

**Convective Heat and Mass Transfer**  
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**Module No. # 01**  
**Lecture No. # 30**  
**Prediction of Turbulent Heat Transfer**

In the previous lecture, we saw how the law of the wall for velocity that is  $U^+$  can be used to evaluate friction coefficient  $C_f$ , for both external boundary layers as well as for pipe flows irrespective of the pressure gradient and also the effect of surface velocity  $u_w$  or the surface roughness. In all those cases, we were able to obtain coefficient of friction as a function of the Reynolds number - the local value of Reynolds number.

Today, we are going to look at how the temperature law of the wall can be used to predict the Stanton number in external boundary layers and Nusselt number in internal flows. Again, most of these methods are essentially analogy methods that is, in similar between heat transfer and momentum transfer is one way or the other assumed. We would be able to predict the variations of Stanton number and Nusselt number as functions of Reynolds number as well as the Prandtl number.

**LECTURE-30 PREDICTION OF  
TURBULENT HEAT TRANSFER**

- ① Prediction of  $St_x$  ( Ext BIs )
  - ① Use of law of the Wall
  - ② Analogy Method
  - ③ Integral Method
  - ④ Effects of  $V_w$  and Roughness
- ② Prediction of  $Nu$  ( Internal Flows )
  - ① Use of Law of the wall
  - ② Analogy Method

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So, I would first deal with external boundary layers in four different ways; one is use of the law of wall for situations in which there is no suction or blowing; I would also use analogy methods. Then, I will show you how to apply integral methods that is, the solution of the integral energy equation to take care of effects of pressure gradients wall temperature variations. Finally, we will extend the method to take care of the effects of roughness and suction and blowing. Likewise, we would look at prediction of Nusselt number in pipe flows - internal flows - and again make use of the law of the wall as well as the analogy methods.

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**Use of Wall Law - Ext BLs - 1 L-30( $\frac{1}{17}$ )**

- From lecture 28, the temperature law of the wall is written as
 
$$T_{\infty}^+ = Pr_T (U_{\infty}^+ + PF_{\infty}) \text{ where}$$

$$U_{\infty}^+ = \frac{U_{\infty}}{u_{\tau}} = \frac{U_{\infty}}{\sqrt{\tau_w/\rho}} = \sqrt{\frac{2}{C_{f,x}}}$$

$$T_{\infty}^+ = \frac{-(T_{\infty} - T_w)}{q_w/(\rho C_p u_{\tau})} = \frac{\rho C_p U_{\infty}}{h_x} \times \frac{u_{\tau}}{U_{\infty}} = \frac{\sqrt{C_{f,x}/2}}{St_x}$$

$$St_x = \frac{\sqrt{C_{f,x}/2}}{Pr_T(\sqrt{2/C_{f,x}} + PF_{\infty})} \quad (Pr_T \approx 0.9)$$
- For  $Pr = 1$ ,  $PF_{\infty} = 0$ . Reynolds used  $Pr_T = 1$ . Hence,  $St_x = C_{f,x}/2$  ( Perfect Reynolds Analogy ). For near-unity  $Pr$ ,  $St_x \approx (C_{f,x}/2) \times Pr^{-0.4}$ . Hence, for zero  $Pr$   $Gr$ ,  $\rightarrow St_x = 0.0286 Re_x^{-0.2} Pr^{-0.4}$ .
- For rough surface , appropriate  $C_{f,x}$  and  $PF_{\infty}$  to be used

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So, let us start with the external boundary layer case and from lecture 28, you will recall that the temperature law of the wall is written as  $T_{\infty}^+ = Pr_T (U_{\infty}^+ + PF_{\infty})$  which is a function of  $U_{\infty}^+$  but, if we write that equation in the infinity state, we would get that  $T_{\infty}^+ = Pr_T T_{\infty}^+ + PF_{\infty}$  which as you recall is a **function of Prandtl number only.**

$U_{\infty}^+$  here, would be simply  $U_{\infty}$  by  $u_{\tau}$  which is  $U_{\infty}$  under root  $\tau_w$  by  $\rho$  or it can be written as under root  $\rho U_{\infty}^2$  divided by  $\tau_w$ , which is nothing but under root  $2$  by  $C_{f,x}$ .  $T_{\infty}^+$  - as you will recall - is defined as  $\frac{-(T_{\infty} - T_w)}{q_w/(\rho C_p u_{\tau})}$ .

So, if I divide and multiply this by  $U_\infty$  and note that  $q_w$  divided by  $T_w$  minus  $T_\infty$  is  $h_x$  - the heat transfer coefficient, then you can see that, this part is  $1$  over Stanton number and  $u_\tau$  over  $U_\infty$  is nothing but under root  $C_f x$  by  $2$ . As a result, this equation simply transforms to Stanton  $x$  equal to  $C_f x$  by  $2$  square root into Prandtl  $T$  into under root  $2$  by  $C_f x$  plus  $Pr_\infty$ .

As you will recall, I said the turbulent Prandtl number is approximately  $0.9$  - some people take it  $0.85$  - but still one can make Prandtl  $T$  also a function of Prandtl number itself, as I indicated what the possible correlation could be. For Prandtl number  $1$ , you would recall  $Pr_\infty$  is  $0$ ; so that is  $0$ , for Prandtl number equal to  $1$ . Reynolds actually use Prandtl  $T$  equal to  $1$  and hence, you will see the Stanton  $x$  would simply be  $C_f x$  by  $2$ , which implies perfect analogy between heat transfer and momentum transfer.

From experiments for near unity Prandtl number, Stanton  $x$  correlates as  $C_f x$  by  $2$  into Prandtl raise to minus  $0.4$  and hence for  $0$ , the pressure gradient boundary layer Stanton  $x$  is equal to  $0.0286$  Reynolds  $x$  to the power of minus  $0.2$  Prandtl to the power of minus  $0.4$ . This was the correlation you use routinely in your under graduate work and we have shown that it can be derived from this equation - the temperature law of the wall. Of course, if you had a rough surface then, one must evaluate  $C_f x$  for a rough surface and  $Pr_\infty$  also to be used for a rough surface, so as to get Stanton number for a rough surface.

In all this evaluation you can see that  $C_f x$  must be evaluated from the methods of the previous lecture. So, whatever the pressure gradient or  $v_w$  or whatever is present, you simply use that to evaluate  $C_f x$  and straight away use **that to** evaluate Stanton  $x$  from that, this is the simplest way to evaluate Stanton number.

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**Analogy Method - 1 - Ext BLs - L30( $\frac{2}{17}$ )**

① Recall that  $Pr_{eff} = dT^+/du^+ = (dT^+/dy^+) \times (du^+/dy^+)$ . Hence, using  $(\tau_{tot}/\tau_w) \simeq 1 = (1 + \nu_t/\nu) (du^+/dy^+)$  gives

$$\frac{dT^+}{dy^+} = \frac{(1 + \nu_t/\nu)(du^+/dy^+)}{Pr^{-1} + (\nu_t/\nu)/Pr_T} = \left[ \frac{1}{Pr} + \left( \frac{1}{du^+/dy^+} - 1 \right) \frac{1}{Pr_T} \right]$$

② Integrating from  $y = 0$  to  $\infty$ , and using the 3-layer law for  $u^+$ , and hence for  $(du^+/dy^+)$  it follows that

$$T_{sl}^+ - 0 = Pr y_{sl}^+ = Pr u_{sl}^+ = 5 Pr$$

$$T_{trl}^+ - T_{sl}^+ = 5 Pr_T \ln \left( 1 + 5 \frac{Pr}{Pr_T} \right) \quad (y_{trl}^+ = 30 \text{ used})$$

$$T_{\infty}^+ - T_{trl}^+ = 2.5 Pr_T \ln \left[ \frac{1 + (Pr/Pr_T)(\delta^+/2.5 - 1)}{1 + 11 (Pr/Pr_T)} \right]$$

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Then, we can also apply somewhat more rigorously analogy method, in which we call that the effective Prandtl number is essentially,  $dT$  plus by  $du$  plus which we can write as  $dT$  plus by  $dy$  plus multiplied by  $du$  plus by  $dy$  plus then, sorry, this should be raised to  $dy$  plus by  $du$  plus it should be not  $du$  plus by  $dy$  plus  $dy$  plus by  $du$  plus.

Hence, using the relation  $\tau_{tot}$  divided by  $\tau_w$  is approximately equal to 1 equal to  $1 + \nu_t/\nu$  by  $du$  plus by  $dy$  plus. You will recall we had derived this equation - gives  $dT$  plus by  $dy$  plus equal to  $1 + \nu_t/\nu$  by  $du$  plus by  $dy$  plus Prandtl raise to minus 1 plus  $\nu_t/\nu$  by  $\nu$  into Prandtl T or simply  $1$  over Prandtl number into  $1$  over  $du$  plus by  $dy$  plus minus  $1$   $1$  over Prandtl T raise to minus 1.

Integrating from  $y$  equal to 0 to infinity and using the 3-layer law for  $u$  plus and hence for  $du$  plus by  $dy$  plus, it follows that. Now, if  $du$  plus by  $dy$  plus for the laminar sub layer is equal to 1 then of course, that quantity vanishes and I simply get  $T$  age of the sub layer plus minus 0 equal to Prandtl  $y_{sl}$  plus equal to Prandtl  $u_{sl}$  plus and as you know,  $u_{sl}$  plus and  $y_{sl}$  plus are 5; therefore, you get  $T_{sl}$  plus is equal to 5 Prandtl.

Extend the integration further from sub layer to transitional layer, so  $T$  transitional layer plus minus  $T_{sl}$  plus. Here,  $du$  plus by  $dy$  plus would become one over  $\kappa y$  plus where  $\kappa$  is 0.2 and therefore, that will become 5; so 5 Prandtl T  $\ln$   $1 + 5$  Prandtl

by Prandtl T and here, I use transitional layer as y plus for transitional as 30 then, you get that relationship.

Then, from the age of the transitional layer till the age of the boundary layer you will get T infinity plus minus T transitional layer plus again du plus by dy plus will be 1 over kappa y plus and therefore, where kappa is 0.4, then you will get that relationship that involving delta plus the boundary layer thickness.

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**Use of Wall Law - Ext BLs - 1 L-30( $\frac{1}{17}$ )**

- From lecture 28, the temperature law of the wall is written as
 
$$T_{\infty}^+ = Pr_T (U_{\infty}^+ + PF_{\infty}) \text{ where}$$

$$U_{\infty}^+ = \frac{U_{\infty}}{u_{\tau}} = \frac{U_{\infty}}{\sqrt{\tau_w/\rho}} = \sqrt{\frac{2}{C_{f,x}}}$$

$$T_{\infty}^+ = \frac{-(T_{\infty} - T_w)}{q_w/(\rho C_p u_{\tau})} = \frac{\rho C_p U_{\infty}}{h_x} \times \frac{u_{\tau}}{U_{\infty}} = \frac{\sqrt{C_{f,x}/2}}{St_x}$$

$$St_x = \frac{\sqrt{C_{f,x}/2}}{Pr_T(\sqrt{2/C_{f,x}} + PF_{\infty})} \quad (Pr_T \approx 0.9)$$
- For  $Pr = 1$ ,  $PF_{\infty} = 0$ . Reynolds used  $Pr_T = 1$ . Hence,  $St_x = C_{f,x}/2$  ( Perfect Reynolds Analogy ). For near-unity  $Pr$ ,  $St_x \approx (C_{f,x}/2) \times Pr^{-0.4}$ . Hence, for zero  $Pr$  Gr,  $\rightarrow St_x = 0.0286 Re_x^{-0.2} Pr^{-0.4}$ .
- For rough surface, appropriate  $C_{f,x}$  and  $PF_{\infty}$  to be used

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Essentially, we got layer by layer contributions to T infinity plus. If I add these three equations - as I show on the next slide - and rearrange if I add these things then you will see T sl plus here gets cancelled with that and this gets cancelled with that, I would get a relationship for T infinity plus which as you recall from the previous slide T infinity plus is nothing but C f x by 2 by Stanton.

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**Analogy Method - 2 - Ext BLs - L30( $\frac{3}{17}$ )**

Adding the last 3 Eqns, it can be shown that

$$\frac{\sqrt{C_{f,x}}/2}{St_x} = 5Pr + 5Pr_T \ln\left(1 + 5\frac{Pr}{Pr_T}\right) + 2.5Pr_T \ln\left\{\frac{1 + (Pr/Pr_T)(\delta^+/2.5 - 1)}{1 + 11(Pr/Pr_T)}\right\}$$

where using the Power law,

$$\delta_+ = \left(\frac{U_{\infty}^+}{8.75}\right)^7 = \left(\frac{\sqrt{2/C_{f,x}}}{8.75}\right)^7$$

The  $C_{f,x}$  is evaluated using Integral method of lecture 29.

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Then, I get  $C_{f,x}$  by 2 divided by Stanton number equal to 5 Prandtl plus all this; this is the transitional layer contribution, this is the laminar sub layer contribution and this is the fully turbulent layer contribution. Now, how do we evaluate delta plus here? Well as you recall, in the outer layers delta power law very well applies here. So, instead of logarithmic law if we apply power law then simply delta plus is equal to  $U_{\infty}^+$  plus divided by 8.75 raise to 7 or that is equal to this quantity.

The  $C_{f,x}$  is evaluated by integral methods of equation of lecture 29, so you substitute delta plus here as  $C_{f,x}$  and  $C_{f,x}$  that appears here. Here are both first evaluated from lecture 29 for a given situation and that gives you the variation of Stanton number as a function of Reynolds number and Prandtl number.

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**Use of Int Energy Eqn - 1 - L30( $\frac{4}{17}$ )**

- When  $U_\infty$  and  $(T_w - T_\infty)$  vary arbitrarily with  $x$

$$\frac{1}{U_\infty (T_w - T_\infty)} \frac{d}{dx} [\Delta_2 U_\infty (T_w - T_\infty)] = St_x$$

- For further analysis, let<sup>1</sup>  $St_x = C Re_x^{-n}$
- Then for const  $U_\infty$  and  $(T_w - T_\infty)$  BL, that is flat plate

$$\frac{d \Delta_2}{dx} = St_x = C \left( \frac{U_\infty x}{\nu} \right)^{-n} \text{ integration gives}$$

$$\Delta_2 = \frac{C}{1-n} \left( \frac{U_\infty}{\nu} \right)^{-n} x^{1-n} \text{ using } \Delta_2 = 0 \text{ at } x = 0$$

$$St_x = C \left( \frac{1-n}{C} \times \frac{U_\infty \Delta_2}{\nu} \right)^{\frac{n}{n-1}}$$

<sup>1</sup>Ambroke G S Sov. Phy. Tech. Phy., vol2, p 1979, 1957

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Now, when  $U_\infty$  and  $T_w - T_\infty$  vary arbitrarily with  $x$  then, one must invoke the integral energy equation which reads as  $\frac{1}{U_\infty (T_w - T_\infty)} \frac{d}{dx} [\Delta_2 U_\infty (T_w - T_\infty)] = St_x$ , so you recall this integral energy equation when  $\nu$  is 0.

For further analysis, let  $St_x$  be equal to  $C Re_x^{-n}$ . Now, this method is often called the Ambroke's methods; Ambroke was a soviet scientist and he published a paper on this. For the moment, we will simply assume that the  $St_x$  will vary as  $C$  times  $Re_x$  to the power of minus  $n$  and that is substituted here.

Then, for constant  $U_\infty$  and  $T_w - T_\infty$  boundary layer, that is let us say flat plate, it will simply mean  $\frac{d \Delta_2}{dx} = St_x = C \left( \frac{U_\infty x}{\nu} \right)^{-n}$ . If we integrate that I will get  $\Delta_2 = \frac{C}{1-n} \left( \frac{U_\infty}{\nu} \right)^{-n} x^{1-n}$ .

If I use the idea that  $\Delta_2$  is 0 at  $x = 0$  that is, right at the leading edge then,  $St_x$  would simply become the function of  $\Delta_2$  raise to  $n$  over  $n - 1$ . Now, this is a very important relationship because we will be subsequently using this relationship for a variety of situations.

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### Use of Int Energy Eqn - 2 - L30( $\frac{5}{17}$ )

- ① We assume validity of the last relationship regardless of the previous history of the BL. Then, TEE becomes
 
$$\frac{d}{dx} [\Delta_2 U_\infty (T_w - T_\infty)] = U_\infty (T_w - T_\infty) C \left( \frac{1-n}{C} \times \frac{U_\infty \Delta_2}{\nu} \right)^{n-1}$$
 or, integration gives
 
$$\Delta_2 = \frac{C \nu^n}{(1-n) U_\infty (T_w - T_\infty)} \left[ \int_0^x U_\infty (T_w - T_\infty)^{1/(1-n)} dx \right]^{1-n}$$
- ② Using  $St_x \sim \Delta_2$  relation from previous slide
 
$$St_x = \frac{C \nu^n (T_w - T_\infty)^{n/(1-n)}}{\left[ \int_0^x U_\infty (T_w - T_\infty)^{1/(1-n)} dx \right]^n}$$

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So, we assume the validity of the last relationship regardless of the previous history of the boundary layer. Then, the integral energy equation becomes  $\frac{d}{dx} [\Delta_2 U_\infty (T_w - T_\infty)] = U_\infty (T_w - T_\infty) C \left( \frac{1-n}{C} \times \frac{U_\infty \Delta_2}{\nu} \right)^{n-1}$ . If we were to integrate this equation you will get  $C \nu^n (T_w - T_\infty)^{n/(1-n)} = \left[ \int_0^x U_\infty (T_w - T_\infty)^{1/(1-n)} dx \right]^n$ .

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### Use of Int Energy Eqn - 1 - L30( $\frac{4}{17}$ )

- ① When  $U_\infty$  and  $(T_w - T_\infty)$  vary arbitrarily with x
 
$$\frac{1}{U_\infty (T_w - T_\infty)} \frac{d}{dx} [\Delta_2 U_\infty (T_w - T_\infty)] = St_x$$
- ② For further analysis, let<sup>1</sup>  $St_x = C Re_x^{-n}$
- ③ Then for const  $U_\infty$  and  $(T_w - T_\infty)$  BL, that is flat plate
 
$$\frac{d \Delta_2}{dx} = St_x = C \left( \frac{U_\infty x}{\nu} \right)^{-n}$$
 integration gives
 
$$\Delta_2 = \frac{C}{1-n} \left( \frac{U_\infty}{\nu} \right)^{-n} x^{1-n}$$
 using  $\Delta_2 = 0$  at  $x = 0$ 

$$St_x = C \left( \frac{1-n}{C} \times \frac{U_\infty \Delta_2}{\nu} \right)^{n-1}$$

<sup>1</sup>Ambrooke G S Sov. Phy. Tech. Phys., vol2, p 1979, 1957

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### Use of Int Energy Eqn - 2 - L30( $\frac{5}{17}$ )

④ We assume validity of the last relationship regardless of the previous history of the BI. Then, TEE becomes

$$\frac{d}{dx} [\Delta_2 U_\infty (T_w - T_\infty)] = U_\infty (T_w - T_\infty) C \left( \frac{1-n}{C} \times \frac{U_\infty \Delta_2}{\nu} \right)^{n-1}$$

or, integration gives

$$\Delta_2 = \frac{C \nu^n}{(1-n) U_\infty (T_w - T_\infty)} \left[ \int_0^x U_\infty (T_w - T_\infty)^{1/(1-n)} dx \right]^{1-n}$$

④ Using  $St_x \sim \Delta_2$  relation from previous slide

$$St_x = \frac{C \nu^n (T_w - T_\infty)^{n/(1-n)}}{\left[ \int_0^x U_\infty (T_w - T_\infty)^{1/(1-n)} dx \right]^n}$$

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In effect then, I can calculate delta 2 for any arbitrary variation of U infinity and T w minus T infinity with respect to x. If I use the previous Stanton x delta 2 relationship of this type then, you can see I can get Stanton x equal to C nu n T w minus T infinity raise to n over n minus 1 divided by this integral and this essentially allows you to calculate Stanton x for any arbitrary variation of free stream velocity and wall temperature.

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### Use of Int Energy Eqn - 3 - L30( $\frac{6}{17}$ )

Assuming flat plate data  $C = 0.0284 Pr^{-0.4}$  and  $n = 0.2$ , expression of previous slide becomes

$$St_x = \frac{0.0284 Pr^{-0.4} \nu^{0.2} (T_w - T_\infty)^{0.25}}{\left[ \int_0^x U_\infty (T_w - T_\infty)^{1.25} dx \right]^{0.2}}$$

$$\simeq 0.0295 Pr^{-0.4} Re_x^{-0.2} \left[ 1 - \frac{165}{St_x} \left( \frac{\nu}{U_\infty^2} \frac{dU_\infty}{dx} \right) \right]$$

These expressions give remarkably good fit to Exptl data for pr gr parameter  $(\nu/U_\infty^2) (dU_\infty/dx) < 10^6$  ( Crawford and Kays )

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### Use of Wall Law - Ext BLs - 1 L30( $\frac{1}{17}$ )

- From lecture 28, the temperature law of the wall is written as
 
$$T_{\infty}^+ = Pr_T (U_{\infty}^+ + PF_{\infty}) \text{ where}$$

$$U_{\infty}^+ = \frac{U_{\infty}}{u_{\tau}} = \frac{U_{\infty}}{\sqrt{\tau_w/\rho}} = \sqrt{\frac{2}{C_{f,x}}}$$

$$T_{\infty}^+ = \frac{-(T_{\infty} - T_w)}{q_w/(\rho C_p u_{\tau})} = \frac{\rho C_p U_{\infty}}{h_x} \times \frac{u_{\tau}}{U_{\infty}} = \frac{\sqrt{C_{f,x}/2}}{St_x}$$

$$St_x = \frac{\sqrt{C_{f,x}/2}}{Pr_T(\sqrt{2/C_{f,x}} + PF_{\infty})} \quad (Pr_T \approx 0.9)$$
- For  $Pr = 1$ ,  $PF_{\infty} = 0$ . Reynolds used  $Pr_T = 1$ . Hence,  $St_x = C_{f,x}/2$  ( Perfect Reynolds Analogy ). For near-unity  $Pr$ ,  $St_x \approx (C_{f,x}/2) \times Pr^{-0.4}$ . Hence, for zero  $Pr$  Gr,  $\rightarrow St_x = 0.0286 Re_x^{-0.2} Pr^{-0.4}$ .
- For rough surface, appropriate  $C_{f,x}$  and  $PF_{\infty}$  to be used

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Now, assuming the flat plate data for C equal to 0.0284 Prandtl raised to minus 0.4 and n equal to 0.2, you will recall - I showed this to be the case in the first slide here - that Stanton x can be written as point 0.286 Reynolds x to the power of **this**. Essentially, if I say C is equal to 0.0286 Prandtl raise to minus 0.4 and n equal to 0.2 then, that is what I have done here.

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### Use of Int Energy Eqn - 3 - L30( $\frac{6}{17}$ )

Assuming flat plate data  $C = 0.0284 Pr^{-0.4}$  and  $n = 0.2$ , expression of previous slide becomes

$$St_x = \frac{0.0284 Pr^{-0.4} \nu^{0.2} (T_w - T_{\infty})^{0.25}}{[\int_0^x U_{\infty} (T_w - T_{\infty})^{1.25} dx]^{0.2}}$$

$$\approx 0.0295 Pr^{-0.4} Re_x^{-0.2} \left[ 1 - \frac{165}{St_x} \left( \frac{\nu}{U_{\infty}^2} \frac{dU_{\infty}}{dx} \right) \right]$$

These expressions give remarkably good fit to Exptl data for pr gr parameter  $(\nu/U_{\infty}^2) (dU_{\infty}/dx) < 10^6$  ( Crawford and Kays )

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Then, I would get a relationship for Stanton x equal to this quantity and you will recall that U infinity is actually the pressure gradient although these constants are strictly valid

only for flat plate but, this Stanton number expression has used the same value of C and n and you get this relationship for Stanton x.

Now, Crawford and Kays have actually experimented with constant wall temperature boundary layers in which the free stream varies arbitrarily that means, only free stream varies arbitrarily not  $T_w - T_\infty$  and their experimental data fit this correlation very well; this is the pressure gradient parameter  $\nu_\infty^2 \frac{dU_\infty}{dx}$  is the pressure gradient parameter less than 10 raise to plus 6.

The Stanton number evaluated from this relation this integration and that evaluated from this agrees extremely well and therefore, we can say that use of C and n in this manner appears to be quite valid even for situations in which  $U_\infty$  is not constant or  $T_w$  is not constant. So with this experience, we now move forward and look at situations in which there is suction and blowing.

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**Effect of  $v_w$  - 1 - L30( $\frac{7}{17}$ )**

- For Flat Plate and  $(T_w - T_\infty) = \text{const}$ , Crawford and Kays show that for finite  $v_w$ ,
 
$$\frac{St_{x,v_w}}{St_{x,v_w=0}} = \frac{\ln(1 + B_h)}{B_h} \rightarrow B_h = \frac{v_w/U_\infty}{St_{x,v_w}}$$
 or  $St_{x,v_w} = 0.0284 Pr^{-0.4} Re_x^{-0.2} \left[ \frac{\ln(1 + B_h)}{B_h} \right]$
- For arbitrary variation of  $v_w$  IEE reads as
 
$$\frac{d\Delta_2}{dx} = St_{x,v_w} + \frac{v_w}{U_\infty} = St_{x,v_w} (1 + B_h) \text{ or}$$

$$= \left[ 0.0284 Pr^{-0.4} \ln(1 + B_h) \frac{(1 + B_h)}{B_h} \right] Re_x^{-0.2}$$

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Now, again for a flat plate and  $T_w - T_\infty$  constant Crawford and Kays show that for finite  $v_w$  Stanton x, when  $v_w$  is finite divided by Stanton x, when  $v_w$  equal to 0 can be written as  $\ln(1 + B_h) / B_h$ .  $B_h$  is nothing but our blowing parameter but this time based on Stanton x.

We shall derive this relationship later on when we consider mass transfer problem in which the suction and blowing would be viewed as a problem of mass transfer and the B

h is defined in this fashion and Stanton x v w then and substituting for Stanton v w equal to 0 which is 0.0284 Prandtl minus 0.4 Reynolds x to the power become into ln 1 plus B h by B h that is what it become.

The energy equation for a flat plate where U infinity is constant and T w minus T infinity equal to constant; energy equation will simply be d delta 2 by dx Stanton x v w plus v w by U infinity but, if I substitute for v w by U infinity is equal to B h into Stanton x v w then, you will see it simply becomes 1 plus B h or the total expression then can be written in this fashion Reynolds x to the power of minus. There is ln 1 plus B h multiplied by 1 plus B h divided by B h Reynolds x to the power of minus 0.2.

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**Effect of  $v_w$  - 2 - L30( $\frac{8}{17}$ )**

- For  $B_h = \text{const}$ , and using  $\Delta_2 = 0$  at  $x = 0$ , integration give
 
$$Re_x^{-0.2} = 1.057 \left[ 0.0284 Pr^{-0.4} \ln(1 + B_h) \frac{(1 + B_h)}{B_h} \right]^{0.25} Re_{\Delta_2}^{-0.2}$$
- Using  $St_{x,v_w} \sim Re_x$  relation from previous slide
 
$$St_{x,v_w} = 0.0125 Pr^{-0.5} Re_{\Delta_2}^{-0.25} (1 + B_h)^{0.25} \left[ \frac{\ln(1 + B_h)}{B_h} \right]^{1.25}$$

We assume validity of this relation even when  $B_h$ ,  $U_\infty$  and  $(T_w - T_\infty)$  vary arbitrarily with  $x$  ( see next slide )

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Now, if B h was constant then the entire term here inside the bracket would be constant and it is not very difficult to integrate this equation using delta 2 equal to 0 at x equal to 0, integration gives Re x to the power of minus 0.2 equal to this relationship multiplied by Re delta 2 raise to minus 0.25 or using again the Stanton x Reynolds Re x relationship of the previous slide here.

The solution can be written as Stanton x for a finite v w a is written as 0.0125 Prandtl raise to minus 0.5 Reynolds delta 2 raise to minus 0.25 1 plus B h raise to 0.25 and then, this factor raise to 1.25.

Of course, this integration was made possible by assuming  $B_h$  equal to constant. Like in the previous case, we shall assume the validity of this relationship between Stanton  $x$  and  $\Delta_2$  even when  $B_h U_\infty$  and  $T_w - T_\infty$  vary arbitrarily with  $x$ , so that is what I have done next slide.

(Refer Slide Time: 18:37)

**Effect of  $B_h$ ,  $U_\infty$  and  $(T_w - T_\infty)$  - L30( $\frac{9}{17}$ )**  
 For this case, IEE will read as

$$\frac{d}{dx} [\Delta_2 U_\infty (T_w - T_\infty)] = 0.0125 Pr^{-0.5} Re_{\Delta_2}^{-0.25} U_\infty (T_w - T_\infty) \times \left[ \frac{(1 + B_h)}{B_h} \ln(1 + B_h) \right]^{1.25}$$

Integration gives

$$St_x = 0.0284 Pr^{-0.4} \times \frac{\nu^{0.2} (T_w - T_\infty)^{0.25} (1 + B_h)^{0.25} \{\ln(1 + B_h)/B_h\}^{1.25}}{\left[ \int_0^x U_\infty (T_w - T_\infty)^{1.25} \{(1 + B_h) \ln(1 + B_h)/B_h\}^{1.25} dx \right]^{0.2}}$$

Crawford and Kays show remarkable good fit to experimental data and predictions using mixing length.

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So, for this case the integral energy equation we will read in this, like this into all this quantity into that. Then, if we were to integrate as in the previous case, you get Stanton  $x$  will be equal to 0.0284 Prandtl raise to minus 0.4 then,  $B_h$  now varies with  $x$  so that is included and you can see that I can perform this integration for any arbitrary variation of  $U_\infty$ ,  $T_w - T_\infty$  and  $B_h$ , so that I can get variation of Stanton  $x$ .

Crawford and Kays shown remarkably good fit to experimental data and predictions using mixing length; so they had a situation in which highly accelerated boundary layer were considered with  $\nu$  present and it was changing arbitrarily. Therefore, the problem was solved by mixing length model and predictions were obtained; experimental data were available for the same case. Then, this approximate expression derived from Ambrose's procedure was used and very good agreement was shown between experimental data and the correlation as well as predictions using mixing length.

(Refer Slide Time: 19:58)

**Similarity Method for TBL - L30(10/17)**

① The governing Eqn for a Temperature TBL is

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = +\nu \frac{\partial}{\partial y} \left[ b_{Pr} \frac{\partial T}{\partial y} \right]$$

where  $b_{Pr} = \alpha/\nu + \alpha_t/\nu = Pr^{-1} + Pr_T^{-1} \nu_t^+$ , where  $Pr_T = \nu_t/\alpha_t$ ,  $\nu^+ = \nu_t/\nu$  and  $\nu_t$  is given by Prandtl's mixing length.

② Using similarity variables defined in lecture 29,

$$\frac{d}{d\eta} (b_{Pr} \theta') + f \theta' + \left(\frac{2n}{m+1}\right) f' (1-\theta) = \frac{2x}{m+1} \left( f' \frac{d\theta}{dx} - \theta' \frac{df}{dx} \right)$$

$$m = \left(\frac{x}{U_\infty}\right) \frac{dU_\infty}{dx}, \quad n = \left(\frac{x}{T_w - T_\infty}\right) \frac{d(T_w - T_\infty)}{dx}, \quad \theta = \frac{T_w - T}{T_w - T_\infty}$$

Iterative solution is required at each x.  
BCs:  $\theta(0) = 0$  and  $\theta(\infty) = 1$ .

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So with this, I end the methods in which law of the wall is used or integral energy equation is used. Now, I turn to the more differential equation based methods; it so happens that you can use similarity type methods for turbulent boundary layers as well. The governing equation for the temperature boundary layer would read as  $u \frac{dT}{dx} + v \frac{dT}{dy} = +\nu \frac{d}{dy} [b_{Pr} \frac{dT}{dy}]$  where  $b_{Pr}$  is  $\alpha/\nu + \alpha_t/\nu$  which is  $Pr^{-1} + Pr_T^{-1} \nu_t^+$  and  $\nu_t^+$  would be given by Prandtl's mixing length as a function of  $y$  the distance from the wall.

So in lecture 29, I introduced the similarity variables to be used for turbulent boundary layers. If we use the same similarity variables then, the equation for turbulent heat transfer boundary layer would be given by this  $\frac{d}{d\eta} (b_{Pr} \theta') + f \theta' + \left(\frac{2n}{m+1}\right) f' (1-\theta) = \frac{2x}{m+1} \left( f' \frac{d\theta}{dx} - \theta' \frac{df}{dx} \right)$  again a function of  $x$  on the right hand side. This was also found in case of momentum equation that you do get things on the right hand side which are functions of  $x$ , where the things on the left hand side are essentially functions of  $\eta$ .

Here,  $m$  is the pressure gradient parameter defined as  $\left(\frac{x}{U_\infty}\right) \frac{dU_\infty}{dx}$ ;  $n$  is the parameter related to wall temperature variation,  $n$  would be 0 of course, if  $T_w$  was constant and  $\theta$  is defined as  $\frac{T_w - T}{T_w - T_\infty}$ .

Again, like in the velocity boundary layer case you need to of course,  $f$  and  $f'$  are available already from the velocity boundary layer solutions. Therefore, solutions for  $\theta$  would be obtained by iterative method just in the manner in which the similarity solution for velocity was solved. The boundary conditions are of course,  $\theta = 0$  at  $\eta = 0$  and  $\theta \rightarrow 0$  as  $\eta \rightarrow \infty$ .

That completes discussion of the external boundary layers and the similarity method. You need to do is, at every  $x$  you solve the left hand side by shooting method; the right hand side is evaluated from values available at the one step before it, so that the right hand side can be formulated as the constant for that step. One simply solves the left hand side again by shooting method and using the intermediate solution for  $\theta$ ; the right hand side is evaluated again till convergence is obtained and we accept the solution at that position  $x$  and move to the next step.

(Refer Slide Time: 23:46)

**Wall law - Pipe Flow - 1 - L30(11/17)**

• Writing wall-law for Pipe centerline  
 $T_{cl}^+ = Pr_T (u_{cl}^+ + PF_\infty) = Pr_T (\bar{u}^+ + 1.5/\kappa + PF_\infty)$  where

$$\begin{aligned} T_{cl}^+ &= \frac{T_w - T_b}{q_w} \times \left( \frac{T_w - T_{cl}}{T_w - T_b} \right) \times \rho C_p u_\tau \\ &= \left( \frac{k}{hD} \right) \times \left( \frac{\bar{u}D}{\alpha} \right) \times \left( \frac{u_\tau}{\bar{u}} \right) \times \left( \frac{T_w - T_{cl}}{T_w - T_b} \right) \\ &= \frac{Re Pr}{Nu} \times \sqrt{\frac{f}{2}} \times \left( \frac{T_w - T_{cl}}{T_w - T_b} \right) \end{aligned}$$

• Hence, Equating for  $T_{cl}^+$ ,

$$Nu = \frac{Re Pr \sqrt{f/2}}{Pr_T (\sqrt{2/f} + 1.5/\kappa + PF_\infty)} \left( \frac{T_w - T_{cl}}{T_w - T_b} \right)$$

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Now, I move to the internal flows, we wish to use the wall-law for pipe flow. So, if I write the  $T^+$  equal to  $u^+$  Prandtl  $T$  into  $u^+ + PF_\infty$  corresponding to central line then,  $T_{cl}^+$  will be that and  $u_{cl}^+$  - you will recall - is nothing but,  $\bar{u}^+ + 1.5/\kappa + PF_\infty$ .

So,  $T_{cl}^+$  would get defined in this manner;  $T_w - T_b$  actually, it will get defined as  $T_w - T_{cl}$  divided by  $q_w$ , but if I multiply and divide by  $T_w - T_b$  then,

you will see this gets  $T_w - T_{bulk}$  over  $q_{wall}$ ;  $T_w - T_{cl}$  over  $T_w - T_{bulk}$  multiplied by  $\rho C_p$  and  $u_{\tau}$  that would be the definition of  $T_{cl}$  plus.

Now, I multiply by  $k$  and divide by  $k$ ; I multiply by  $\bar{u}$  and divide by  $\bar{u}$  then, you will see this can be written as **this**, this nothing but  $1/h$  - the heat transfer coefficient - into  $k$  divided by diameter which I have divided by and again multiplied by. So, I get  $\bar{u} D / \alpha$  multiplied by  $u_{\tau}$  over  $\bar{u}$  multiplied by  $T_w - T_{cl}$  divided by  $T_w - T_{bulk}$ .

Now,  $k / h D$  is nothing but one over Nusselt number;  $\bar{u} D / \alpha$  is nothing but Peclet number or product of Reynolds number and Prandtl number. So, these two factors are nothing but, Reynolds Prandtl divided by Nusselt number;  $u_{\tau}$  over  $\bar{u}$  would be simply under root  $f$  by 2 the friction factor for a pipe flow, multiplied by  $T_w - T_{cl}$  divided by  $T_w - T_{bulk}$ .

Hence, equating  $T_{cl}$  plus from this expression and from this expression that is, the law of the wall then, you will see I can write Nusselt number equal to Reynolds Prandtl under root  $f$  by 2 divided by Prandtl  $T_w$  into under root  $2$  by  $f$  plus  $1.5 \kappa$  plus  $PF$  infinity and this becomes  $T_w - T_{cl}$  over  $T_w - T_{bulk}$ .

To use this relationship we would of course, need estimate of  $T_w - T_{central\ line}$  divided by  $T_w - T_{bulk}$  which we expect to be somewhat greater than one in a turbulent pipe flow because, remember the temperature profiles are very flat inside the core of the flow and there are sharp gradients of temperature near the wall and therefore, the ratio of  $T_w - T_{cl}$  divided by  $T_w - T_{bulk}$  would be slightly greater than 1, that is what we shall show.



(Refer Slide Time: 26:55)

**Wall law - Pipe Flow - 2 - L30( $\frac{12}{17}$ )**

- To evaluate temperature ratio, we use **Power laws**

$$\left(\frac{T - T_w}{T_{cl} - T_w}\right) = \left(\frac{y}{R}\right)^{1/7} = \frac{u}{u_{cl}}$$

Then, using definition of  $T_b$ , it can be shown that

$$\left(\frac{T_w - T_{cl}}{T_w - T_b}\right) = \frac{6}{5} \simeq 1 \quad \text{and} \quad \frac{u_{cl}}{\bar{u}} = \frac{60}{49} \simeq 1.22$$

- The most widely used correlation due to Gnielinski is

$$Nu = \frac{(Re - 1000) Pr \sqrt{f/2}}{\sqrt{2/f} + 12.7 (Pr^{2/3} - 1)}$$

valid for  $0.5 < Pr < 2000$  and  $2300 < Re < 5 \times 10^6$

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Now, many times quite analogous to what is power law for velocity you assume a validity of the power law even for temperature  $T$  minus  $T_w$  divided by  $T$  central line minus  $T_w$  equal to  $y$  by  $R$  will rise to 1 by 7 and which we said it would be equal to  $u$  over  $u_{cl}$ .

Then, using definition of  $T$  bulk which is simply  $\int_0^r u t dr$  divided by  $\int u r dr$ . It is easy to show that  $T_w$  minus  $T$  central line divided by  $T_w$  minus  $T$  bulk would be about 6 by 5 which is close to 1, let us say, 1 to 1.2. Remember this relationship is not absolutely exact but, we can take it to be between 1 and 1 is a function of Reynolds number. You will notice that for higher Reynolds number you need to take this as 1 over 9 whereas, below 50000 it can be taken as 1 over 7.

(Refer Slide Time: 28:10)

### Wall law - Pipe Flow - 1 - L30(<sup>11</sup>/<sub>17</sub>)

- Writing wall-law for Pipe centerline  
 $T_{cl}^+ = Pr_T (u_{cl}^+ + PF_\infty) = Pr_T (\bar{u}^+ + 1.5/\kappa + PF_\infty)$  where

$$\begin{aligned}
 T_{cl}^+ &= \frac{T_w - T_b}{q_w} \times \left( \frac{T_w - T_{cl}}{T_w - T_b} \right) \times \rho C_p u_\tau \\
 &= \left( \frac{k}{hD} \right) \times \left( \frac{\bar{u}D}{\alpha} \right) \times \left( \frac{u_\tau}{\bar{u}} \right) \times \left( \frac{T_w - T_{cl}}{T_w - T_b} \right) \\
 &= \frac{Re Pr}{Nu} \times \sqrt{\frac{f}{2}} \times \left( \frac{T_w - T_{cl}}{T_w - T_b} \right)
 \end{aligned}$$

- Hence, Equating for  $T_{cl}^+$ ,

$$Nu = \frac{Re Pr \sqrt{f/2}}{Pr_T (\sqrt{2/f} + 1.5/\kappa + PF_\infty)} \left( \frac{T_w - T_{cl}}{T_w - T_b} \right)$$

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(Refer Slide Time: 28:18)

### Wall law - Pipe Flow - 2 - L30(<sup>12</sup>/<sub>17</sub>)

- To evaluate temperature ratio, we use Power laws

$$\left( \frac{T - T_w}{T_{cl} - T_w} \right) = \left( \frac{y}{R} \right)^{1/7} = \frac{u}{u_{cl}}$$

Then, using definition of  $T_b$ , it can be shown that

$$\left( \frac{T_w - T_{cl}}{T_w - T_b} \right) = \frac{6}{5} \simeq 1 \quad \text{and} \quad \frac{u_{cl}}{\bar{u}} = \frac{60}{49} \simeq 1.22$$

- The most widely used correlation due to Gnielenski is

$$Nu = \frac{(Re - 1000) Pr \sqrt{f/2}}{\sqrt{2/f} + 12.7 (Pr^{2/3} - 1)}$$

valid for  $0.5 < Pr < 2000$  and  $2300 < Re < 5 \times 10^6$

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As a result  $T_w - T_{cl}$  over  $T_w - T_{bulk}$  is actually a function of Reynolds number; this factor turns out to be function of Reynolds number. The higher the Reynolds number closer it gets to the value of 1. Likewise,  $U_{cl}$  by  $\bar{u}$  is 1.22 but that is too high an estimate but, nonetheless it can be taken to be approximately solved close to 1.

Then, you can substitute these values for  $u_{cl}$  by  $\bar{u}$  and I mean, mainly you want this value but, that evaluation of that value requires this value so that is why I have coated it.

The most widely used correlation for pipe flow which is considered to be very accurate with the experimental data is 1 by Gnielenski and it reads like this  $Nu = \frac{Re - 1000}{Pr} \sqrt{f} \times \frac{Pr}{Pr + 2.5}$  for  $Pr > 0.5$  and  $2300 < Re < 5 \times 10^6$  and it is valid for gases to heavy oils 0.5 to 2000 and 2300 to 5 million Reynolds number.

(Refer Slide Time: 29:27)

**Wall law - Pipe Flow - 1 - L30( $\frac{11}{17}$ )**

① Writing wall-law for Pipe centerline  
 $T_{cl}^+ = Pr_T (u_{cl}^+ + PF_\infty) = Pr_T (\overline{u^+} + 1.5/\kappa + PF_\infty)$  where

$$\begin{aligned} T_{cl}^+ &= \frac{T_w - T_b}{q_w} \times \left( \frac{T_w - T_{cl}}{T_w - T_b} \right) \times \rho C_p u_\tau \\ &= \left( \frac{k}{hD} \right) \times \left( \frac{\overline{u} D}{\alpha} \right) \times \left( \frac{u_\tau}{\overline{u}} \right) \times \left( \frac{T_w - T_{cl}}{T_w - T_b} \right) \\ &= \frac{Re Pr}{Nu} \times \sqrt{\frac{f}{2}} \times \left( \frac{T_w - T_{cl}}{T_w - T_b} \right) \end{aligned}$$

② Hence, Equating for  $T_{cl}^+$ ,

$$Nu = \frac{Re Pr \sqrt{f/2}}{Pr_T (\sqrt{2/f} + 1.5/\kappa + PF_\infty)} \left( \frac{T_w - T_{cl}}{T_w - T_b} \right)$$

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Notice the similarity between the equation we have derived here, so instead of Reynolds the Gnielenski correlation has Reynolds minus 1000 Prandtl number is still there under root f by 2 is very much, there Prandtl T is perhaps taken as 1 and then, there is a factor of 2 by f and remember this is 3.66 in our calculation and then, the PF infinity which we use from our correlation, but the Gnielenski takes a Prandtl T equal to 1; he is also taking the ratio if it is like this, **this** to be nearly 1 and then this is the function of the Prandtl number.

(Refer Slide Time: 29:55)

### Wall law - Pipe Flow - 2 - L30( $\frac{12}{17}$ )

- To evaluate temperature ratio, we use **Power laws**

$$\left(\frac{T - T_w}{T_{cl} - T_w}\right) = \left(\frac{y}{R}\right)^{1/7} = \frac{u}{u_{cl}}$$

Then, using definition of  $T_b$ , it can be shown that

$$\left(\frac{T_w - T_{cl}}{T_w - T_b}\right) = \frac{6}{5} \approx 1 \quad \text{and} \quad \frac{u_{cl}}{u} = \frac{60}{49} \approx 1.22$$

- The most widely used correlation due to Gnielenski is

$$Nu = \frac{(Re - 1000) Pr \sqrt{f/2}}{\sqrt{2/f} + 12.7 (Pr^{2/3} - 1)}$$

valid for  $0.5 < Pr < 2000$  and  $2300 < Re < 5 \times 10^6$

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This is most widely accepted correlation has a form which is very similar to what we derive it from the temperature law.

(Refer Slide Time: 30:21)

### Analogy Method - Pipe Flow - 1 - L30( $\frac{13}{17}$ )

- In the **FD Pipe flow**,  $dp/dx = \text{const}$ . Hence, the axial momentum eqn and its consequences are

$$\frac{1}{r} \frac{d(r \tau_{tot})}{dr} = -\frac{dp}{dx} \rightarrow \frac{\tau_{tot}}{\tau_w} = \frac{r}{R} = 1 - \frac{y}{R}$$

But  $\tau_{tot} = \rho(\nu + \nu_t) \frac{du}{dr} = -\rho(\nu + \nu_t) \frac{du}{dy}$

$$\left(1 + \frac{\nu_t}{\nu}\right) = \frac{1 - y^+/R^+}{du^+/dy^+}$$

- Then form Slide 2,

$$\frac{dT^+}{dy^+} = \left(1 - \frac{y^+}{R^+}\right) \left[ \frac{1}{Pr} + \left(\frac{1 - y^+/R^+}{du^+/dy^+} - 1\right) \frac{1}{Pr_T} \right]^{-1}$$

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We can also apply analogy method for pipe flow, so for fully developed pipe flow for example,  $dp/dx$  is equal to constant hence, the axial momentum equation and its consequences are that  $1/r \cdot dr \tau_{tot}$  by  $dr$  would be equal to minus  $dp$  by  $dx$ . Integration would give  $\tau_{tot}$  by  $\tau_w$  equal to  $r$  by  $R$  which means, the total stress divided by the wall stress is a linear function of  $r$  by  $R$ . This we had shown from the

experimental data in an earlier slide and replacing  $r$  equal to  $r$  minus  $y$  the capital radius minus  $y$  as a distance from the wall it will be  $1 - y/R$ .

(Refer Slide Time: 31:32)

### Analogy Method - 1 - Ext BLs - L30( $\frac{2}{17}$ )

- Recall that  $Pr_{eff} = dT^+/du^+ = (dT^+/dy^+) \times (du^+/dy^+)$ . Hence, using  $(\tau_{tot}/\tau_w) \simeq 1 = (1 + \nu_t/\nu) (du^+/dy^+)$  gives

$$\frac{dT^+}{dy^+} = \frac{(1 + \nu_t/\nu)(du^+/dy^+)}{Pr^{-1} + (\nu_t/\nu)/Pr_T} = \left[ \frac{1}{Pr} + \left( \frac{1}{du^+/dy^+} - 1 \right) \frac{1}{Pr_T} \right]$$

- Integrating from  $y = 0$  to  $\infty$ , and using the 3-layer law for  $u^+$ , and hence for  $(du^+/dy^+)$  it follows that

$$T_{sl}^+ - 0 = Pr y_{sl}^+ = Pr u_{sl}^+ = 5 Pr$$

$$T_{trl}^+ - T_{sl}^+ = 5 Pr_T \ln \left( 1 + 5 \frac{Pr}{Pr_T} \right) \quad (y_{trl}^+ = 30 \text{ used})$$

$$T_{\infty}^+ - T_{trl}^+ = 2.5 Pr_T \ln \left[ \frac{1 + (Pr/Pr_T)(\delta^+/2.5 - 1)}{1 + 11 (Pr/Pr_T)} \right]$$

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So,  $\tau_{tot}$  is equal to  $\rho \nu_t \frac{du}{dr}$  and that would become  $-\rho \nu_t \frac{du}{dy}$ . Therefore,  $1 + \nu_t/\nu$  can be shown to be  $1 - y/R + \nu_t/\nu$  divided by  $du/dy$ . So, that is what I shall use to substitute in from slide 2 here I would use that to substitute for  $du/dy$  and  $\nu_t/\nu$ , sorry, I would use that to do this  $1 - y/R + \nu_t/\nu$ .

(Refer Slide Time: 31:45)

### Analogy Method - Pipe Flow - 1 - L30( $\frac{13}{17}$ )

- In the **FD Pipe flow**,  $dp/dx = \text{const}$ . Hence, the axial momentum eqn and its consequences are

$$\frac{1}{r} \frac{d(r \tau_{tot})}{dr} = -\frac{dp}{dx} \rightarrow \frac{\tau_{tot}}{\tau_w} = \frac{r}{R} = 1 - \frac{y}{R}$$

But  $\tau_{tot} = \rho(\nu + \nu_t) \frac{du}{dr} = -\rho(\nu + \nu_t) \frac{du}{dy}$

$$\left( 1 + \frac{\nu_t}{\nu} \right) = \frac{1 - y^+/R^+}{du^+/dy^+}$$

- Then from Slide 2,

$$\frac{dT^+}{dy^+} \simeq \left( 1 - \frac{y^+}{R^+} \right) \left[ \frac{1}{Pr} + \left( \frac{1 - y^+/R^+}{du^+/dy^+} - 1 \right) \frac{1}{Pr_T} \right]^{-1}$$

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Then, you will see that  $dT^+ / dy^+$  is equal to  $(1 - y^+) / Pr^+$  over  $du^+ / dy^+$  minus  $1 / Pr^+$ .

(Refer Slide Time: 32:10)

**Analogy Method - Pipe Flow - 2 - L30<sup>(14/17)</sup>**

Integrating from  $y = 0$  to  $R^+$ , and using the 3-layer law for  $u^+$ , and hence for  $(du^+ / dy^+)$  it can be shown that

$$T_{sl}^+ - 0 = 5 Pr^+ (y_{sl}^+ = 5)$$

$$T_{trl}^+ - T_{sl}^+ = 5 Pr_T \ln \left( 1 + 5 \frac{Pr^+}{Pr_T} \right) (y_{trl}^+ = 30)$$

$$T_{cl}^+ - T_{trl}^+ = 2.5 Pr_T \ln \left( \frac{R^+}{30} \right) \text{ for } Pr \geq 1^2$$

where

$$T_{cl}^+ = \frac{Re Pr}{Nu} \sqrt{\frac{f}{2}} \left( \frac{T_w - T_{cl}}{T_w - T_b} \right)$$

Therefore, adding the three equations ( see next slide )

<sup>2</sup>For  $Pr \ll 1$ , closed form soln cannot be obtained

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Now, we will evaluate  $du^+ / dy^+$  for each of the three layers that this sub layer the transitional layer and the fully turbulent layer. This is what the integration gives you for 3-layer law,  $T_{sl}^+ - 0$  equal to that as before; this is also everything is as before for an external boundary layer.  $T_{cl}^+ - T_{trl}^+$  would give you  $2.5 Pr_T \ln(R^+ / 30)$  again for  $Pr \geq 1$  but, also gases can be included which is very close to 1. If I add these three, I would get  $T_{sl}^+$  before adding of course,  $T_{cl}^+$  by definition is  $Re Pr / Nu$  over this; we showed on the previous slide.

(Refer Slide Time: 33:02)

**Analogy Method - Pipe Flow - 3 - L30** <sup>(15/17)</sup>

① With  $R^+ = (Re/2) \sqrt{f/2}$ , addition gives

$$Nu = \frac{Re Pr \sqrt{f/2} (T_w - T_{cl}) / (T_w - T_b)}{5 Pr + 5 Pr_T \ln \left( 1 + 5 \frac{Pr}{Pr_T} \right) + 2.5 Pr_T \ln \left\{ \left( \frac{Re}{60} \right) \sqrt{\frac{f}{2}} \right\}}$$

② Dittus Boelter Correlation -  $Nu = 0.023 Re^{0.8} Pr^n$ ,  $n = 0.4$  for heating and  $n = 0.3$  for cooling.

③ Sieder and Rouse Correlation

$$Nu = 5 + 0.015 Re^a Pr^b, (0.1 < Pr < 10^4), (10^4 < Re < 10^6)$$

$$a = 0.88 - \frac{0.24}{4 + Pr} \quad b = 0.333 + 0.5 \exp(-0.6 Pr)$$

For Liquid Metals,  $Nu = a + b Re^{0.85} Pr^{0.93}$  where  
 (  $a = 6.3$  and  $b = 0.0167$  for  $q_w = \text{const}$  ) and  
 (  $a = 4.8$  and  $b = 0.0156$  for  $T_w = \text{const}$  )

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If I equate this equal to the summation of the three, I would get the expression for the Nusselt number where  $R^+$  is expressed as  $Re/2$  under root  $f/2$ . Therefore, Nusselt number would become  $Re Pr \sqrt{f/2} (T_w - T_{cl}) / (T_w - T_b)$  divided by the entire quantity here. This will be the second expression we have got now for representing Nusselt number.

Of course, you have been using quite routinely the Dittus Boelter Correlation for very simple as  $Nu = 0.023 Re^{0.8} Pr^n$ , where  $n$  is equal to 0.4 for heating case and  $n$  equal to 0.3 for cooling case.

In chemical engineering literature most often the correlation due to Sieder and Rouse is used. It says, Nusselt number is equal to 5 plus 0.015 Reynolds raise to  $a$  and Prandtl raise to  $b$  and this is valid for 0.1 to 10000. Reynolds number from 10000 to a million,  $a$  is made a function of Prandtl number and  $b$  is made again a function of Prandtl number, these are some of the experimental correlations which are routinely used.

Now as I said, all our analysis is actually for Prandtl number greater than 1 but, for liquid metals Nusselt number is correlated in this fashion  $a + b Re^{0.85} Pr^{0.93}$  and where  $a$  and  $b$  take these values. Remember, I said in my lectures when I introduced turbulent flow that Nusselt numbers do not respond to the boundary conditions like in laminar flow; Nusselt numbers in turbulent flow are relatively

insensitive to whether it is a constant wall flux boundary condition or constant wall temperature.

That argument applies only to gases and situations in which Prandtl number is much greater than 1. When you come to liquid metals, they develop character the thermal boundary layer thickness enters the transitional layer and also enters the inner part of the turbulent layer many time depends on the value of Prandtl number. In such situations, the boundary condition begins to influence events even in turbulent flow and that is what is shown here, the constants a and b should be modified as shown here, for q wall equal to constant and T wall equal to constant. So, what it shows is again the Nusselt number for constant wall heat flux will be greater than the Nusselt number for constant wall temperature a circumstance very similar to that found in laminar flows.

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**Comparison of Correlations - L30(<sup>16</sup>/<sub>17</sub>)**

Re	3000	10000	50000	10 <sup>5</sup>	10 <sup>6</sup>
<b>Pr = 0.5, Temp rat = 1.1, Pr<sub>T</sub> = 0.943</b>					
Gin	8.13	25.2	84.5	147	883
DB	10.5	27.6	100	174	1100
SR	11.9	23.7	75.6	130	845
Anal	10.3	24.4	81.0	139	880
<b>Pr = 5.0, Temp rat = 1.1, Pr<sub>T</sub> = 0.887</b>					
Gin	19.2	70.1	287	524	3750
DB	26.5	69.4	251	438	2760
SR	29.7	74.1	278	498	3520
Anal	28.1	76.5	293	531	3860
<b>Pr = 25.0, Temp rat = 1.1, Pr<sub>T</sub> = 0.882</b>					
Gin	33.2	126	545	1020	7780
DB	50.4	132	479	834	5260
SR	52.1	139	552	1010	7540
Anal	40.3	114	455	842	6500

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$$Pr_T = 0.85 + 0.0309 \left( \frac{Pr + 1}{Pr} \right)$$
$$\approx 0.91 \text{ for } Pr \geq 1$$
$$\approx 0.88 \text{ for } Pr \rightarrow \infty$$
$$\approx \underline{31.0} \text{ for } Pr \leq 0.001$$

By taking temperature ratio as 1.1 and Prandtl T equal to 0.943, 887. You recall, I had given you the relationship that Prandtl T actually can be modeled as, Prandtl T equal to 0.85 plus 0.0309 into Prandtl divided by Prandtl plus 1, sorry, this should be Prandtl 1 divided by Prandtl.

You will see that this relationship assumes for Prandtl greater than or equal to 1 will give you for example, at Prandtl equal to 1 this will simply make it about 0.06 and Prandtl T will be about 0.91 for Prandtl equal to 1 and for very large Prandtl numbers also it will be around 0.88 because this ratio would be 1.

Imagine for liquid metals where Prandtl number is, let us say, of the order of 0.001 then, you will see this quantity becomes 1.01 divided by 0.001 or nearly 1000. Therefore, this will become almost equal to 31.0, so Prandtl T can be very large for very small Prandtl number that is liquid metal.

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### Comparison of Correlations - L30(<sup>16</sup>/<sub>17</sub>)

Re	3000	10000	50000	10 <sup>5</sup>	10 <sup>6</sup>
<b>Pr = 0.5, Temp rat = 1.1, Pr<sub>T</sub> = 0.943</b>					
Gin	8.13	25.2	84.5	147	883
DB	10.5	27.6	100	174	1100
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<b>Pr = 25.0, Temp rat = 1.1, Pr<sub>T</sub> = 0.882</b>					
Gin	33.2	126	545	1020	7780
DB	50.4	132	479	834	5260
SR	52.1	139	552	1010	7540
Anal	40.3	114	455	842	6500

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$$Pr_{YT} = .85 + 0.0307 \left( \frac{Pr+1}{Pr} \right)$$
  
$$\approx 0.9 \text{ for } Pr \geq 1$$
  
$$\approx 0.88 \text{ for } Pr \geq 1$$
  
$$\approx \underline{\underline{0.8}} \text{ for } Pr \leq 0.001$$

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**Comparison of Correlations - L30(<sup>16</sup>/<sub>17</sub>)**

Re	3000	10000	50000	10 <sup>5</sup>	10 <sup>6</sup>
<b>Pr = 0.5, Temp rat = 1.1, Pr<sub>T</sub> = 0.943</b>					
Gin	8.13	25.2	84.5	147	883
DB	10.5	27.6	100	174	1100
SR	11.9	23.7	75.6	130	845
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Anal	40.3	114	455	842	6500

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I am presently considering cases in which say from gases to organic liquids, so Prandtl 0.5 to 5 and 25. I have calculated Prandtl T using the relationship that I just showed, so it is 0.943 for 0.5, Prandtl T equal to 0.887 for Prandtl equal to 5 and 0.882 for Prandtl equal to 25. The temperature ratio I have always taken as 1.1 in each of those cases then just see what happens. At Reynolds number 3000, Gnienlenski correlation predicts 8.13 whereas, Dittus Boelter predicts 10.5, Schleicher and Rous produce 11.9 whereas, the analogy method produces 10.3; similarly, at 10000, 50000, 1 lakh and 1 million.

Then, you will see that Gnienlenski and other correlation predict very well are comparable Nusselt number, but Dittus Boelter is way out as you can see, for a high Reynolds number Dittus Boelter relationship predicts much higher than that predicted by any of the other correlations for gasses.

What about water? Something very similar, you will see that sufficiently higher Reynolds number **say tenth** this correlations are close to each other but again at higher Reynolds number Dittus Boelter under predicts the Nusselt number compared to other correlations; whereas, for gases it was over predicting and for water it is under predicting that trend continues even for organic liquids.

So, for Prandtl greater than 1, apparently Dittus Boelter under predicts whereas, for Prandtl less than 1 it over predicts compared to the correlations that are well accepted - Gnielenski in particular is very well accepted correlation.

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**Summary - L30(17/17)**

- The correlations for Pipe flow can be applied to non-circular ducts by evaluating  $f$ ,  $Re$  and  $Nu$  based on hydraulic diameter
- The easy-to-use Dittus-Boelter correlation overpredicts  $Nu$  for  $Pr < 1$  and underpredicts  $Nu$  for  $Pr > 1$
- For complete description of flow and heat transfer involving complex ducts, strong and changing strain rates due to body forces etc, it is best to adopt CFD techniques with two-eqn or stress-flux eqn models.
- This completes discussion of Turbulent flow and Heat Transfer. In the remaining lectures, we shall discuss Convective Mass Transfer

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From this relative comparisons, we say that the correlations for pipe flow can be applied to of course, a non-circular ducts by evaluating  $f$ , Reynolds and Nusselt number based on hydraulic diameter; this you have routinely done in your under graduate work. Of course, the theory to support this assumption requires solution of Reynolds stress equations, so that the secondary flow is predicted in the cross section by the Reynolds's stress model that can actually explain why hydraulic diameter concept works for non-circular ducts.

**It is easy to use** Dittus Boelter correlation, actually over predicts Nusselt number for Prandtl greater than 1 and under predicts for Prandtl greater than 1. For complete description of flow and heat transfer involving complex ducts, strong and changing strain rates due to body forces and it is best to use CFD techniques with two stress equation models. This completes our discussion on turbulent flow and heat transfer. Hence forth, I will begin with convective mass transfer.