Convective Heat and Mass Transfer

Prof. A. W. Date
Department of Mechanical Engineering
Indian Institute of Technology, Bombay

Module No. # 01

Lecture No. # 05

Laminar Boundary Layers

In the previous lectures, having derived the equations of bulk mass conservation, the momentum equations, the mass transport equation and the energy transport equation, we are now ready to take up their application to a special class of flows which are called boundary layer flows.

I will be beginning with boundary layer flows, which are in laminar state. Today's lecture is to derive appropriate equations for laminar velocity bounded errors and scalar bounded errors.

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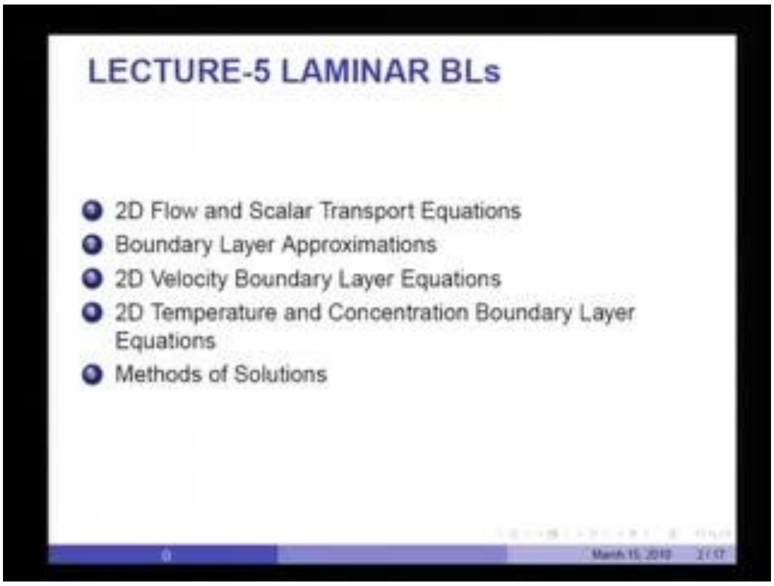

The purpose is to derive 2 dimensional flow and scalar transport equations to invoke the boundary layer approximations, to write out 2 dimensional velocity boundary layer equations and to write out 2 dimensional temperature and concentration boundary layer equations.

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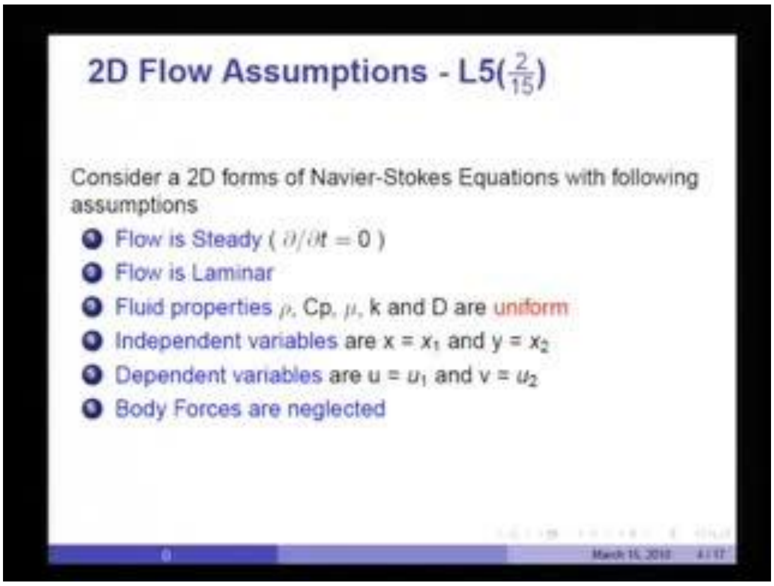

Then very briefly mention the methods of solution, which we shall be developing in the course of lectures. Recall that our 3D Navier Stokes equations were written in this manner d rho m by dt d rho m U j by dx j equal to 0.

Then there were 3 momentum equations; one each in direction x 1 x 2 and x 3 and it comprised of unsteady term, the convection term, the pressure gradient term, the diffusion term and the body force term.

This is the second part of the stress term that gives rise to d by dx j mu d U j by dx i. Further, we are going to make certain assumptions because as I said we are we want to avoid use of numerical methods and try to achieve as much as possible by analytical means.

This requires that we make certain assumptions- the first assumption is the flow is steady d by dt equal to 0. I am not at all suggesting that analytical solutions to unsteady flow are not possible but then our main interest in this course is to deal with steady flows in equipments.

Therefore, I will take assume that d by dt equal to 0. I would say that the flow is laminar and perhaps the most important assumption here is that all properties the intrinsic properties, density and specific heat and the transport properties mu k and d are uniform.

You will recall mu arose out of Stokes's stress and strain loss, k arose out of the Fourier's law of heat conduction and d arose out of the Fick's law of mach diffusion. We are going to say that they are all uniform.

That means they do not vary with position in the flow and therefore for all practical purposes they are constants in space. I will also now, since we are dealing with 2 dimensions, I will instead of writing x 1 x 2, I shall write x and y by saying x is equal to x 1 and y is equal to x 2.

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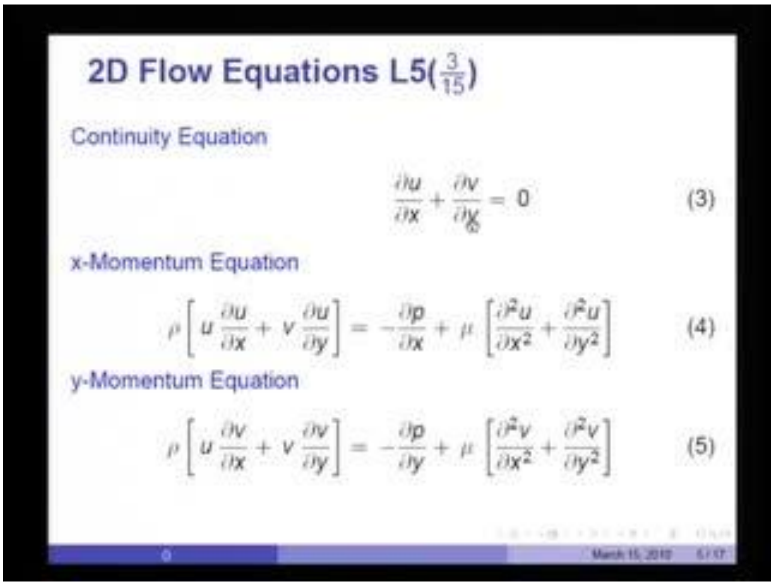

And the dependent variables, I would change to u equal to u 1 and v equal to u 2 and body forces which are essentially problem specific are presently to be ignored. So with this assumption you will notice our mass conservation equation would simply because density is constant would reduce to du by dx plus dv by dy.

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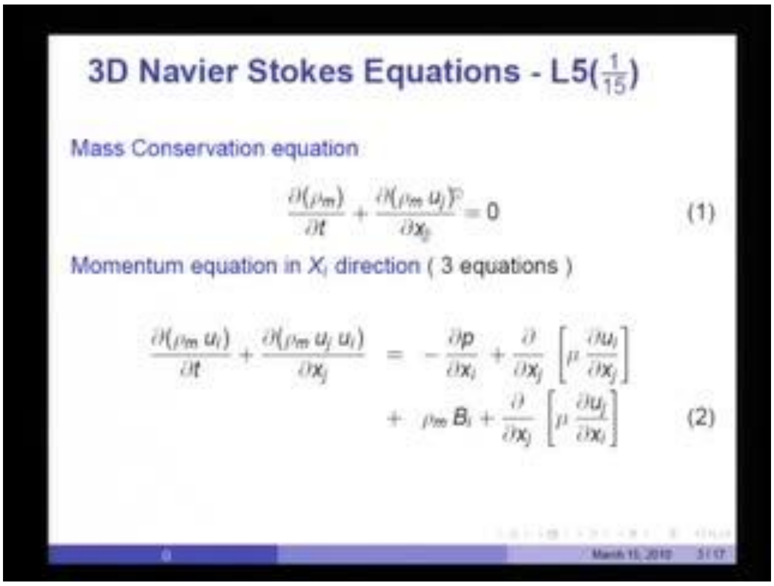

I can go back a little to see, if density is constant then that term is 0 and this term rho m du j by dx j equal to 0 would simply be du 1 by dx 1 plus du 2 by dx 2 equal to 0. Therefore, that will simply read as du by dx plus dv by dy equal to 0.

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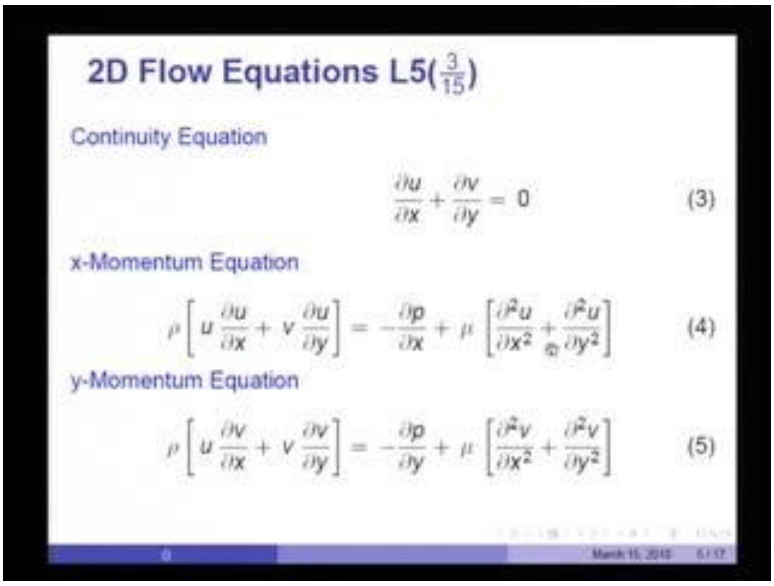

The x momentum equation or the momentum equation in x direction would read as rho times u du by dx plus v du by dy equal to minus dp dx plus mu d 2 u dx square and d 2 u dy square.

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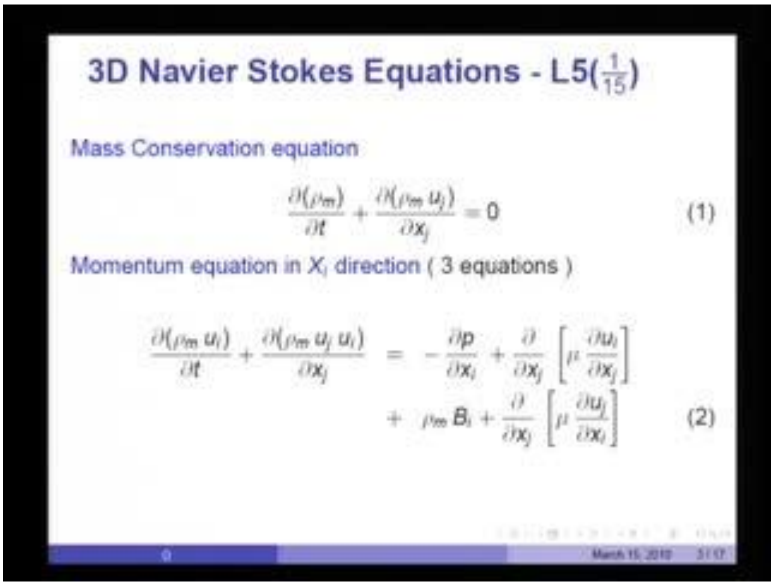

This requires little explanation. For example, if mu is constant then, d by dx j mu du j dx i would simply be written as mu times or if I may use the paper you will see that, d by dx j of mu times du j dx i would be mu times d 2 U j by dx j dx i, that will be equal to mu times d by dx i of du j by dx j and that is 0 by continuity equations for constant density du j by dx j is equal to 0.

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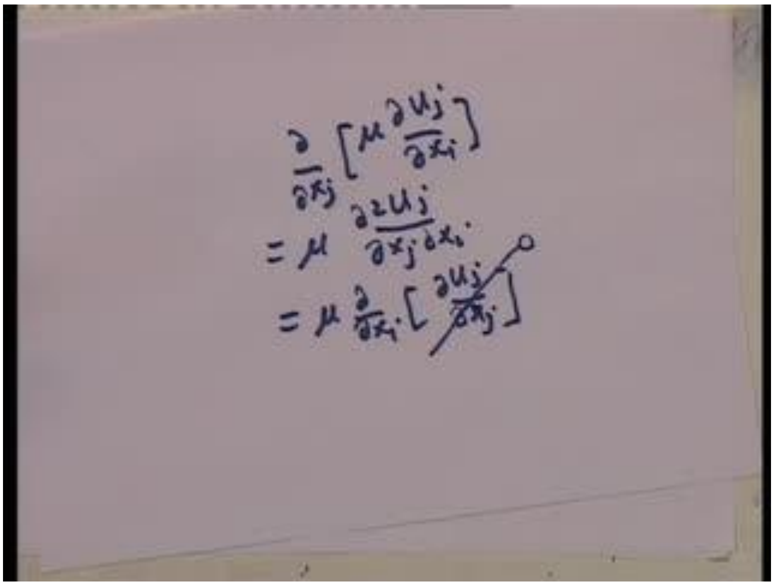

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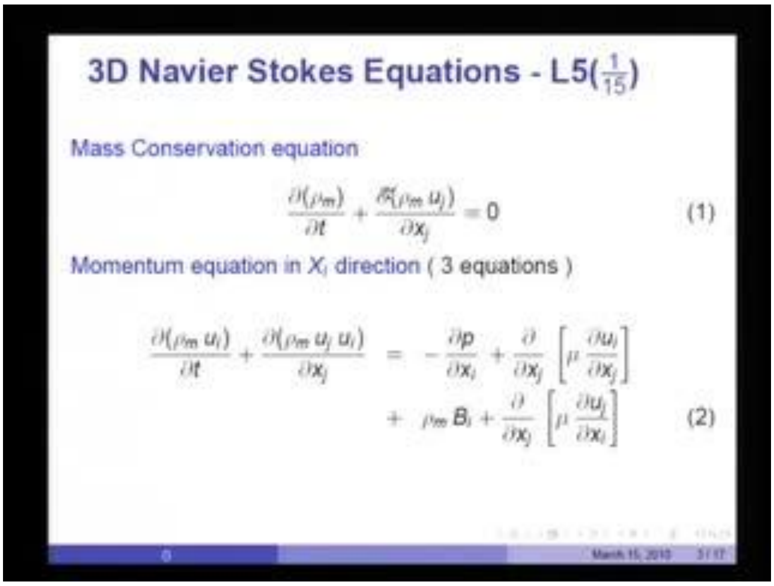

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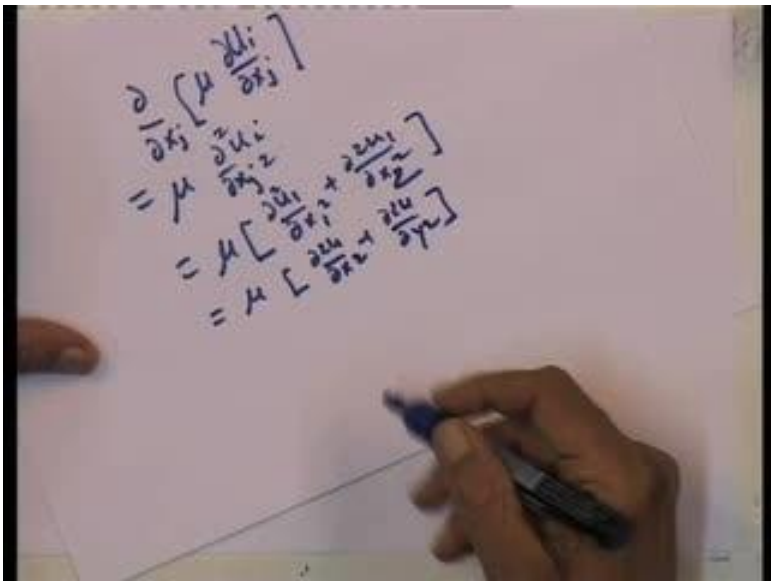

Therefore, this term simply vanishes. Now, let us look at this term - says that d by dx j of mu du i dx j for constant viscosity. This will be d by dx j u i and in 2 dimension; this will simply be du 1 by dx 1 d 2 u 1 by dx 1 square plus d 2 u by u 1 by dx 2 square and with our replacements mu this will be d 2 U by dx square plus d 2 u by dy square. So that explains, how that term would modify, we will look at this term now.

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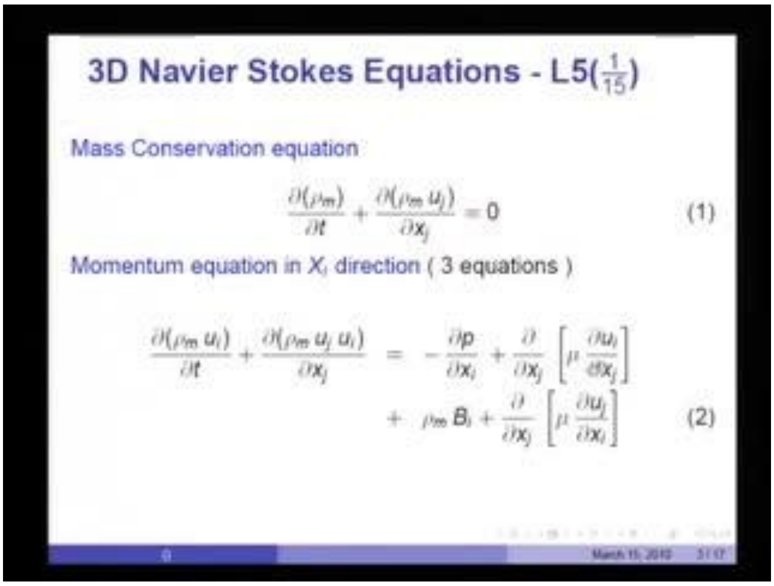

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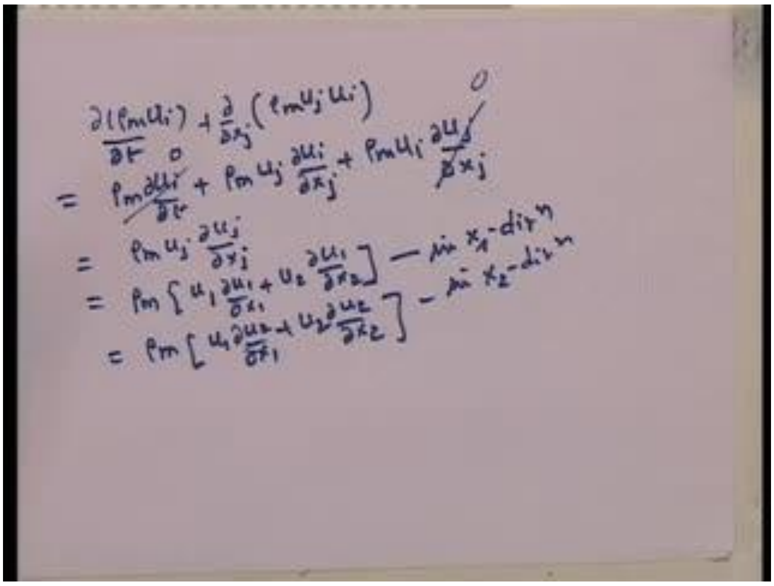

For constant density and steady flow, you have d rho m u i by dt plus d by dx j of rho m u j u i and for constant density, this will simply become rho m du i by dt plus rho m u j du i by dx j plus rho m u i du j by dx j.

Now, in a steady flow that is 0 and due to mass conservation equation du j dx j is 0. Therefore, this will become d rho m u j du j by dx j or in 2 dimension; this will become simply u 1 du 1 by dx 1 plus u 2 du 1 by dx 2 in x direction.

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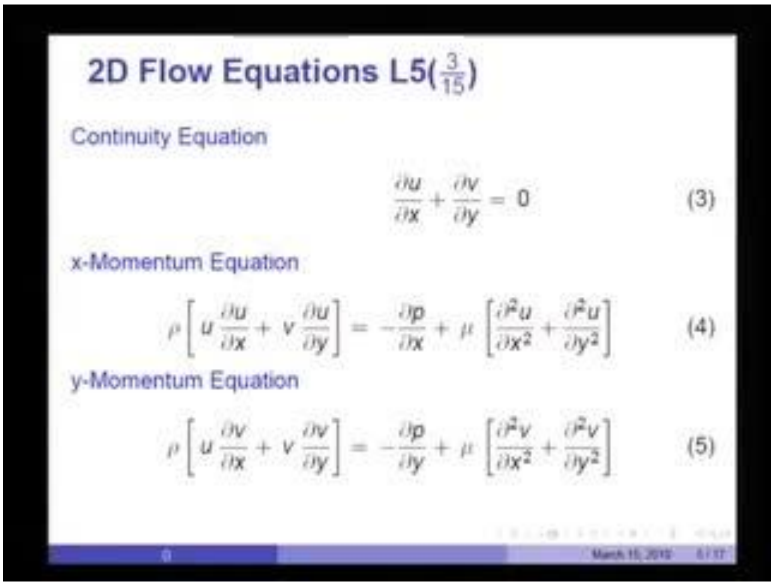

It will return rho m u 1 du 2 by dx 1 plus u 2 du 2 by dx 2 in x 2 direction. You will see now, that how these terms have been modified to read like that. So, you will see in x 1 direction for example there is rho m u 1 du 1 by dx 1 which is rho u du by dx plus u 2, which is v du by dy equal to minus dp by dx and the 2 diffusion terms that I mentioned.

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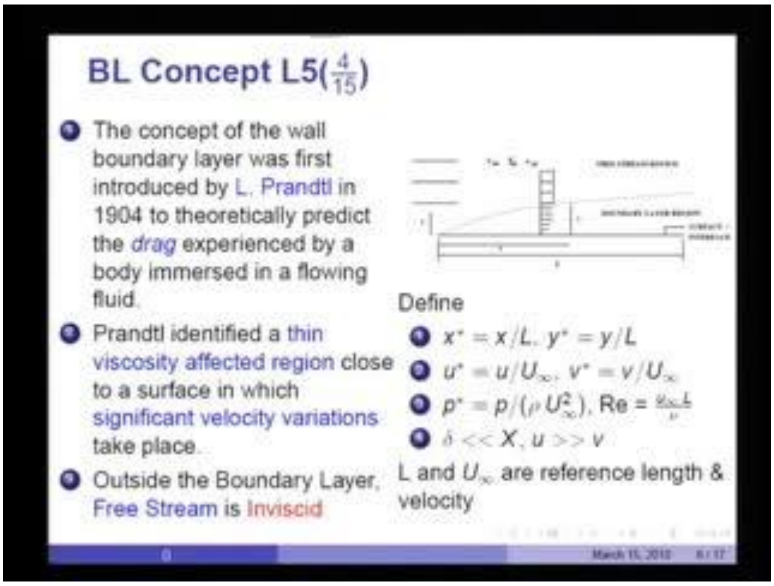

Similarly, in the y direction you would have that term. So, these are the equations of flow of a uniform property laminar 2 dimensional flow. We shall now invoke the boundary layer concept and this is perhaps the most important slide for you to remember.

Now, the concept of boundary layer near a wall was first introduced by Ludwig Prandtl in 1904, to theoretically predict the drag experienced by a body when it was immersed in a flowing fluid.

Ships experience drag, motor cars driving in a motorway experience drag, aircrafts experience drag and these drags are substantial. You have to expand energy to overcome them and therefore it is very important that how much is the drag offered by a body when it when the fluid flows past it.

Prandtl suggested that you do not need to consider the total flow around a body, but nearly concentrate on a thin region very close to the wall as shown here, which he called the boundary layer region. In this region that significant velocity variations take place.

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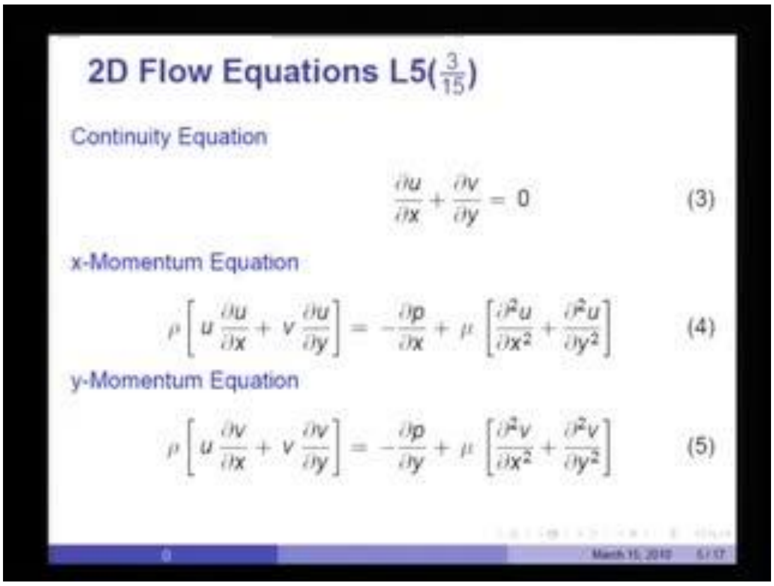

And it is the region, in which viscosity of the fluid is dominates the determinant of flow. In other words, viscosity viscous terms dominate very close to the wall. As you move away from the wall, the importance of these terms becomes even more negligible and it is these terms which mainly dominate in the far away from a solid surface.

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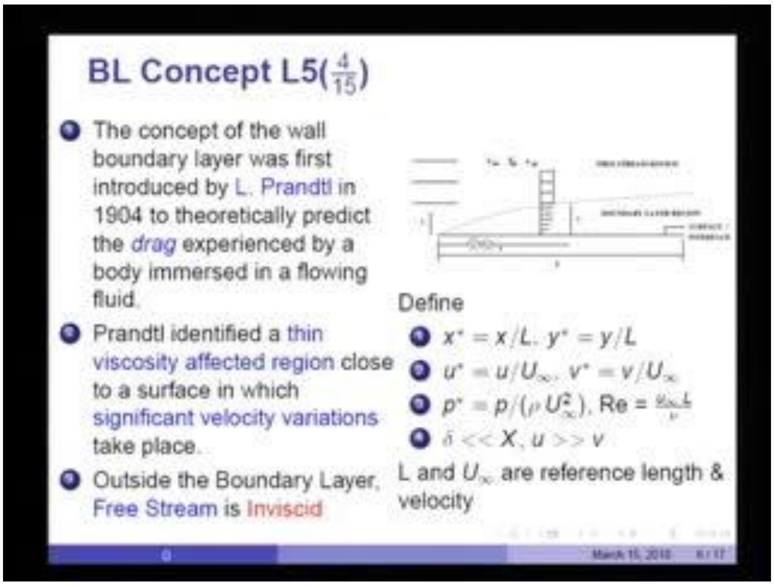

Prandtl called the thin and long region near a wall as the boundary layer region, by long and thin may mean that the dimension lateral to the flow delta at any point x in the flow delta is much smaller than x.

That is what we mean by long and thin flows passed the surface or an interface the region outside where a lateral velocity gradient are almost negligible is called the free stream region. In the free stream, there is velocity u infinity t infinity and a conserved property phi infinity.

Now, to develop this mathematical interpretation of this concept of long and thin flow close to a wall, we are going to introduce dimensionless variables. So, x star would be written as x divided by L, y star would be written as y divided by l, u star would be written as u divided by U infinity, v star would be written as v divided by u infinity, p star as p divided by rho infinity square, Re as U infinity, L by nu this is the reynolds number corresponding to reference length L and assumption is delta is much less than X and u the velocity in x direction is much greater than v.

It is this condition, which ensures that, in a boundary layer conditions of the properties of the flow at one cross section are influenced only by the upstream conditions. The conditions downstream have no influence on the conditions at a given cross section.

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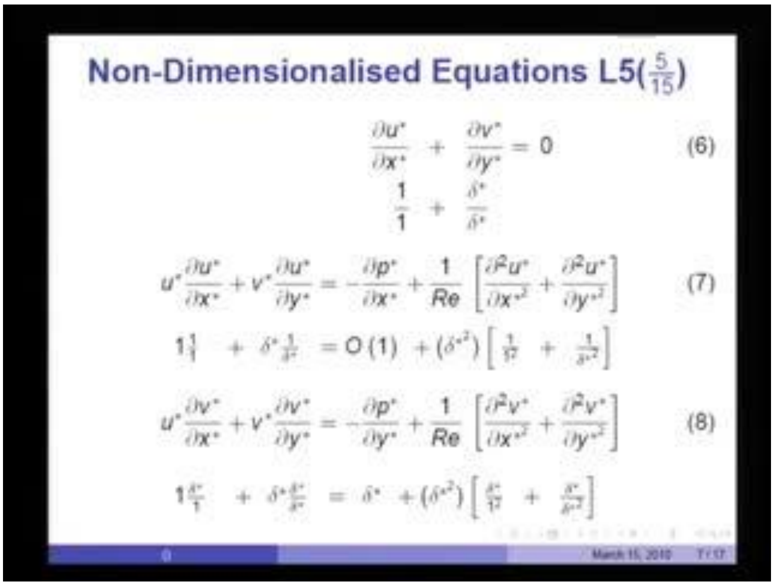

L and U infinity are simply the reference length and reference velocity. Each of these quantity start quantities is a dimensionless quantity and therefore the equations would read like this. So, let us look at the first equation after non dimensionalization.

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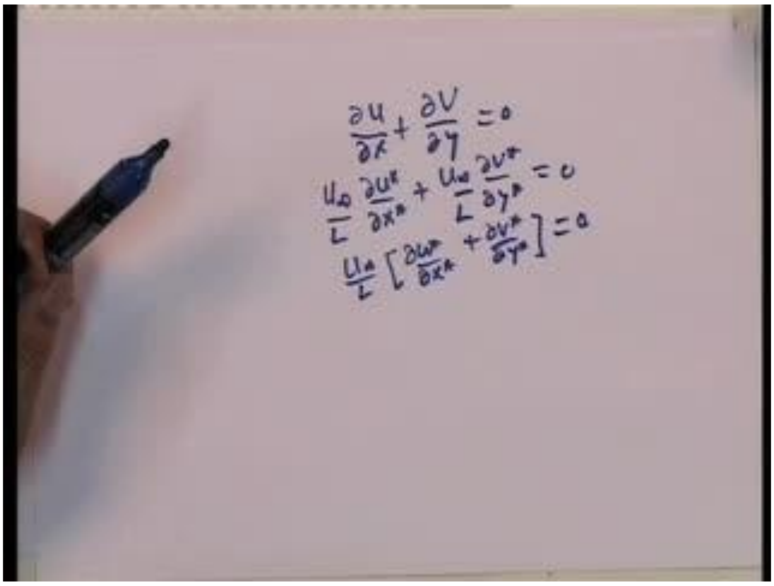

You will see that du by dx plus dv by dy equal to 0. If I make this u to u star, it will become u's infinity du star by dx star, which means divided by L and likewise u infinity dv star by dy star divided by L equal to 0.

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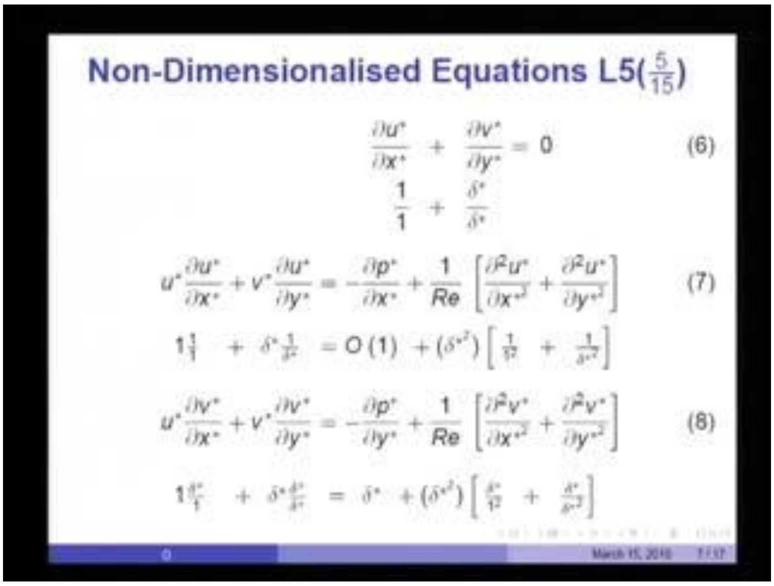

So, it is u infinity by L du star by dx star plus dv star by d y star equal to 0 and that is what I have written in equation number 6 here du star by dx star plus dv star by dy star equal to 0. Likewise, let us look at the momentum equation.

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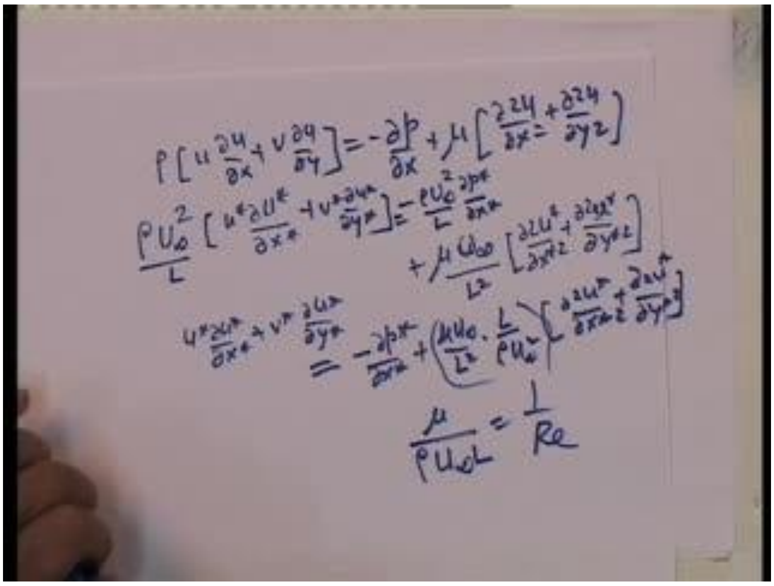

The Momentum equation was written as rho times u du by dx plus v du by dy equal to minus dp by dx plus mu times d2u by dx square plus d2u by dy square. If, I non dimensionalize this equation I will have rho u infinity square giving me u star du star by dx star, which is L plus v star du star by dy star minus rho u infinity square dp star divided by L dx star.

All this is equal to plus mu times u infinity divided by L square d2u star by dx star square plus d2u star by dy star square. If, I divide through by this quantity, you will see I will get u star du star by dx star plus v star du star by dy star equal to minus dp star by dx star plus mu U infinity by L square into L by rho u infinity square into d2u star by dx star square plus d2u star by dy star square. So, what is this group? This is simply mu divided by rho u infinity L and that is nothing but 1 over Reynolds number.

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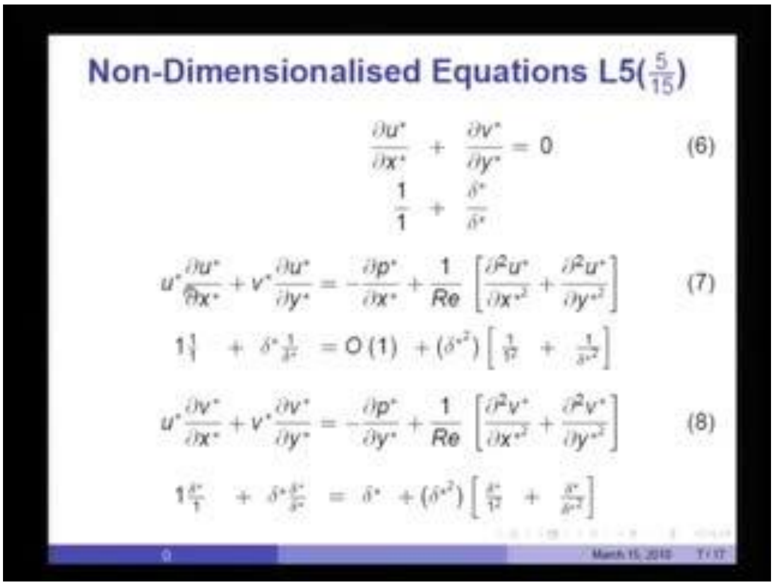

So, this equation is equation 7, u star du star by dx star plus v star du star by dy star minus dp star by dx star plus 1 over Reynolds plus d2u star by dx star square plus d2u star dy square. If, I have to non dimensionalize the v momentum equation, it would appear very similar to that equation with dp star by dy star, here the dependent variable would be v star and again divided by Reynolds number.

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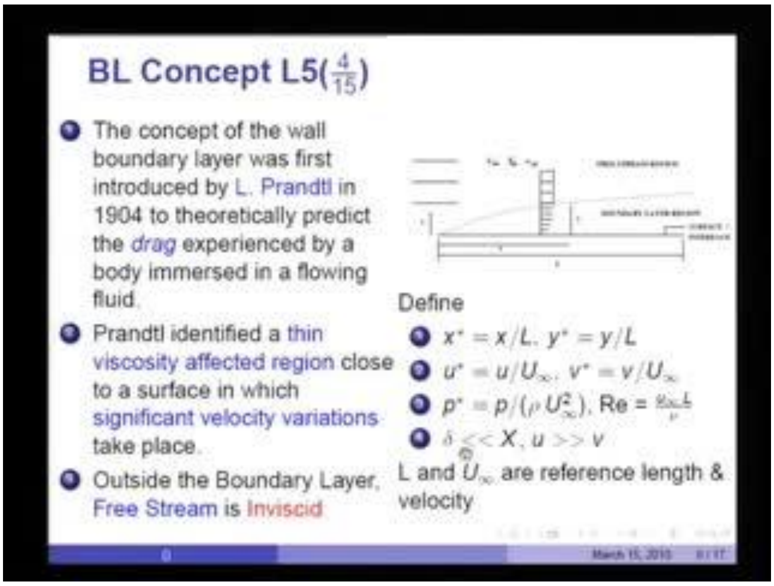

Now, I am going to do an order of magnitude analysis of this equation. So, we said u is considered of the order of 1 and x is considered of the order 1. Then, according to this assumption y dimension would be of the order of delta.

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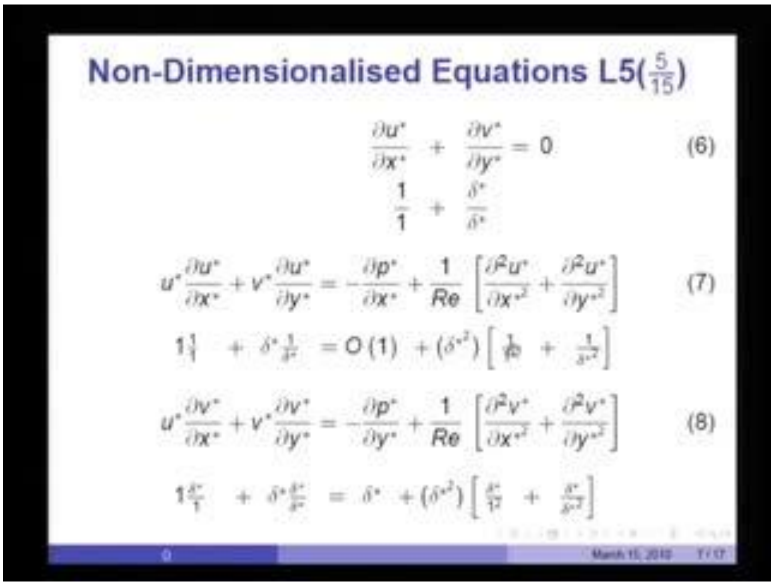

The v velocity will also be of the order of delta and that is what I have done. So, 1 divided by 1 plus delta star by delta star order of delta star and what this shows is that both the terms are of the order unity. Therefore, neither could be ignored in this equation.

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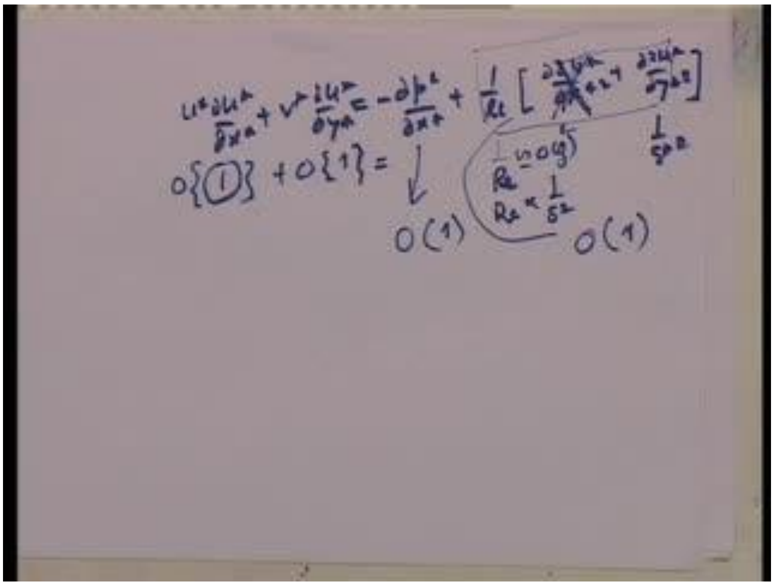

So, du star by dx star and dv star by dy star is equal to 0. Remember, here both numerator and denominators are of the order 1. Here, both are of the order delta but it is the derivative, which is most important and both derivatives are of the same order.

Likewise, let us do this for momentum equation. Then, u star is of order 1, u star dx star is 1 by 1. This is of the order delta star u star is of the order 1 divided by y star, which is delta star and therefore each term on the left hand side is of order.

I will now turn my attention straightaway to the diffusion terms. Then, you will see this term is order of 1 by 1 square. Whereas, this term is 1 divided by delta square and it is quite obvious that this term would usually dominate over this term because delta is so much smaller than 1. In this equation, u star du star by dx star plus v star du star by dy star equal to minus dp star by dx star plus 1 over Reynolds number into d2u star by dx star plus square plus d2u star by dy star square.

This term is much bigger than this term and therefore I will drop that term. Now, this term as a whole is of the order of 1. This is although of the order 1 and this term is of the order delta square delta square

But Prandtl did say that, you must have effect of viscosity included if you wanted to predict the drag, which means one of these two terms must be included. We have already decided to drop this term. So, if this term is included, if an all terms are of order 1 then

Reynolds number or 1 over Reynolds number must be of order 1 or I mean order delta square.

So that, Reynolds number is proportional to one over delta square, this is a large quantity. So that, the delta square would get canceled with this delta square and the total term would then be of order 1 and likewise I can say therefore by deduction this term would also be of order 1.

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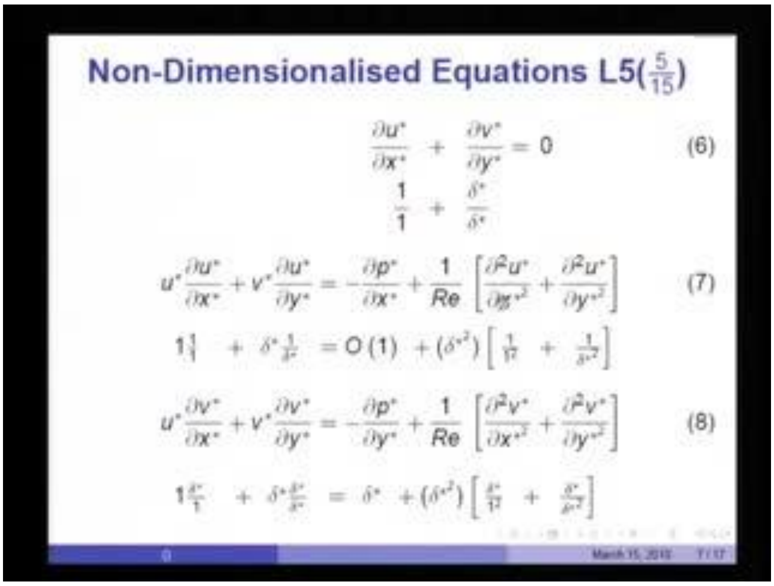

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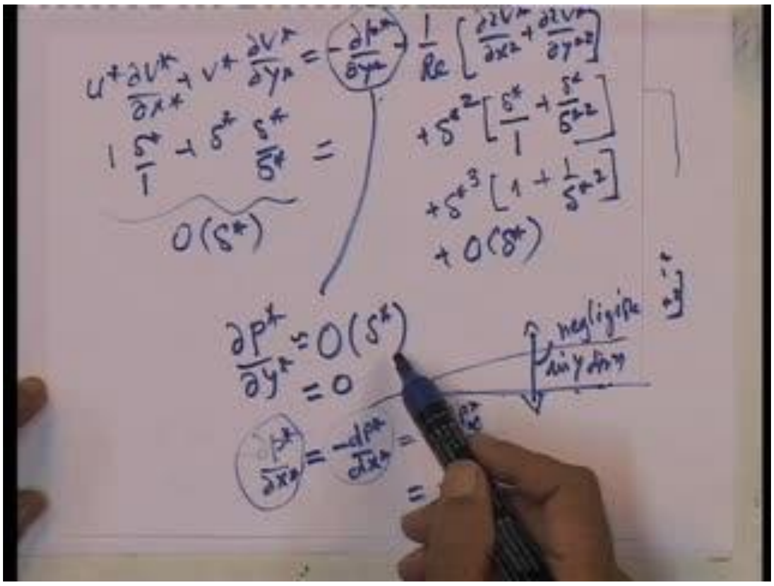

So, the only term that is dropped now, that can be dropped is dt u star by dx star square, all other terms are of order 1. Let us examine likewise the v momentum equation, then you will see that the v momentum equation is u star dv star by dx star plus v star dv star by dx star dy star equal to minus dp star by dy star plus 1 over Reynolds number into d2v star by dx star plus d2v star by dy star square.

This is 1 delta star by 1 plus delta star delta star by delta star. So, you will notice that the left hand side is of order delta square. We have agreed that 1 over Reynolds number shall be of order delta square multiplied by 1 plus delta star by delta star square.

And again you will see that, if I take delta star common then I have delta star cube is a 1 plus 1 over delta star square. So, you will see that the first term is again much smaller than the second term because the first term is of order 1 the second term is of order delta square and therefore that can be neglected but the total term again is of order delta star.

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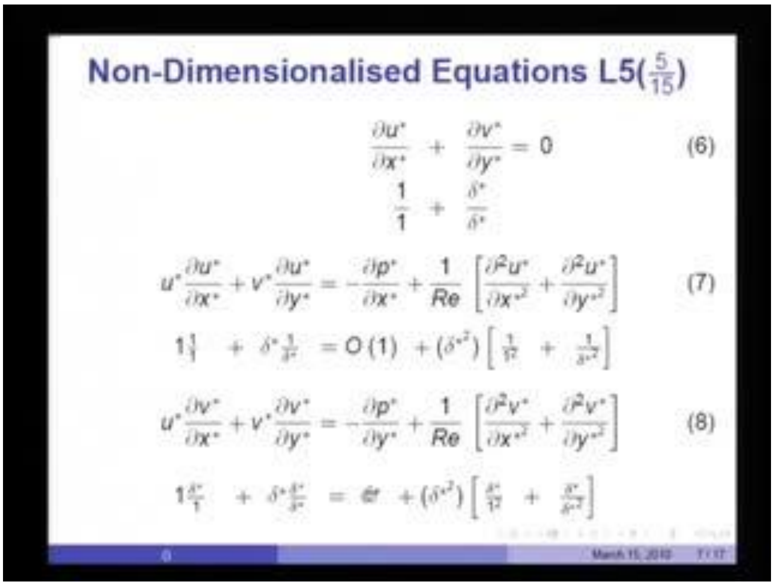

So, each term in this equation is of the order delta star and therefore dp star by dy star will be of order delta star square. So, we conclude that in the x momentum equation each term is of order one.

Whereas, in the v momentum equation or the y momentum equation each term is of order delta star. Therefore, this equation as the whole can be ignored in preference to that equation, which is much bigger so and the principle deduction from this is that dp star by dy star is of the order delta star or can be taken as 0.

What this says is that, in a boundary layer in a long and thin flow the pressure variations is negligible in y direction. In fact, I can say that from this it follows that minus dp star by dx star now would essentially be minus dp star by dx star, which I can also write as minus dp star infinity by dx star. Which is also equal to minus dp whole star by dx. I can measure the pressure at the wall in a given flow and I would get the quantity dp star by dx star.

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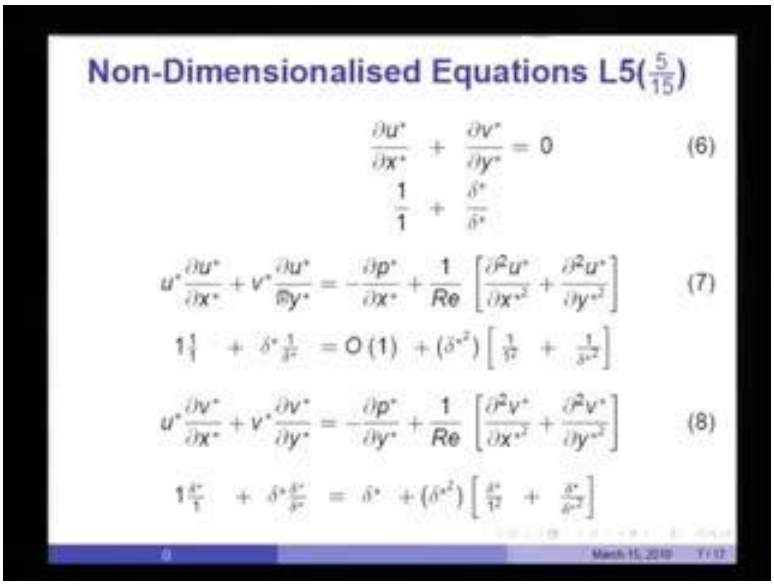

And the partial derivative is now replaced by total derivative simply because the variation in y direction is 0. The conclusion from this slide is each term in this equation is of order 1 therefore retained.

Each term in this equation is of order 1, the only term which is not is, this term the actual second derivative in actual direction and ignore each term in this equation is of order delta square and of the order delta star. Therefore, the equation as a whole is neglected.

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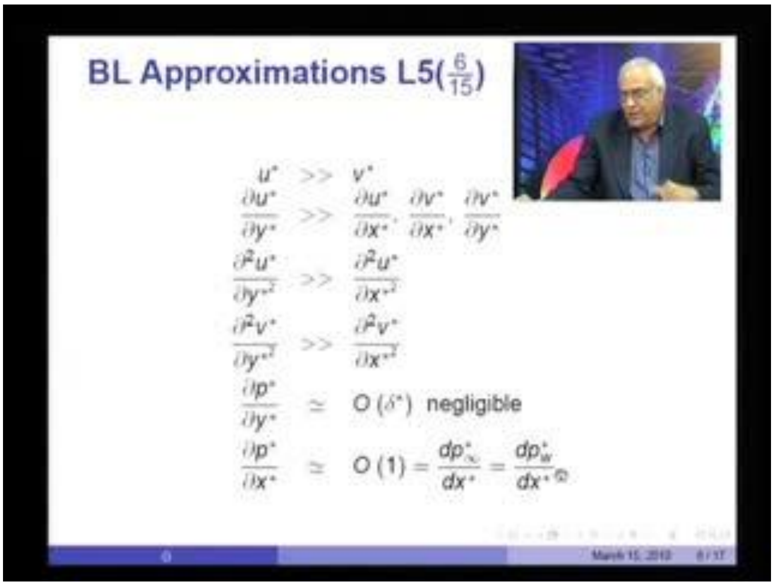

You will then see the boundary layer approximations that emerged. So, what does it say? It says that u star will be much greater than v star, which we had already postulated the gradient of velocity in a y direction.

The u velocity in y direction would be much greater than the gradient of velocity because would be one over delta star, whereas this is one over one this is delta star by one and this is delta star by delta star.

Therefore, this would dominate over all these. We already shown that the second derivative in y direction will be much greater than the second derivative in x direction both for u and v. We also shown that the pressure gradient in y direction would be almost negligible.

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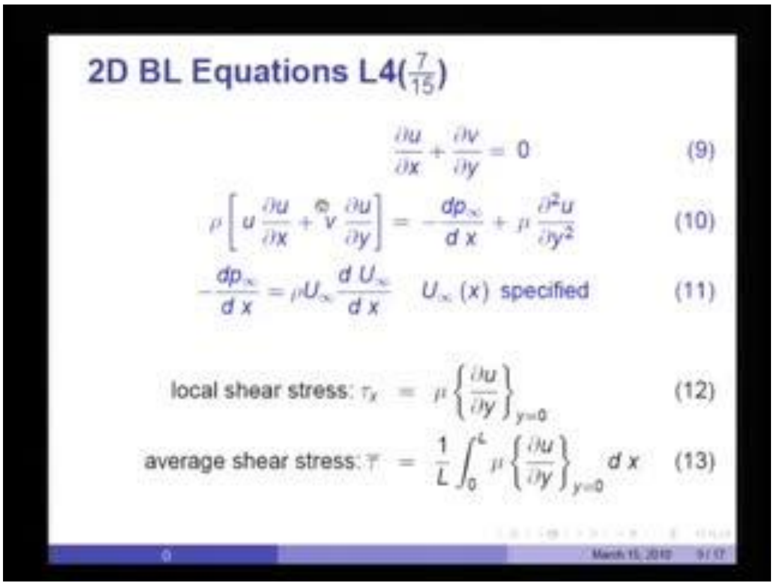

Therefore, the pressure gradient in x direction which is of the order 1 can be replaced by either the pressure gradient in the infinity state that is the free stream region or it can be evaluated at the wall state dp star wall dx star. The reduced equations also called the Boundary layer equations are simply du by dx plus dv by dy equal to 0.

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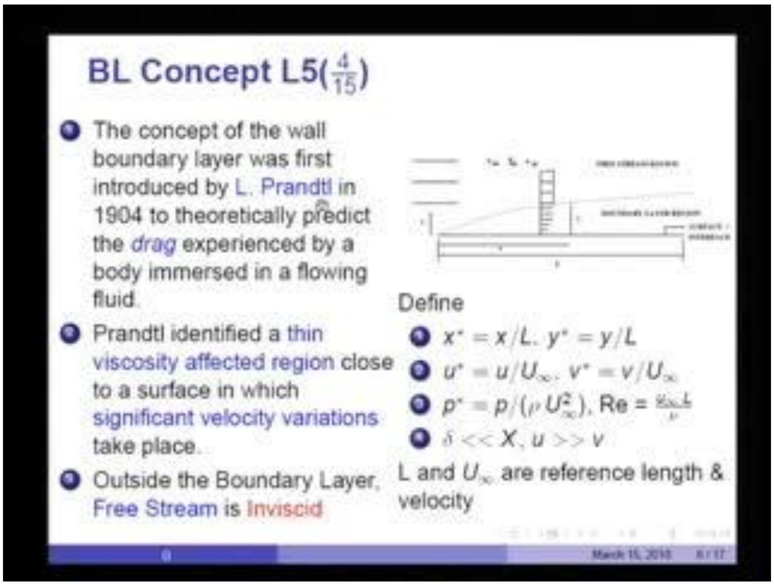

Rho into u du by dx plus v du by dy equal to minus dp infinity by dx plus mu d2u by dy square. Now, just see for a moment, if you look at this figure you will see outside in the free stream region the velocity variation with respect to y is negligible or 0.

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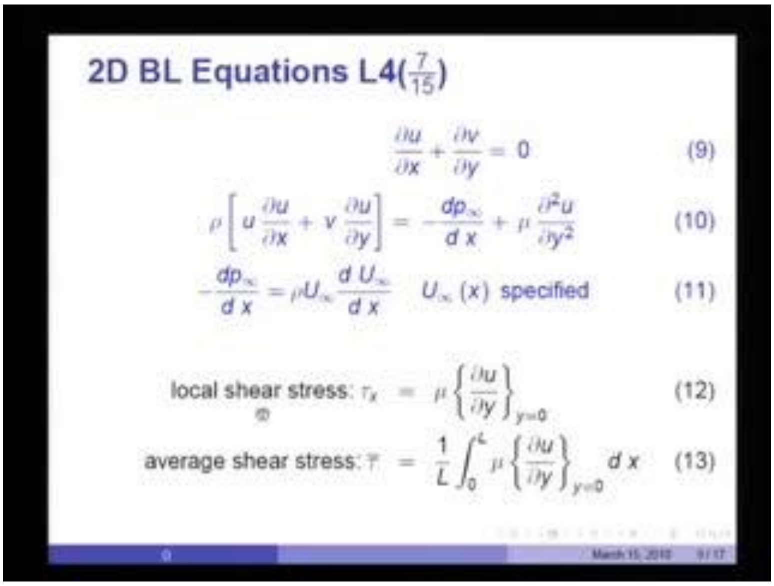

Because u remains equal to u infinity outside the boundary layer. So, if I want to write this equation in the infinity state that is in the free stream region, I will have this will become rho infinity u infinity du infinity by dx but this term would be 0 because du dy is 0 and also d2u dy square is also 0, because this term vanishes in the free stream. The free stream region is sometimes called in viscid region because viscosity is not allowed to play any part.

The momentum equation written for the free stream region would simply be minus dp by infinity by dx equal to rho u infinity du infinity by dx and in fact, I can replace that term by this term.

These equations must be solved with n boundary conditions because there is a second order derivative in y direction. Therefore, you need boundary condition at y equal to 0 which is y equal to wall and y tending to infinity which is the free stream condition.

If u infinity x is specified, one could readily replace that by this condition. When the equations are solved with appropriate boundary condition, you would have u as a function of x and y and v as a function of x and y as a solution because this term is specified have two equations and two unknowns u and v. These can be readily obtained and what you are interested is the shear stress at the wall, which is simply mu times du dy at y equal to 0 which is the local shear stress tx.

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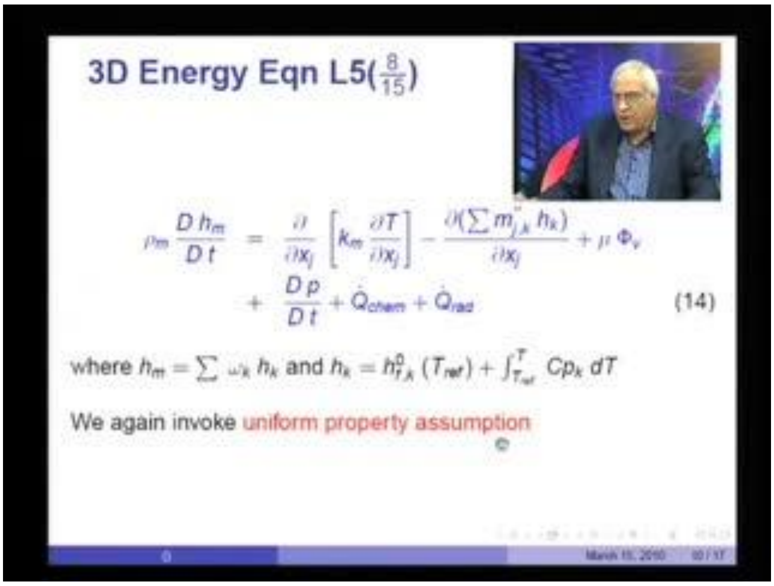

The average shear stress which is given as 1 over L 0 to L mu du dy by dx divided by 1 over L will give you the total drag over a surface L of length L. So, we have finished our discussion on velocity boundary layer equation.

Now, we will turn our attention to energy equation and you will recall the energy equation that we wrote on the last slide of the previous lecture. So, this is the rate of generation of rate of change of enthalpy plus conduction heat transfer first diffusion heat transfer the heat transfer due to mass diffusion.

This is the viscous dissipation, this is the pressure work terms and these are the chemical energy and this is the radiation, h m as you know is omega k h k, where h k is the specific enthalpy of spaces k and h k is also given as heat of formation at sometimes at T ref plus sensible heat Cp k dT.

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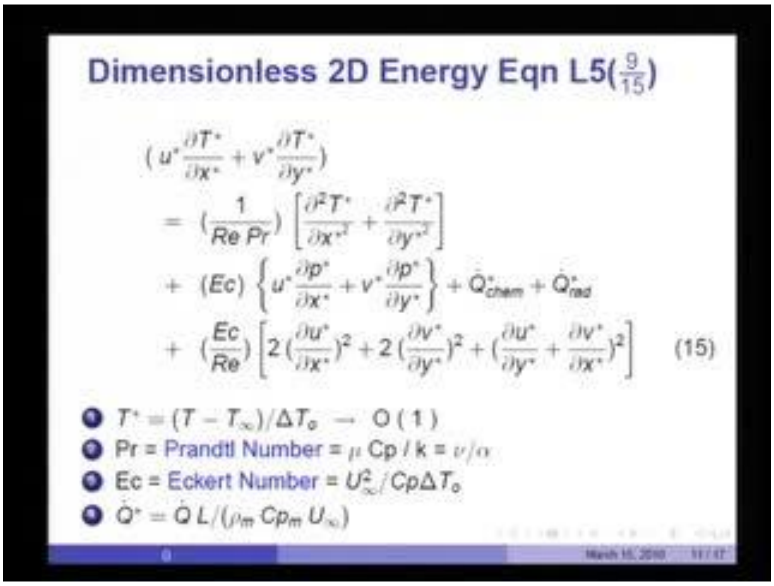

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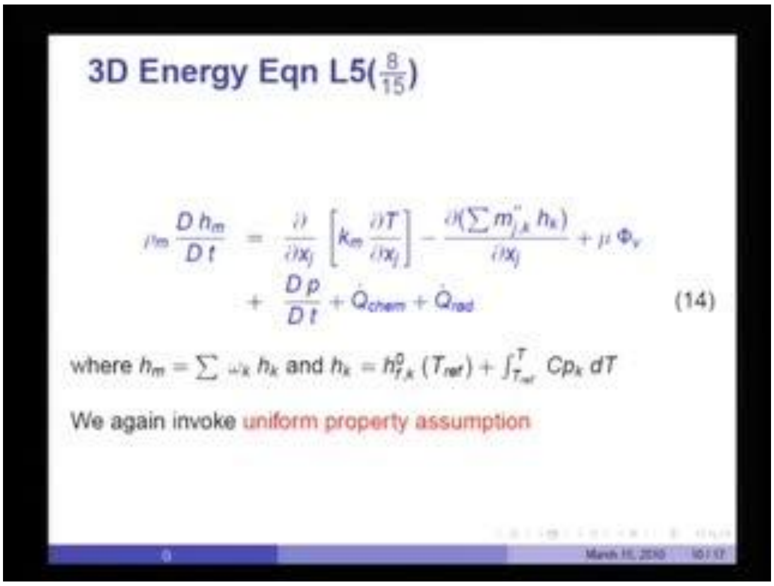

We again invoke uniform property assumption in 2 dimensions. The equation would read something like that; let me go back a little so the first equation would read in 2 dimensions.

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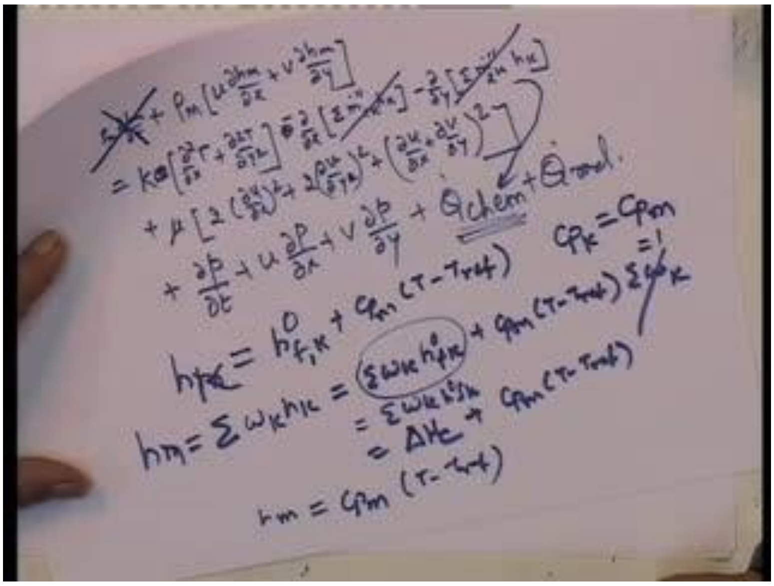

It would read rho m dhm by dt plus rho m into u dhm by dx plus v dhm by dy equal to and with uniform property d2t dx square plus d2t dy square plus d by dx of sum 1k h k minus d by dy of sigma m naught 2k h k plus mu times 2 into du by dx square

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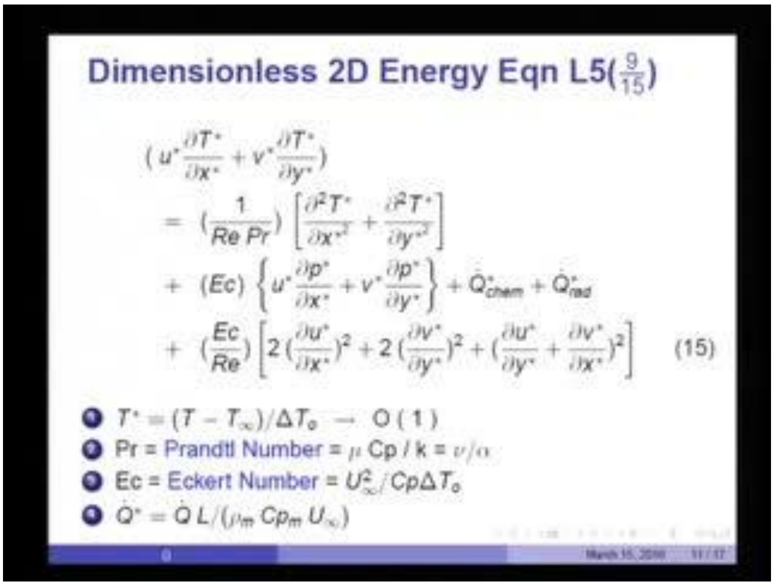

plus 2 times dv by dy square plus dp by dt plus u dp by dx plus v dp by dy plus Q dot chem plus Q dot rad. So, you will see in 2 dimensions the equation takes this form and now, if I first of all make all our assumptions that the flow is steady therefore that is 0.

h m which is h naught f h k which is h naught f k and Cp k equal to Cp m then Cp m T minus T ref because all spaces have the same property of the mixture and therefore Cp k is equal to Cp m.

It is a uniform mixture you have k times d2T dT square and if I ignore for the time being the diffusion equation the diffusion heat transfer for a single phase flow non reacting flow then or absorb that in this.

Then you will see I get terms like this. Now, I am going to non dimensionalize these terms. This as the whole can sigma omega k h k, which is h m would become sigma omega k h naught fk plus Cp m T minus T ref into sigma omega k and which you know is equal to 1.

So, you get that term sigma omega k h naught fk and that is equal to heat of combustion plus Cp m T minus T ref and these term essentially is accounted by the chemical reaction and I can replace h m here for is non reacting flow Cp m T minus T ref.

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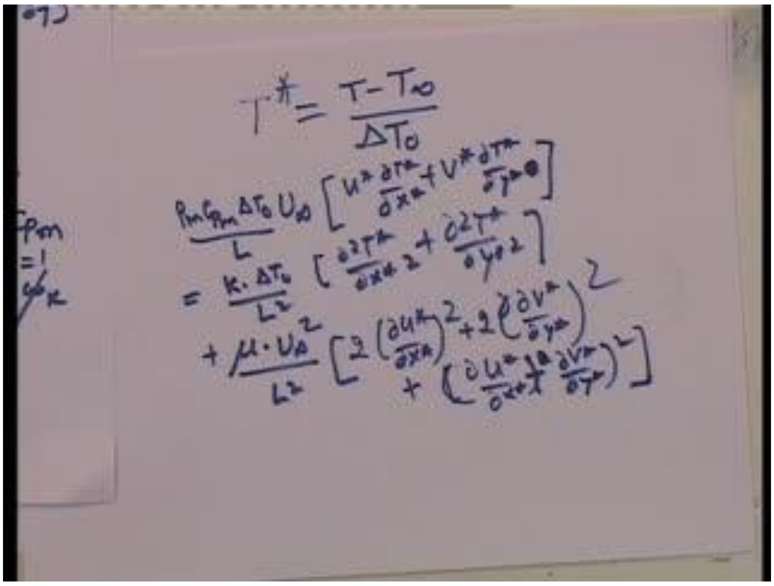

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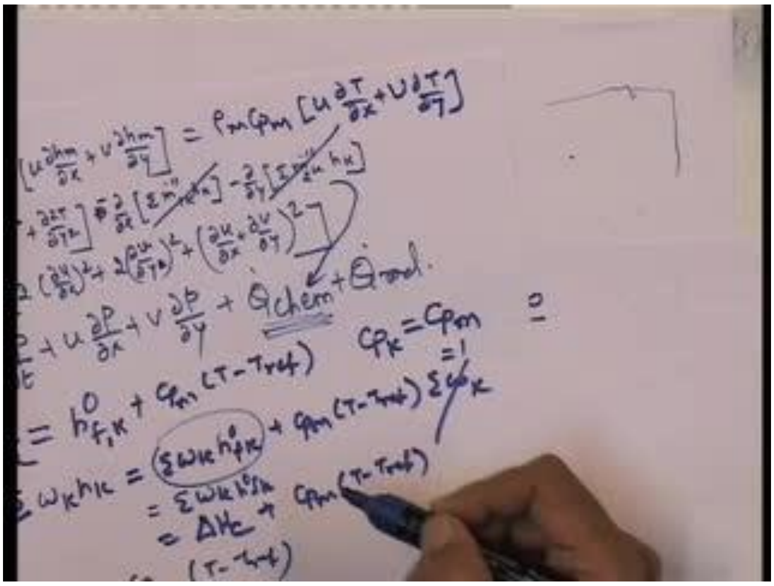

What I am done now is, I have said define T star equal to T minus T infinity divided by some reference temperature difference. Then, you will see that this term would simply become rho m Cp m into U dT by dx plus v dt by dy.

If I have to non dimensionalize the first term here, it would read as rho m Cp m delta T naught divided by L u infinity equal to u star dT star by dx star plus v star dT star by dy star square and that would equal k times delta T naught divided by L square d 2 T star by dx star square plus d 2 T star by dy star square plus mu times u infinity square divided by L square 2 times du star by dx star whole square plus 2 times dv star by dy star whole square

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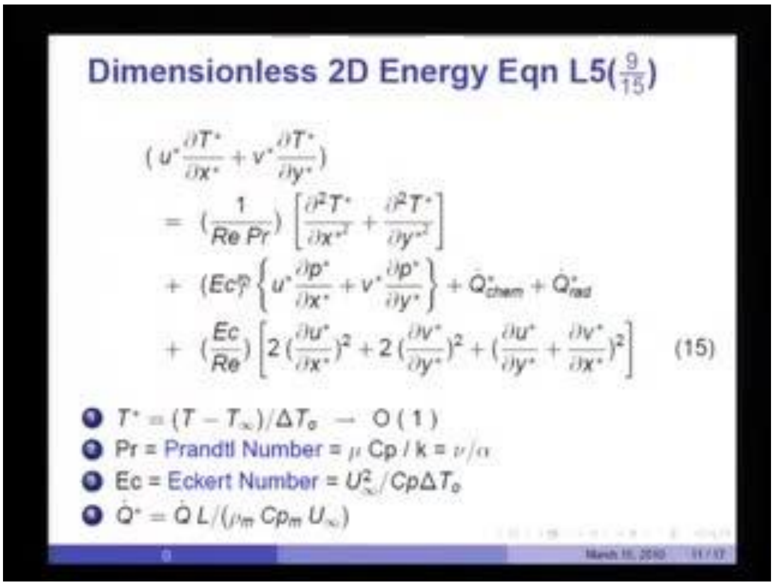

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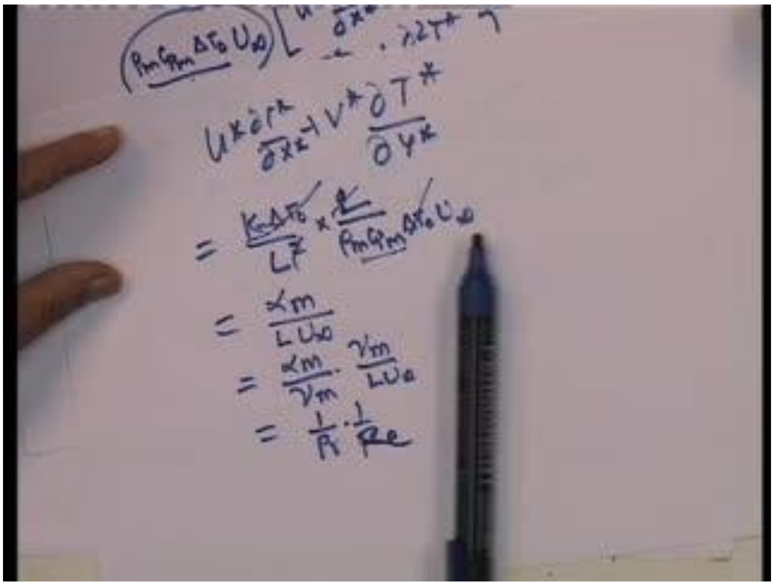

plus du star by dx star whole square plus dv star by dy star whole square and so on. So, that is what you see here the terms are written like that. Now, if I divide through by this quantity you will readily see that, I will get u star dT star by dx star plus v star dt star by dy star equal to k delta t naught by L square into rho into L divided by rho m Cp m delta T naught u infinity. So, you will see delta T naught gets cancelled with delta T naught L gets cancelled with one of these L’s and k divided by rho m Cp m k m. You will be simply the thermal diffusivity alpha m divided by L into u infinity, which I can also write as alpha m divided by nu m nu m divided by LU infinity and that is nothing but 1 over Prandtl number into 1 over Schmidt number and that is what you see I mean one over Reynolds number beg your pardon

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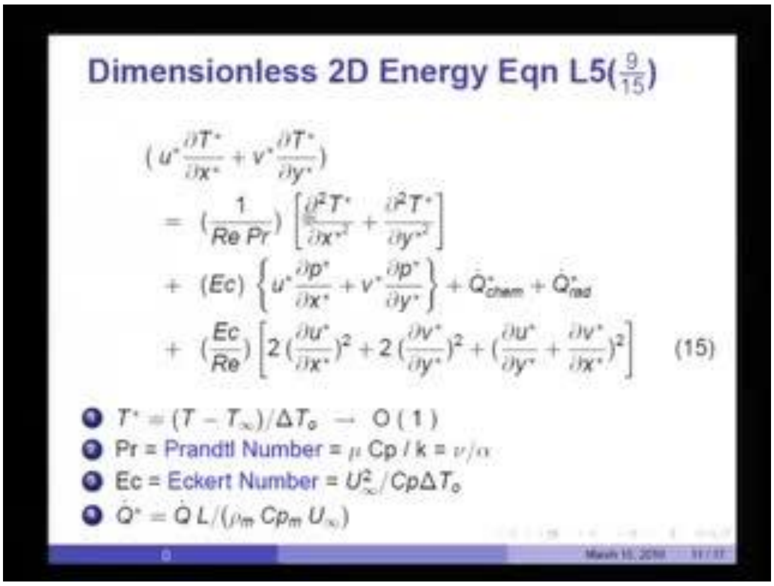

That is what you see here, the multiplier of d 2 T star by dx star square plus d 2 T star by dy star square would simply become 1 over Reynolds Prandtl. The third term, the pressure work term , he viscous dissipation term is somewhat important.

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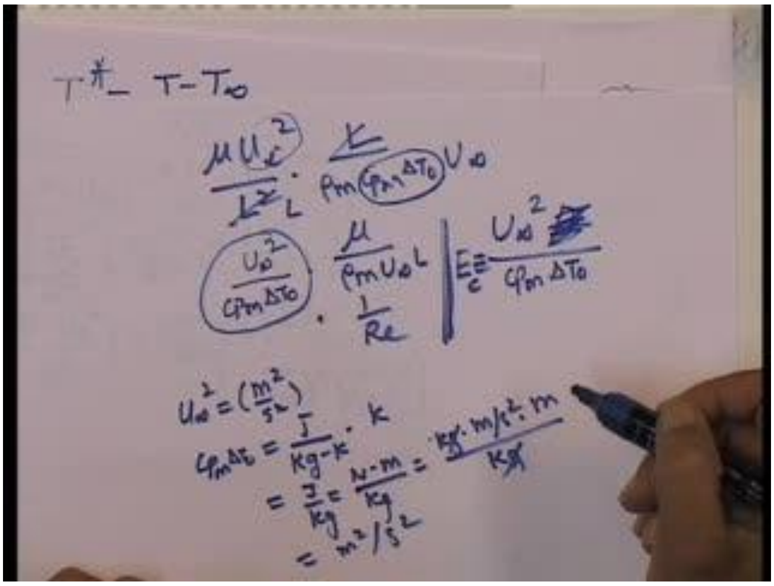

In the sense that you will see now that if I have to divide through again. Then, you will see that mu u infinity square divide by L square divided by 1 over rho m Cp m delta T naught u infinity into L.

Now, you will see this becomes equal to u infinity square divided by Cp m delta T naught. I have taken care of that term and this term then I have mu times rho m U infinity L and therefore this is nothing but 1 over Reynolds number.

This term is dimensionless as we can see u infinity square is meter square per Second Square. What about this term Cp m delta T naught, Cp m is joules per kg kelvin multiplied by Kelvin. So, essentially it is joules per kg.

Joule is Newton meter by kg. Newton is kg into meters per Second Square into meter divided by kg. So, kg kg gets cancelled and you again get meter square per Second Square. Remember, Cp m delta T naught and u infinity square both have same units and therefore this is a dimensionless quantity.

Professor Eckert defined a quantity u infinity square by 2, which is the kinetic energy divided by Cp m delta T naught, which is the sensible energy as Eckert number E c. E c is called the Eckert number E c and therefore you will see that it is at the moment U infinity square by 2.

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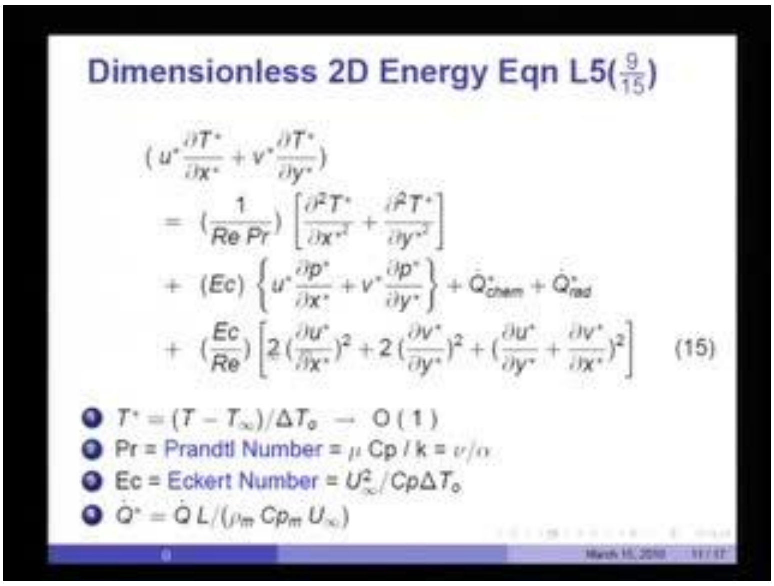

So, you get here Eckert number divided by Reynolds into all that. What about the pressure gradient terms? Here you will see only a multiplier E c. In other words, the dimensionless equation tells us that, there are now three parameters associated with heat transfer.

One is the Prandtl number, the other one is the Reynolds number and the third one is Eckert number. We can readily see, if the Eckert number was large both these terms will be important. If the Reynolds number is high and Eckert number is small then we can ignore these two term.

The pressure work term and the viscous dissipation term. So, with this equation I can do order of magnitude analysis because it is now in dimensionless form. So, what is the order of magnitude analysis that I do?

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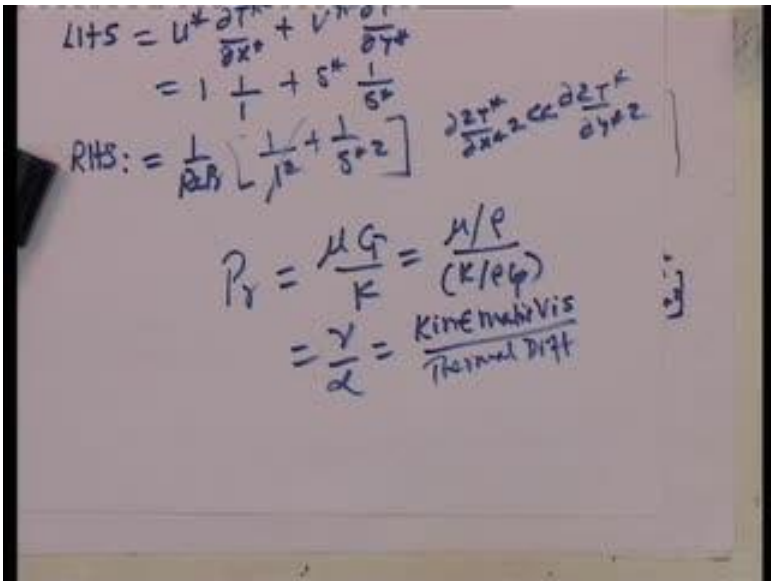

So, the left hand side which is equal to u star dt star by dx star plus v star dt star by dy star will simply become 1 1 1 plus delta star 1 by delta star. So, both the terms on the left hand side are important.

Right hand side remember this is 1 over Reynolds number Prandtl number into 1 divided by 1 square plus 1 divided by delta star square and therefore that term would be d 2 T star by dx star square will be much smaller than d 2 T star by dy star square.

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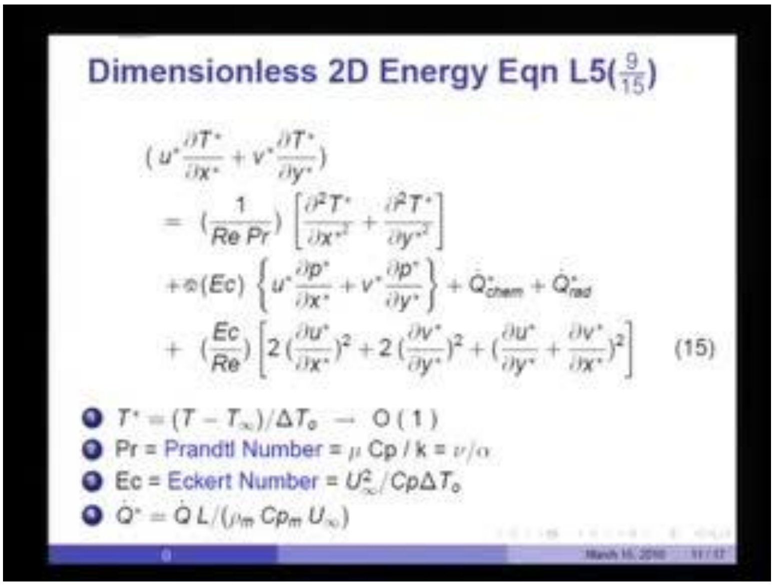

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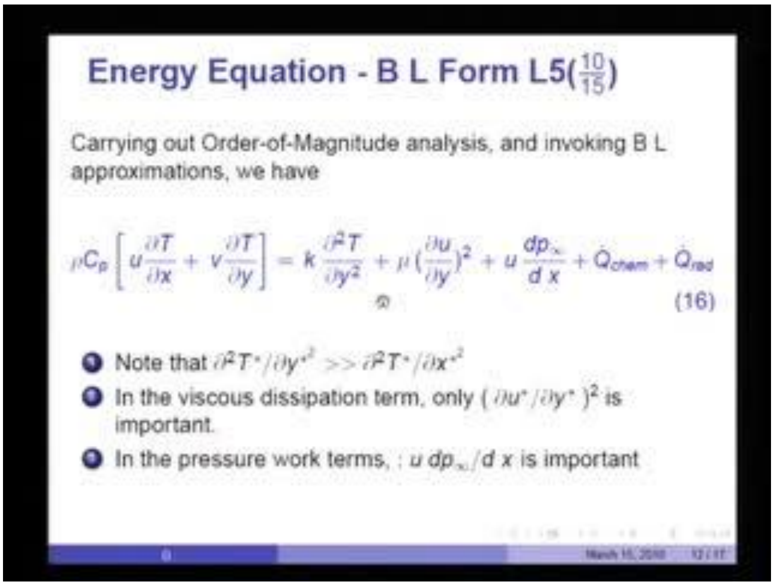

So, if you follow through in this manner you can carry out that order of magnitude analysis and the resulting equation would be simply this rho Cp u dT by dx plus v dT by dy equal to diffusion only in y direction plus viscous dissipation due to velocity gradient in y direction plus u d infinity by dx plus Q dot chem plus Q dot rad.

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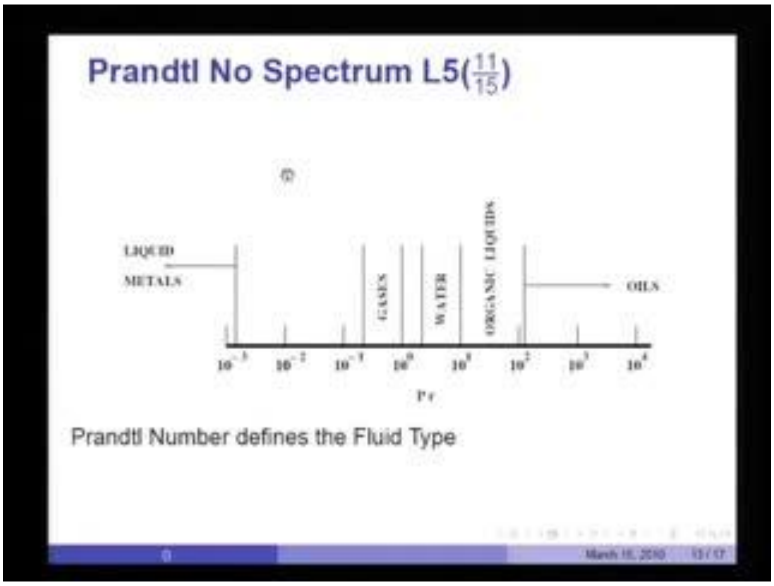

This is the boundary layer form of the energy equation. A note on Prandtl number. Prandtl number as you know is the ratio of Cp mu k, which is also if I divide both the numerator and denominator by rho than it is mu by rho divided by k by rho Cp and that is equal to nu divided by alpha or kinematic viscosity divided by thermal diffusivity.

It is in a way the ratio of the rate at which momentum is transferred divided by the rate at which heat is transferred. In boundary layers, this diffusion is taking place only across the boundary layer.

Then, you also notice that Prandtl number is the property of the fluid. It has nothing to do with the flow and therefore we can classify fluids according to their Prandtl numbers. Close to one and little below say from about 0.5 to 1, you usually get gases, but 3 to 10 or little over 10 is water but if you extend that to about 100 you will get many organic liquids included in this range. If the Prandtl number is much greater than 100, then usually you will encounter very viscous oils because Prandtl number has viscosity in the numerator. Usually, oils have a very large Prandtl numbers. On the other hand, liquid metals like mercury, sodium, liquid potassium, which are used for high heat flux heat transfers as obtained in for example breeder reactors liquid metals are preferred.

In those cases, their conductivities are so high compared to their viscosity that liquid metals have very low Prandtl number. There are not many fluids in this range.

But from about let us say 5 or into 10 is to minus 3 to below, you essentially get liquid metals. So, you have liquid metal range is very low Prandtl number gases very close to unity Prandtl number.

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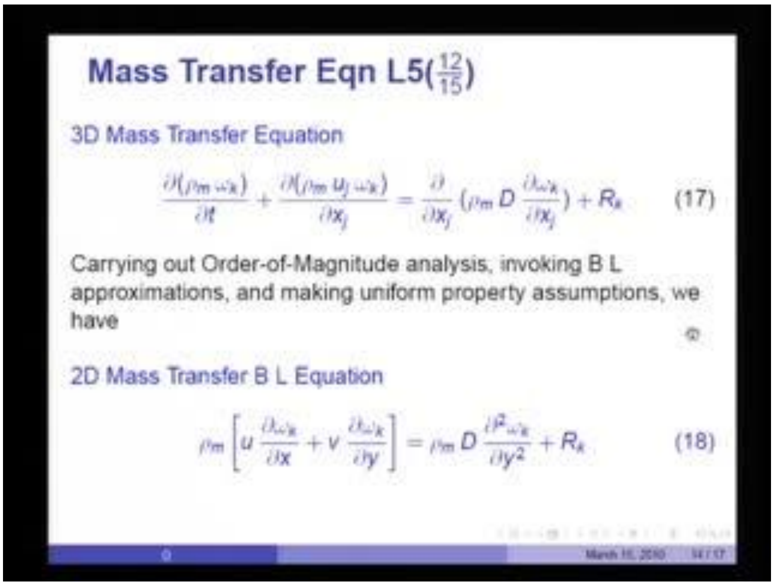

Water and organic liquids between 3 and about 100 and oils much greater than 100. The mass transfer equation can likewise be derived I am wont going to the details and it reads in like this.

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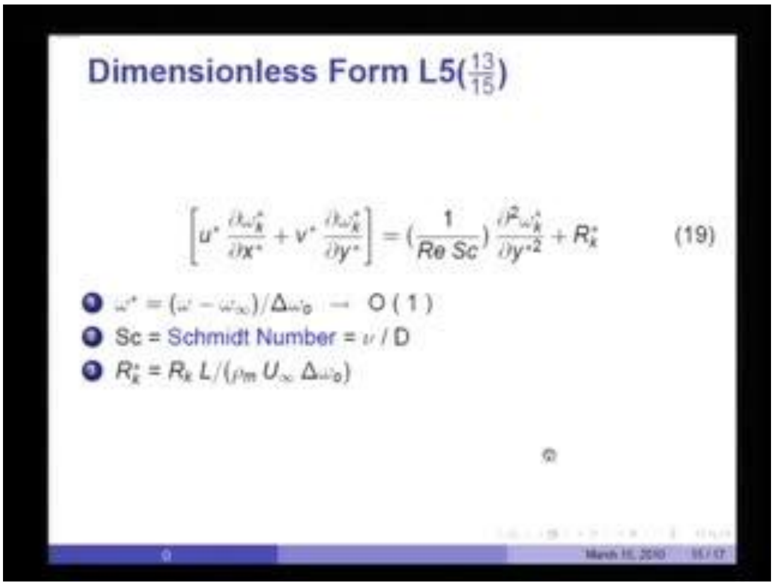

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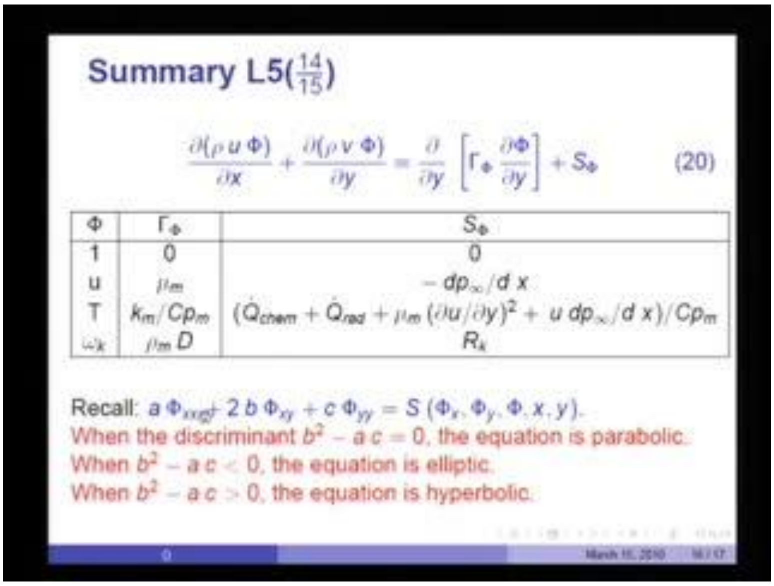

It can be non dimensionalised and instead of Prandtl number you will have a Schmidt number. We will take up all this, when we come to mass transfer. Therefore, I can summarize now what we did in this lecture.

We started with the derived 3 dimensional forms of the equations of bulk mass transfer momentum and energy and reduce them to the boundary layer form which contains two convection terms.

One diffusion term in y direction and one source term. It is good to get used to this form of generalization of the equations, you will see for phi equal to 1 you will simply have d rho U by dx plus d rho v by dy.

And that term would be 0 and S phi is 0 it is essentially the mass conservation equation for u phi equal to u, you will see that this term is simply the viscosity and this term is minus dp infinity by dx.

For temperature, this term can be taken for phi equal to temperature this term can be taken as k m divided by Cp m and all the right hand side will be divided by through by Cp m and for spaces k this.

Now, you will recall from your undergraduate studies that any 2 dimension on differential equation written in the form a into phi xx plus 2 b into phi xy plus c into phi yy equal to sum source term right hand side which may contain gradients of phi in so on and so forth.

Then, the discriminant b squared minus ac; if it is 0, then the equation is called parabolic. If b squared minus ac is less than 0 the equation is elliptic and b squared minus ac is greater than 0, then the equation is hyperbolic.

Of course, we are going to assume that gamma is constant because we are dealing with uniform property. So, we have only d 2 phi by dy square term that means c is finite, but in our equation a and b are 0 and discriminant b square minus ac is actually 0.

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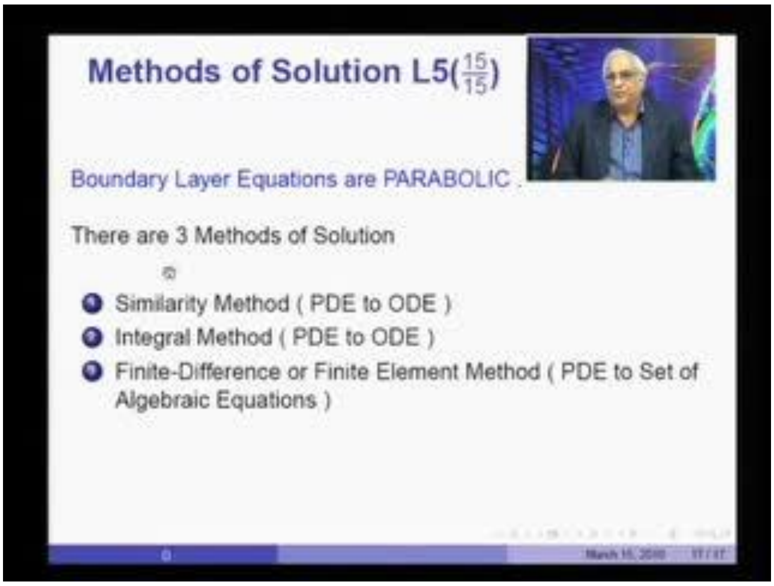

Therefore, our boundary layer equations are parabolic and for such parabolic equation there are 3 methods of solution - the first one is called the Similarity method in which the partial differential equations are converted to ODE’s Ordinary differential equation integral method. Similarly, the numerical method finite difference of finite element. We will take up these methods in the next lecture.