eta bar was defined as y S X, which is shown as y U infinity by n. If I substitute for n here, then you will see y under root U infinity would (( ))

Then similarly, psi, which was Z eta bar into n x would read like that (Refer Slide Time: 23:58) and Z 0 will be V w x over dn by dx equal to constant. So, now we have found the variable eta bar in terms of beta 1 and beta 2, U infinity in terms of beta 1 and beta 2, and psi again in terms of beta 1 and beta 2. Z 0 again likewise and that must be a constant.

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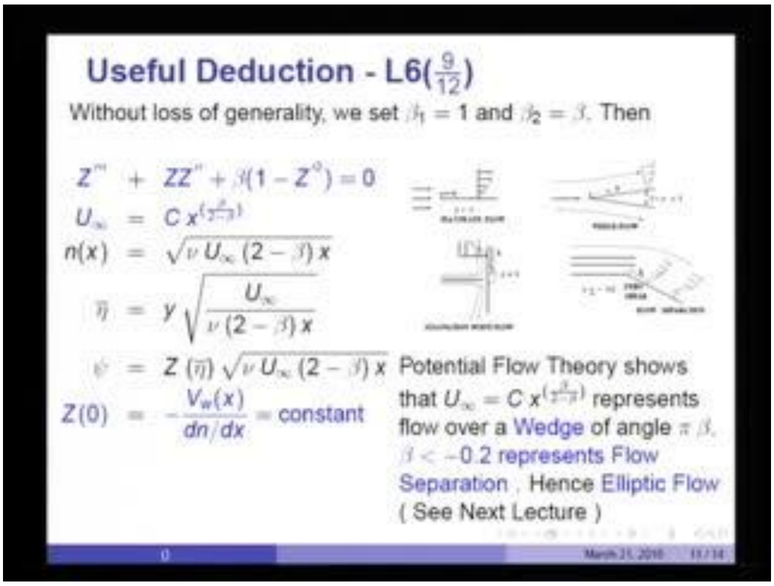

Now, there is something very useful that we can deduce.

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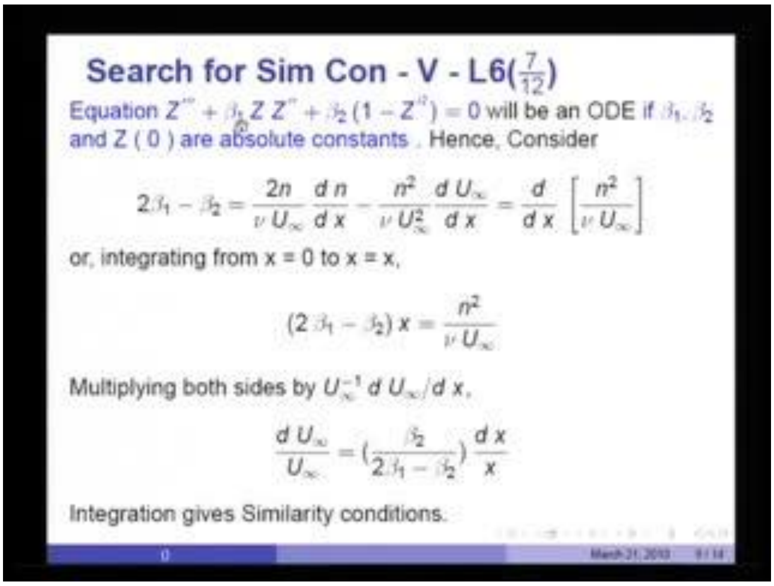

Remember: the equation is an ordinary differential equation as long as beta 1, beta 2 and Z 0 are constants. So, I can arbitrarily choose these values.

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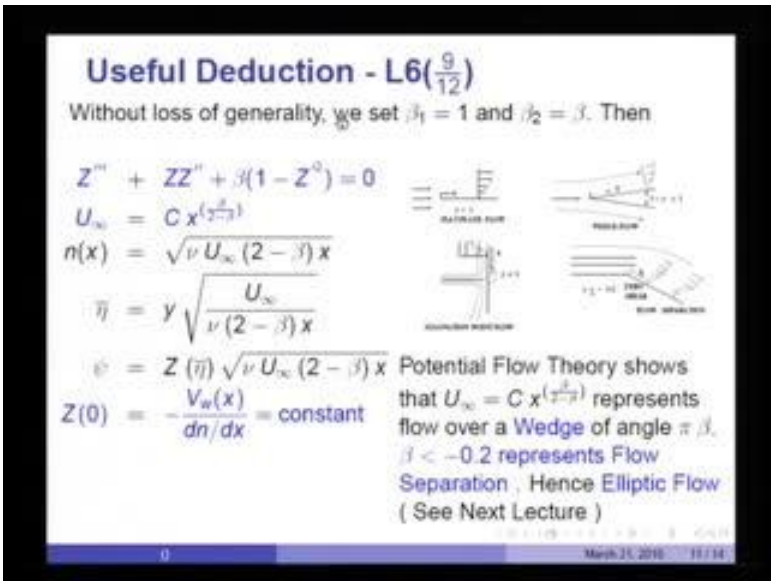

For example, I can set beta 1 equal to 1 and beta 2 equal to beta; that is what I have done. Without loss of generality, I can set beta 1 equal to 1 and beta 2 equal to beta. Then, the equation would read as Z triple prime plus Z Z double prime plus beta 1 minus Z square equal to 0. U infinity will simply read as C x raise to beta over 2 minus beta.

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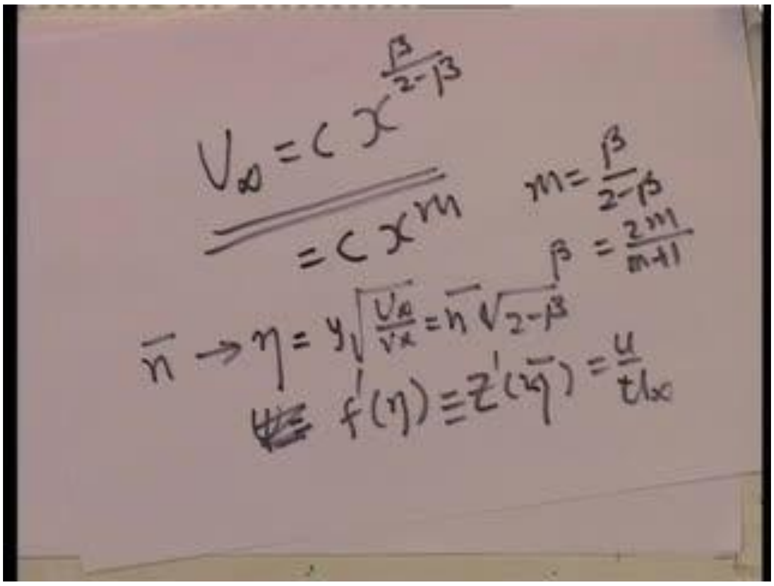

Now, this type of variation of U infinity as a function of C x raise to beta; U infinity equal to C times x times raise to beta over 2 minus beta - Has a very special significance in fluid dynamics and that is what I have shown here (Refer Slide Time: 25:33). The potential flow theory shows that U infinity equal to C x raise to beta over 2 minus beta represents flow over wedges of included angle pi beta. As you can see, (Refer Slide Time: 25:48) this is a wedge of included angle pi beta and the flow is this way. So, the free stream velocity variation would follow U infinity equal to C x raise to beta over 2 minus beta. This is what you get from the potential flow theory. If beta is equal to 0, that is U infinity equals constant; that could straightaway give you the flow over a flat plate, which is a wedge with included angle 0.

However, I can also open up the wedge fully and make it into pi beta equal to 1 or beta equal to 1. I will have what is called as a stagnation point flow because the flow would hit a plate perpendicular to it and would tend to go this way (Refer Slide Time: 26:39) as well as this way. In the positive x direction, the velocity profiles would be like I have shown here.

However, I can also have negative values of beta and that is what is shown here (Refer Slide Time: 26:52). Now, quite intuitively you can see that when a flow jumps over a hump, it is quite possible that downstream of the hump - beginning of the hump, the velocity profile will go through a point of 0 here. Then, the flow actually will recirculate with a negative velocity close to the wall now. That actually happens for beta less than minus 0.2 as we shall show (Refer Slide Time: 27:23).

Now, when recirculation occurs, our boundary layer theory falls flat in the sense that is no longer applicable because we enter the regime of elliptic flows. So, beta equal to minus 0.2 has a very special significance because that is the point at which the flow at best would separate; that is, have a 0 share velocity. If you increase beta beyond that value, then there will be recirculation. So, this generalization beta 1 equal to 1 and beta 2 equal to beta renders this equation (Refer Slide Time: 28:01) with some physical rigor and Z 0 equal to minus V w x over dn by dx must be constant. So, the set of equation that you see here tells you how U infinity should vary, how n should vary with x, how eta bar be defined, how psi be defined, and how Z 0 or in other words, V w should vary with x so that the equation is truly an ordinary differential equation.

For further discussion, I am going to change the definition I am going to say (Refer Slide Time: 28:43) U infinity equal to C x raise to beta over 2 minus beta will be written as c x raise to m, where m is equal to beta over 2 minus beta or beta is equal to 2 m over m plus

1. eta therefore; I am changing eta bar to eta. I am defining eta as y under root U infinity by nu x, which will be equal to eta bar times under root 2 minus beta.

Remember: f dash eta equal to Z dash eta bar equal to u over U infinity. The reason I am doing this is to make further analytical development more elegant. There is no other purpose other than making the analysis more elegant.

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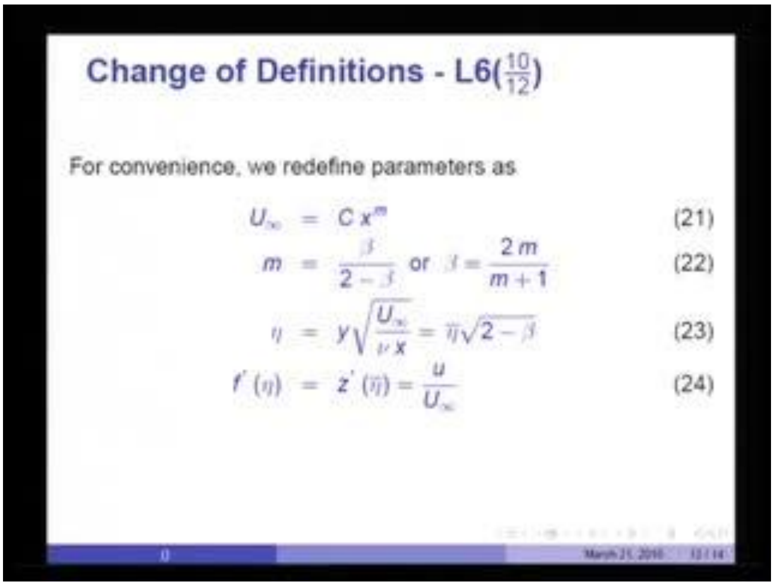

Having to write beta over 2 minus beta every time makes life somewhat difficult. So, I have simply defined m equal to beta over to 2 minus beta or beta is equal to 2 m divided by m plus 1. I have simplified eta by removing 2 minus beta into it. Now, the new variable is eta, I am replacing z by f and saying f dash eta will be equal to z dash eta bar equal to u over U infinity, as before.

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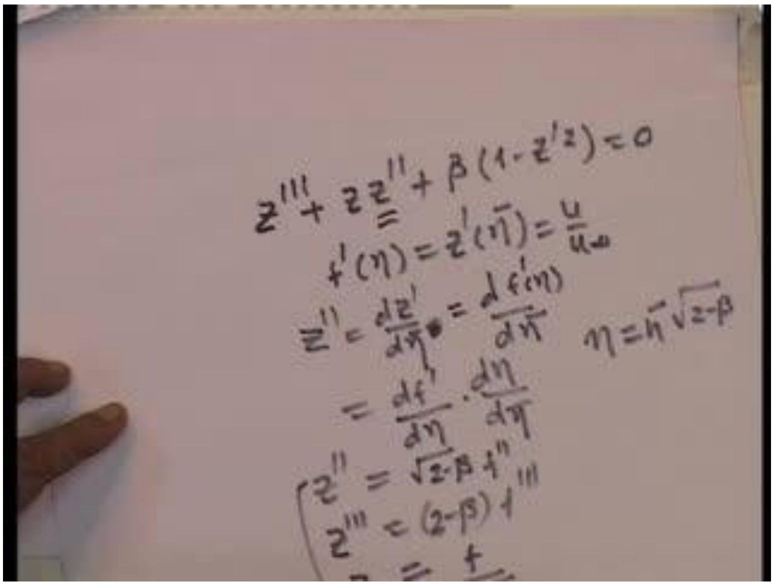

With this substitution, then you will notice that our equation Z triple prime plus Z Z double prime plus beta into 1 minus Z prime square equal to 0 and I am changing this. Remember: I have said f dash eta equal to Z dash eta bar equal to u over U infinity.

Now, you can see what would be Z double prime; Z double prime will be d z prime by d eta bar, but Z prime is f prime eta, which I can also write as df prime by d eta into d eta by d eta bar. However, remember: eta is equal to eta bar under root 2 minus beta. Therefore, Z double prime will be simply under root 2 minus beta into f double prime; that is equal to Z double prime. Likewise, you can show Z triple prime will be 2 minus beta f triple prime and you can also show that Z will be equal to f by under root 2 minus beta.

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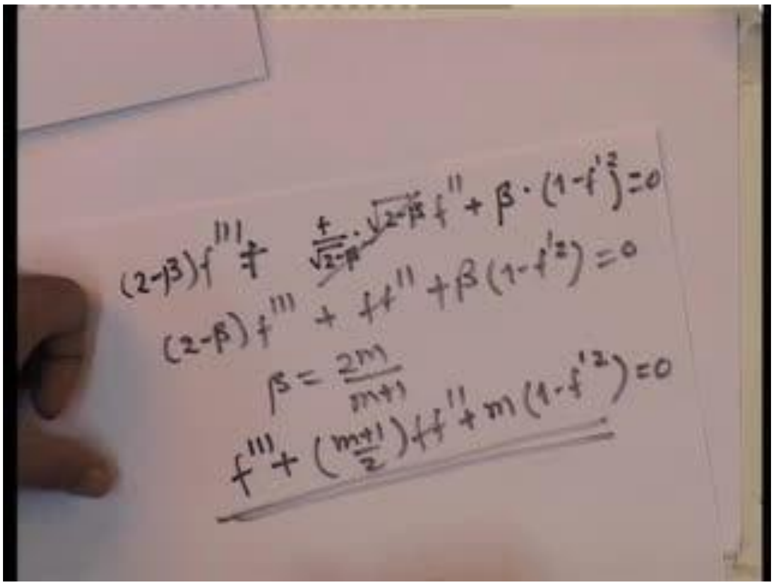

With these three substitutions in this equation (Refer Slide Time: 31:54). So, I will have 2 minus beta f triple prime plus f divided by under root 2 minus beta into under root 2 minus beta f double prime plus beta into 1 minus f prime square equal to 0. You will see this and this get cancelled. So, you get essentially 2 minus beta f triple prime plus ff double prime plus beta into 1 minus f prime square equal to 0.

Now, as we know, we already define beta equal to 2 m over m plus 1. If you have to substitute for 2 minus beta and beta, you will see this equation reduces to f triple prime plus m plus 1 by 2 ff double prime plus m times 1 minus f prime square equal to 0.

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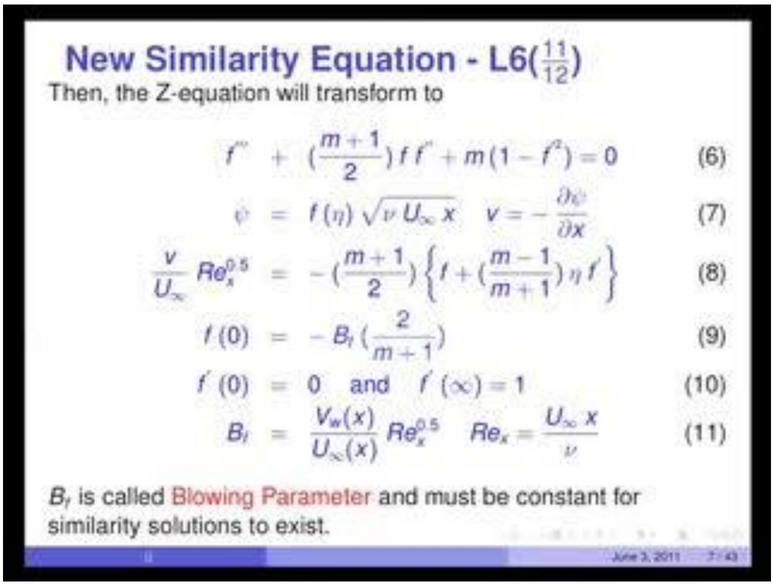

In the new similarity variable f and eta, appropriate Z equation will transform to this equation in f. psi is equal to f eta under root nu U infinity x and therefore, v will be equal to d psi by dx.

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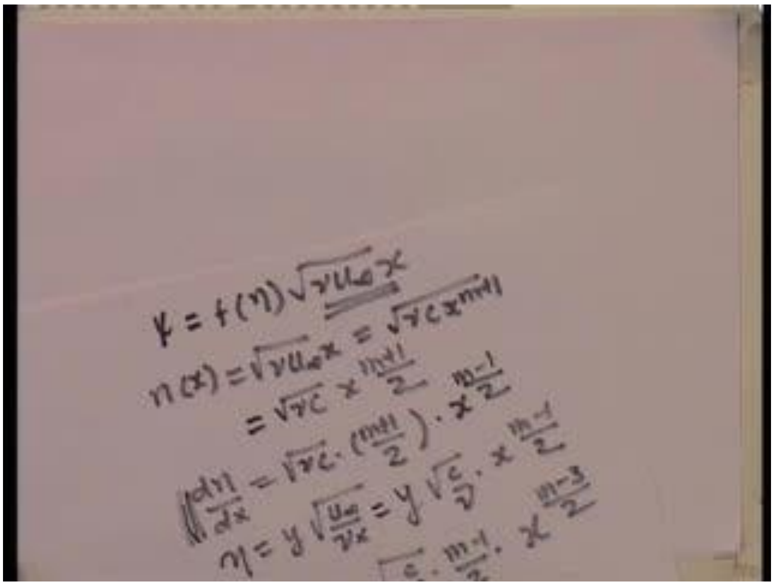

If you differentiate this equation, you will see psi equal to f eta under root nu U infinity x, which is really the n x. So, n x is equal to under root nu U infinity x, but U infinity is c x raise to m. Therefore, nu c x raise to m plus 1. That will be equal to nu c x raise to m plus 1 by 2. dn by dx will be under root nu c into m plus 1 by 2 into x raise to m minus 1 by 2. Similarly eta, which is y times under root U infinity by nu x will become equal to y times under root c by nu into x raise to m minus 1 by 2. This is very important: dn by dx. Therefore, d eta by dx will be simply y times under root c by nu m minus 1 by 2 into x raise to m minus 3 by 2. This is another equation (Refer Slide Time: 35:03) of great value.

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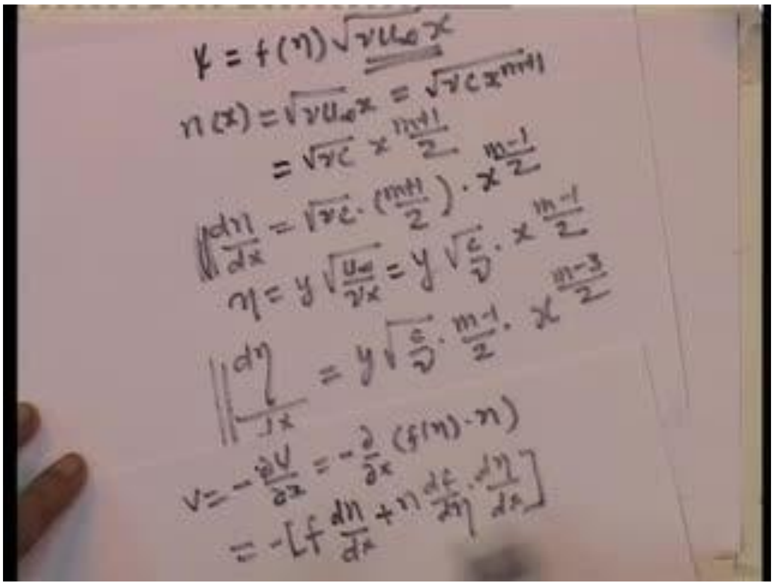

V, which is equal to minus d psi by dx, which is minus d by dx of f eta n. Then, you will see this will be simply equal to minus into f dn by dx plus n into df by d eta into d eta by dx.

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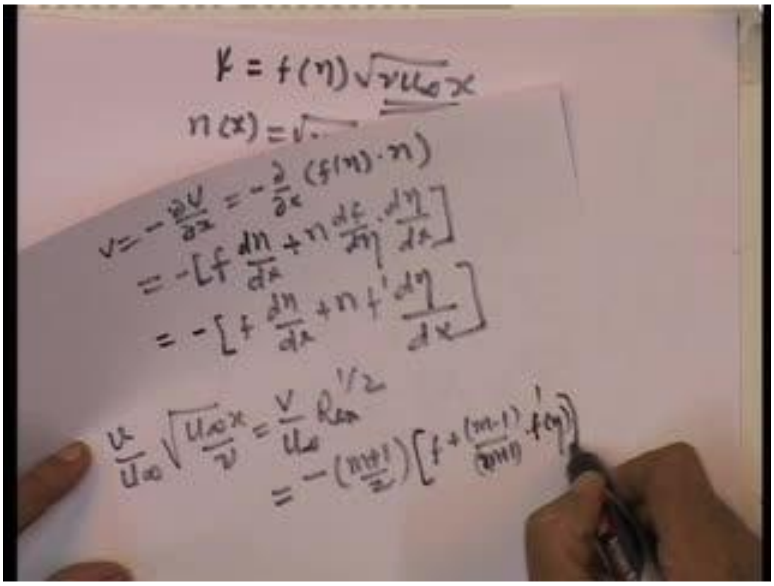

If you substitute the last two derivations; this quantity for dn by dx (Refer Slide Time: 35:45) and this quantity for d eta by dx and replace this by f dash, then this is nothing but f dn by dx plus n f dash d eta by dx. Then, simply substitute these 2 quantities (Refer Slide Time: 36:07) and you can show after some algebra that v by U infinity under root U infinity by x by nu, which is nothing but v over U infinity Rex to the power half. It is equal to minus m plus 1 by 2 into f plus m minus 1 over m plus 1 into f dash eta and that is what I shown here.

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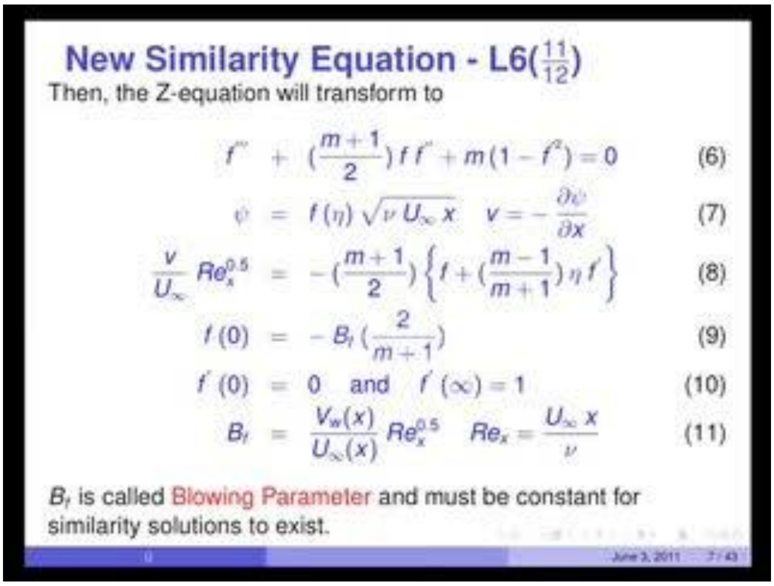

You will now get a solution to v in terms of f, f dash and eta. So, v is a function of eta is known in terms of the pressure gradient parameter m. At f 0, you have V w by dn by dx so that it will transform into V w x by U infinity Re x by because all I do is if f equal to 0, I have simply put f 0 and f dash 0. f dash 0 is 0 already; as you now, will become the no slip condition.

Essentially, f 0 then becomes V w divided by U infinity by x Re x to the power half, which is the B f, which is called the blowing parameter. Re x to the power half and Re x is U infinity x by nu.

We have an equation: f triple prime m plus 1 by 2 into f double prime plus m into 1 minus f dash square equal to 0 with the 3 boundary conditions f 0, f dash 0 and f dash infinity.

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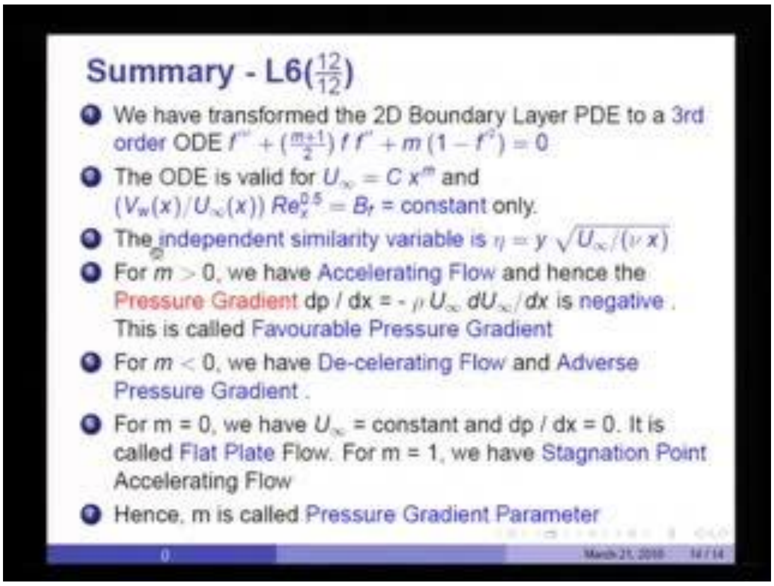

Let me summarize. We have transformed the 2D Boundary Layer partial differential equation to a third order ordinary differential equation f triple prime plus m plus 1 half f f double prime plus m 1 minus f double prime square equal to 0.

The ODE is valid for U infinity equal to C x raise to m and V w x by U infinity x into Re x 0.5 equal to B f equal to constant only. So, it is a solution for any arbitrary variations of U infinity and V w. The equation is not an ODE then. These are called the similarity conditions. So, the solutions can only be obtained for these restricted variations of U infinity and V w. The independent similarity variable is eta equal to y under root U infinity divided by nu x.

Because U infinity equal to C x raise to m, for m greater than 0, U infinity will increase with the x or we say we have an Accelerating Flow or a Negative Pressure Gradient. Likewise, if m less than 0, we say we have a De-celerating Flow or a Positive Pressure Gradient or an Adverse Pressure Gradient. For m equal to 0, we already notice that U infinity is a constant and therefore, the Pressure Gradient is 0. It is called the Flat Plate Flow. For m equal to 1, we have a Stagnation Point Flow or an Accelerating Flow. Overall, we say m is a Pressure Gradient Parameter.

In the next lecture, I will show you how to solve these ordinary differential equations.