Convective Heat and Mass Transfer

Prof. A. W. Date

Department of Mechanical Engineering

Indian Institute of Technology, Bombay

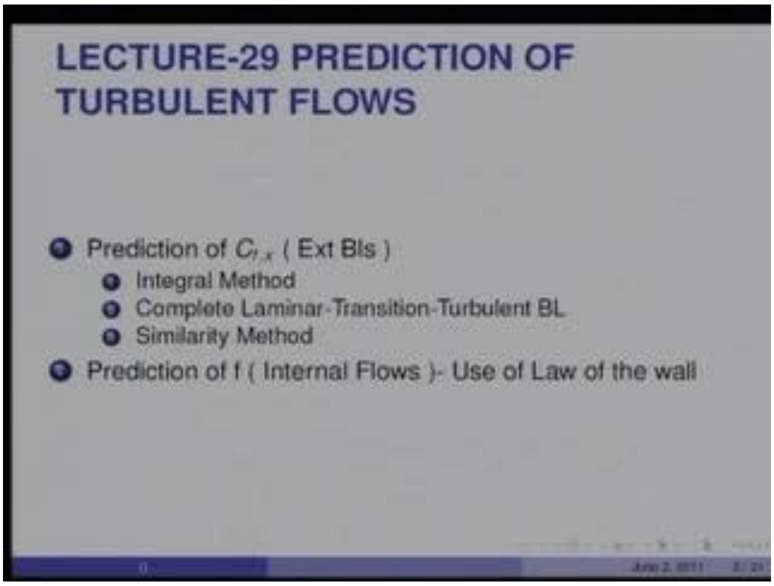
Module No. # 01

Lecture No. # 29

Prediction of Turbulent Flows

In the previous two lectures by using phenomenology we were able to derive near universal laws for both velocity as well as the temperature variations as distance from the wall. The task now is to use these laws to predict friction factors and Nusselt numbers.

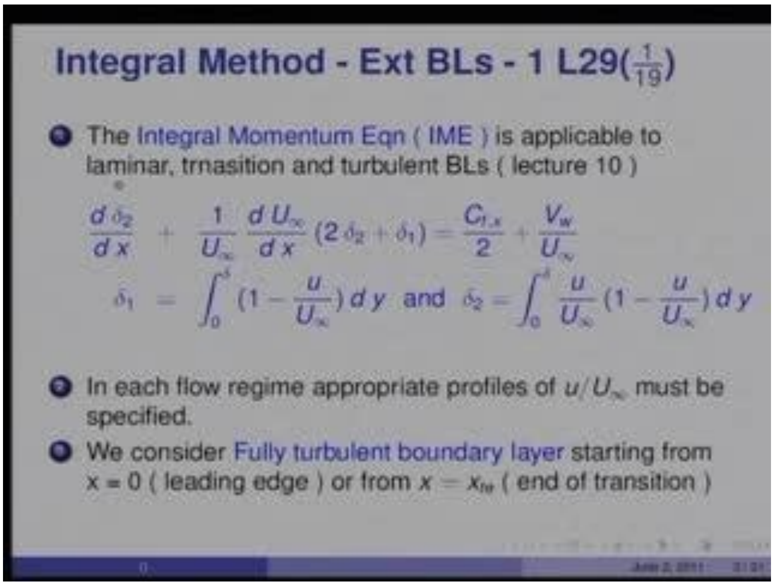
(Refer Slide Time: 00:58)



In today's lecture however, I will concentrate only on the friction factor. I will do firstly to predict the friction factor or the coefficient of friction as you say in external boundary layers, using integral method in purely turbulent boundary layers. Then, our interest would turn to use the integral method for complete boundary layer development that is, starting from laminar through transition to turbulent boundary layers.

We shall also see how complete laminar to turbulent and through transition boundary layers can also be predicted by similarity method. Then, I will turn my attention to the internal flows like flow in a pipe for example and again use the law of the wall there to predict the friction factor.

(Refer Slide Time: 01:49)



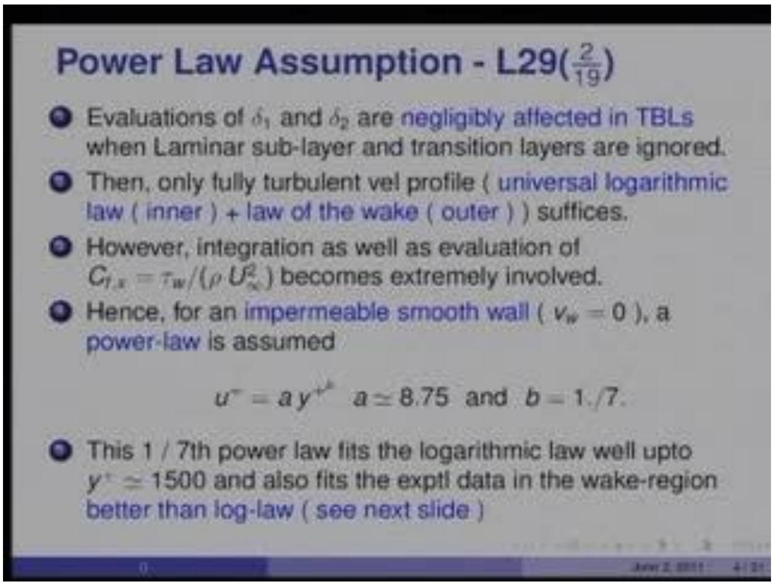
Let us begin with prediction of C f x for a purely turbulent boundary layer using integral method. We begin of course, with the integral momentum equations which as you know is applicable to laminar transition and turbulent boundary layers I mean, there is no separate integral equation because the differential equations have already been integrated from the wall to the infinity state. The integral equation reads as d delta 2 by d x plus 1 over U infinity d U infinity by d x, which is the pressure gradient parameter.

Delta 2 is the momentum thickness, delta 1 the displacement thickness, C f x is the coefficient of friction and V w is the wall velocity. Now of course, definitions of delta 1 and delta 2 - you will readily recall - that the delta 1 for example, is 0 to delta 1 minus u over U infinity d y. Delta 2 likewise is 0 to delta u over U infinity into 1 minus U infinity d y for a constant property boundary layer. In order to solve really the complete problem of laminar transition and turbulent boundary layers what is needed are the expressions for u over U infinity in each of these three regimes.

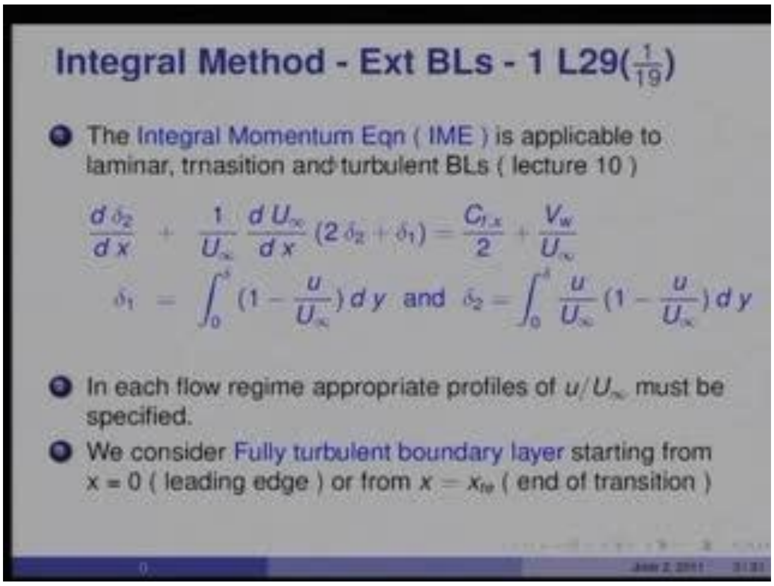
Of course, the laminar one that you already know it was given by polynomial. We shall see how transitional one can be handled and the turbulent one also is something I shall elaborate on. So, before going to complete laminar transitional and turbulent boundary layers what I am going to do first of all is to consider only a fully turbulent boundary layer as if it was originating from x equal to 0 that is the leading edge.

For example, if the leading edge was rough then, there is no possibility of laminar flow developing immediately and the flow becomes turbulent boundary layer from the origin itself. On the other hand, it may originate from the end of transition with the finite thickness of the boundary layer, that case we will see little later.

(Refer Slide Time: 04:12)



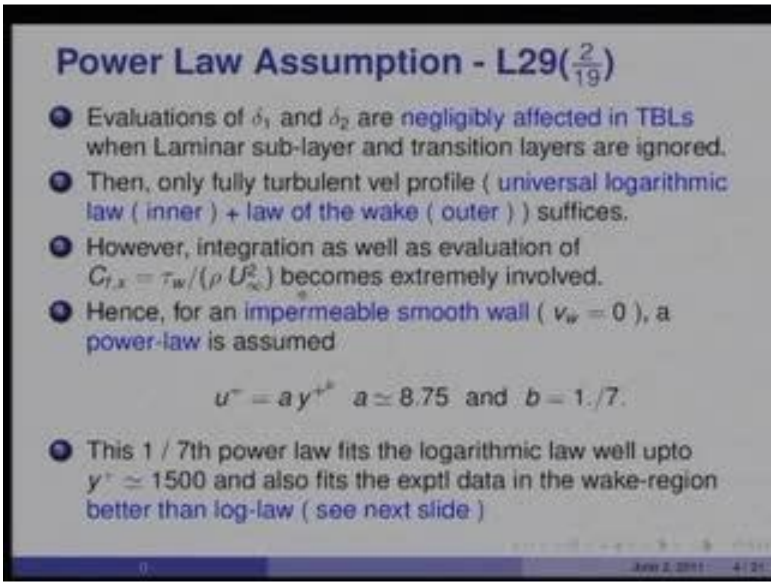
(Refer Slide Time: 04:21)



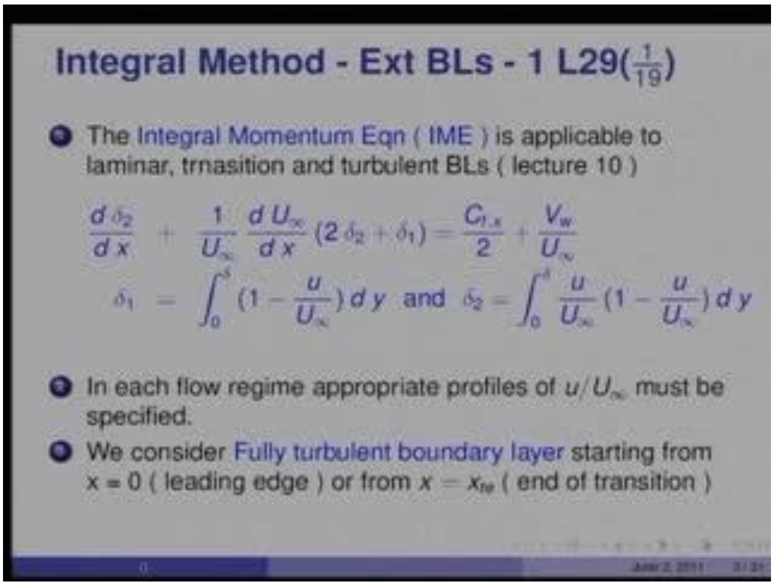
Let us turn our attention on the next slide to the issue of calculating delta 1 and delta 2. First of all, you must recognize that delta 1 and delta 2 as you will now require integration of these from 0 to delta. As you already remember that delta plus would be of the order of 800 to 1000 or 1200 whereas, laminar sub level would extend only up to y plus of 5 and transitional layer would not extend up to y plus of 30. The fully turbulent layer begins at 30 and goes up to 800 to 1000 or 1200 or whatever depends on the Reynolds number at that location.

Therefore, even if I ignore this 0 to 30 region it would not matter to the evaluation of delta 1 and likewise of delta 2; in other words, the low velocity regions are dropped out. Therefore, we essentially compute delta 1 and delta 2 using turbulent velocity profiles only.

(Refer Slide Time: 05:12)

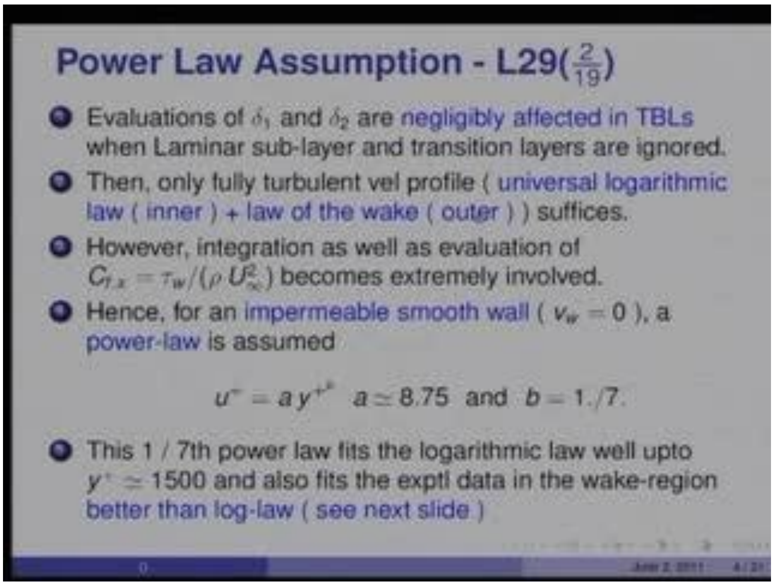


(Refer Slide Time: 05:36)



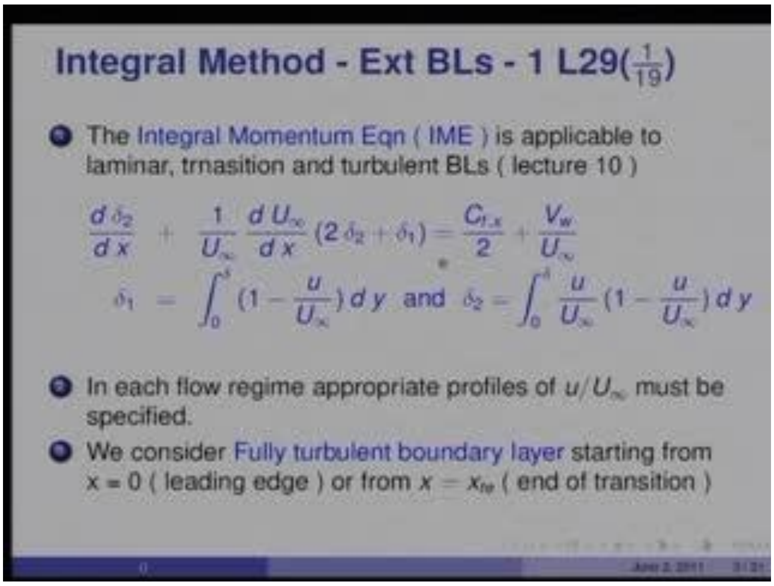
The fully turbulent profile is a logarithmic law for the inner layer - inner part of the turbulent layer - and plus the law of the wake as we saw in the previous lecture. However, if we were to use this law there are two difficulties, first of all, because of the logarithmic expression the evaluation of integration becomes somewhat difficult.

(Refer Slide Time: 05:44)



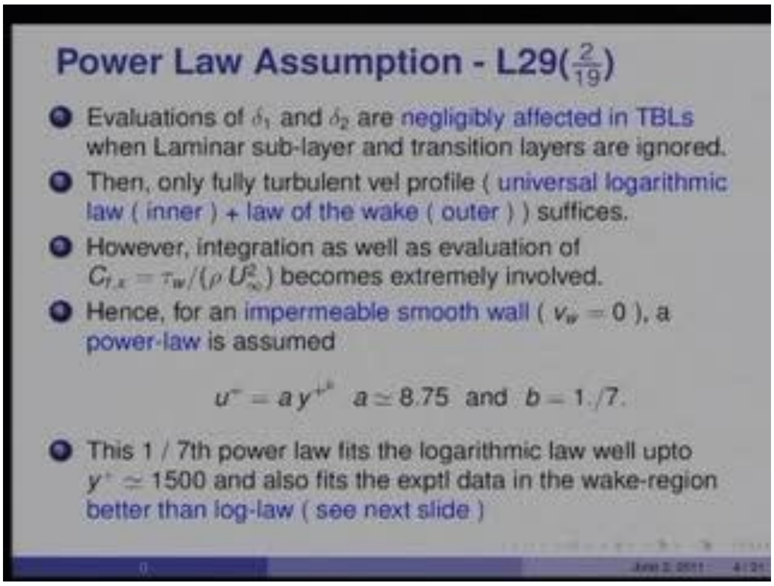
Secondly, C f x is not very easy to extract from the universal logarithmic law of the wall. So, there are two difficulties simply because we cannot use that law to evaluate shear stress as mu d u d y equal to 0. Therefore, shear stress has to be specially evaluated.

(Refer Slide Time: 06:06)



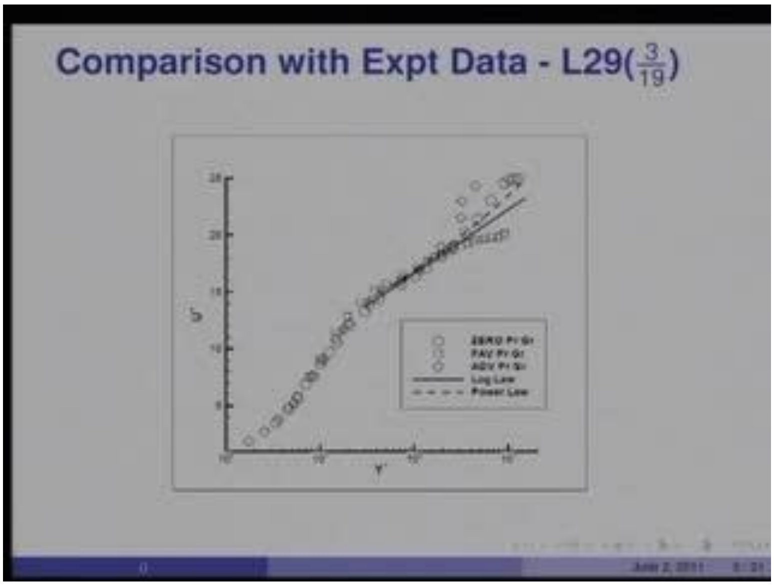
On both these accounts the use of the universal logarithmic law plus law of the weight is somewhat less fruitful and therefore, what is done is a shortcut.

(Refer Slide Time: 06:10)



Now, instead or a smooth impermeable wall attended it that is, V w equal to 0; we assume a power law. Instead of a logarithmic law plus law of the weight, we assume a power law as u plus equal to 8 times y plus rise to b; a and b can be the functions of Reynolds number, but I have given here some values of Reynolds number - some values of a and b - which are usually found to be quite good in practically encountered turbulent boundary layers. So, a is 8.75 and b is 1 by 7 and this is called the 1 /7th power law. Now, this law actually fits the experimental data up to y plus of 1500 quite well in fact better than the log law, as you will see in the next slide.

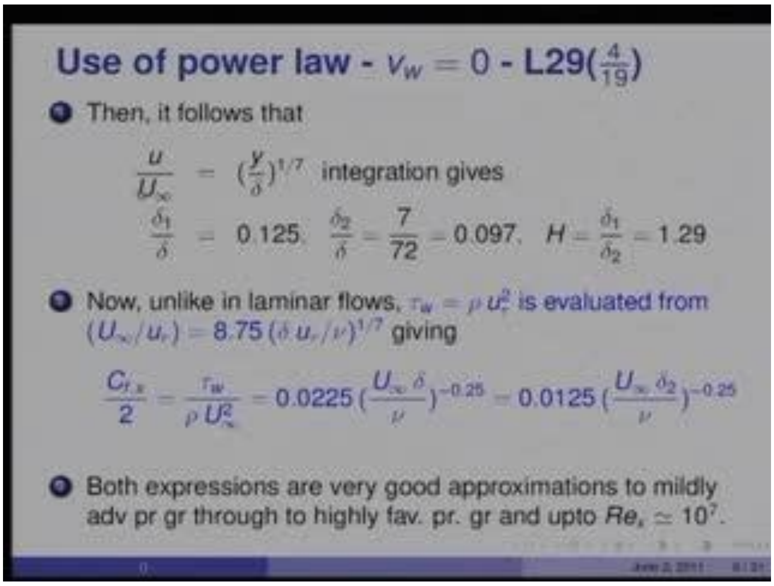
(Refer Slide Time: 07:13)



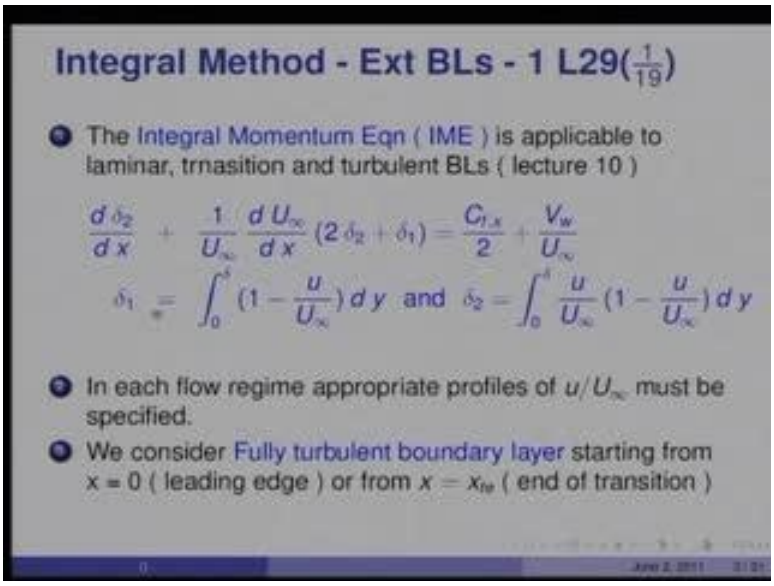
Here is the plot of u plus versus y plus and y plus is on the log scale. Here are the experimental data for zero, favorable and adverse pressure gradients. Of course, our interest is now only in turbulent boundary layers because that is where we are going to integrate things.

You can see that the dash line is the power law expression and the solid line is the logarithmic law plus the law of the weight. Now, you can see that the dash line in fact predicts the velocity profile at least in 0 pressure gradient, quite well right up to the wake region whereas, in the wake region logarithmic law does not really predict that well - the velocities. What we see is that we have some confidence in use of the power law as much as it will not influence the calculations of delta 1 and delta 2 in spite of these little departures that you see here.

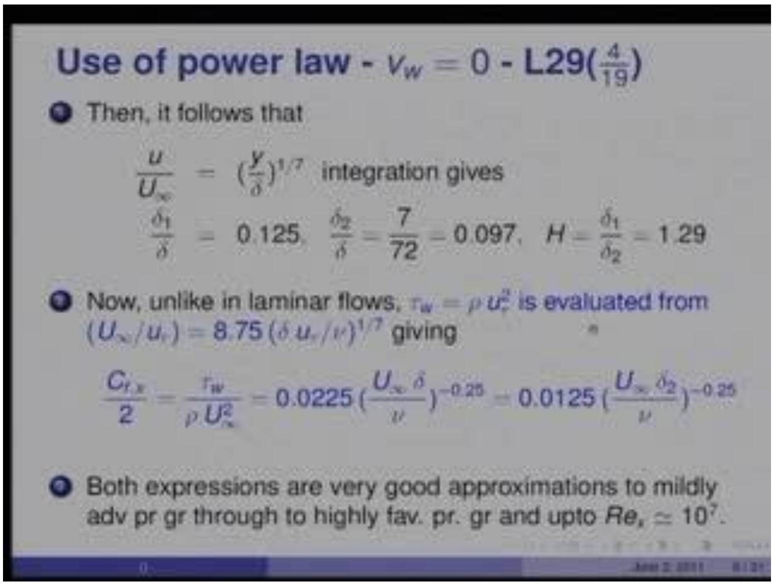
(Refer Slide Time: 08:10)



(Refer Slide Time: 08:24)



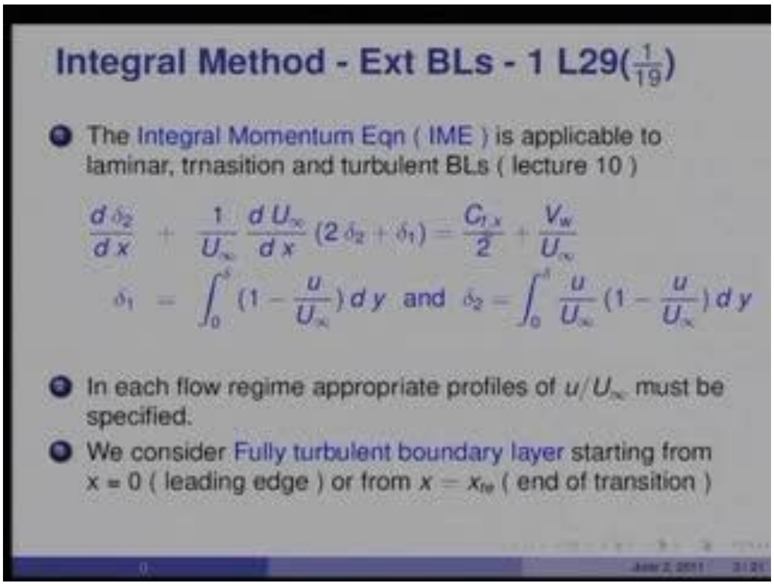
(Refer Slide Time: 08:28)



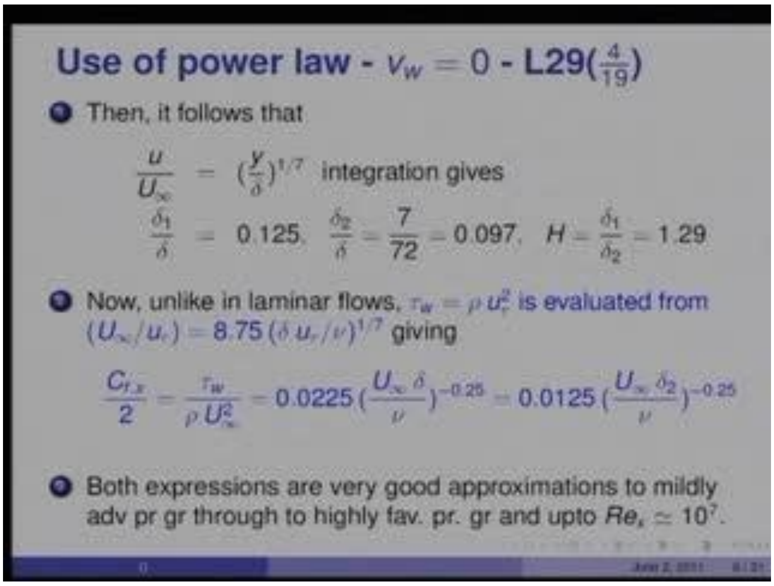
Essentially, our law u plus equal to a y plus raise to b is u over U infinity equal to y by delta 1 by 7. If I substitute this expression in our expressions for delta 1 and delta 2, then you gets ready evaluations as shown here, delta 1 by delta 2 will be 1 by 8 or 0.125; delta 2 by delta would be 7 by 72 or equal to 0.097. The shift factor which is the ratio of delta 1 by delta 2 is equal to 1.29.

Now, this is quite characteristic of turbulent boundary layer that they usually have much smaller shift factor, then in laminar boundary layers - you will recall - that in laminar boundary layers we had H of the order of 2.5 whereas, in turbulent boundary layers it is of the order of 1.3 or so.

(Refer Slide Time: 09:09)



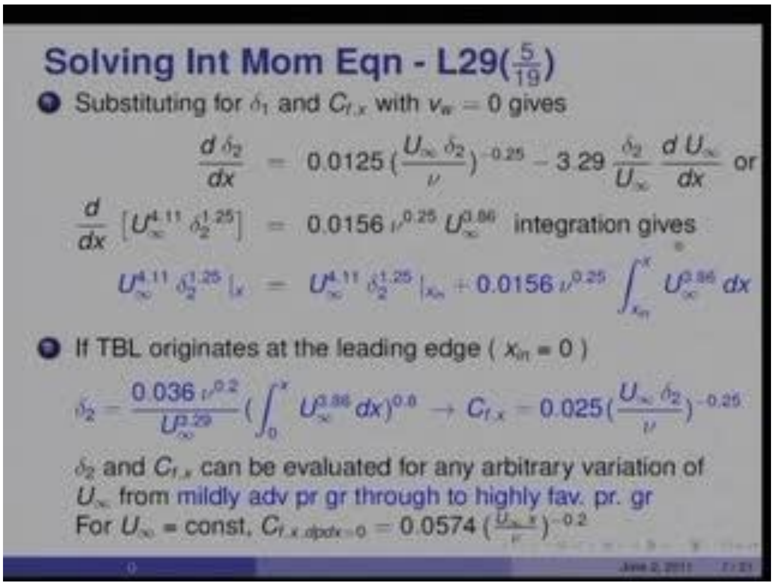
(Refer Slide Time: 09:17)



In order to evaluate the C f x part C f x that you need here, essentially that the heat equals C f x by 2 equals tau wall by rho U infinity square and that can be evaluated quite easily from this expression U over u tau equal to 8.75 delta u tau by nu. You will see, if I multiply this both sides by u tau, I get u tau raise to 8 by 7 equal to the rest. Therefore, I can reform or reorganize this equation because u tau as we know a square root of tau wall by rho.

Therefore, C f x by 2 tau wall by rho infinity square would simply become 0.0225 U infinity delta by nu raise to minus 0.25. Now, if I replace delta in terms of delta 2 using this relationship 7 by 72. Then, it will become 0.0125 U infinity delta 2 by nu raise to minus 0.25. The both these expressions where either in terms of delta or delta 2 are very good approximations to mildly adverse pressure gradients through very highly favorable pressure gradients and up to about Reynolds x of 10 raise to 7 or 10 million.

(Refer Slide Time: 10:47)



We can confidently use C f x by 2 in this manner and use H delta 1 by delta 2 equal to 1.29. Then, you will see that the momentum equation would become like this, d delta 2 by dx this is the C f x term in terms of delta 2. This is really 2 plus delta 1 by delta 2, which is H. So, which was 1.29, that becomes 3.29 into delta 2 by U infinity d U infinity by dx.

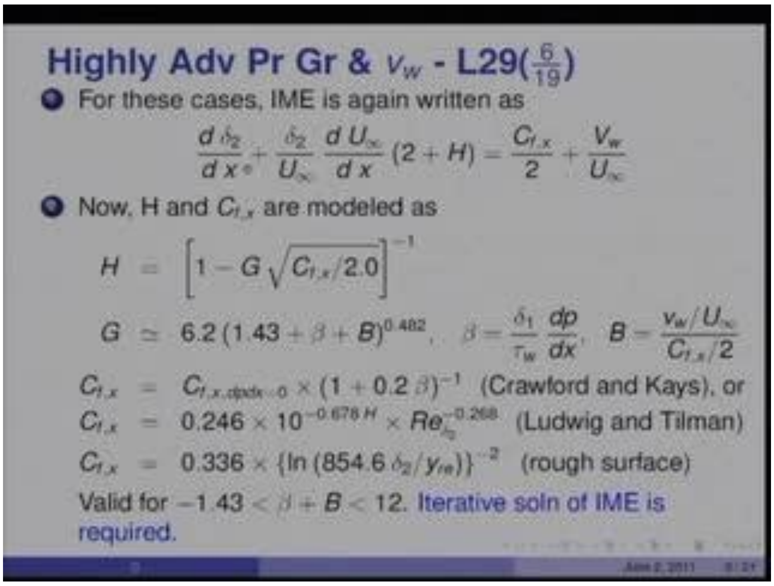
If I transfer this term on the left hand side then, you will see I can reorganize it as d by dx of U infinity raise to 4.11 into delta 2 raise to 1.25 equal to 0.0156 nu 0.25 U infinity raise to 3.86. If I integrate that then I would get U infinity 4.11 delta 2 raise to 1.25 at any x would be equal to the same quantity at the lower limit of integration plus 0.0156 nu raise to 0.25 0 to x U infinity 3.86 dx.

Now, if the turbulent boundary layer originates at the leading edge, so that the x is 0 - the lower limit of integration 0 - then, I can readily evaluate delta 2 for any arbitrary variation of U infinity with x, because this integration can easily be done either by n or by numerical integration knowing of course, the variation of U infinity with respect to x. Once you know delta 2 then, you can evaluate recover your del C f x from this expression that we have already written on the previous slide.

As long as the pressure gradient is mildly adverse through to highly favorable pressure gradient then of course, all this derivations are applicable. Now, suppose U infinity was constant that is, flat plate boundary layer or pressure gradient being equal to 0 which means that term is 0. Then, you will see U infinity raise to 3.86 would simply come out of the integration and you can readily show that C f x would be equal to 0.0574 Reynolds x raise to minus 0.2.

Now, this is an expression here routinely used in your under graduate work for calculating skin friction coefficient variation with x in a flat plate boundary layer and we have recovered that through integral method.

(Refer Slide Time: 13:47)



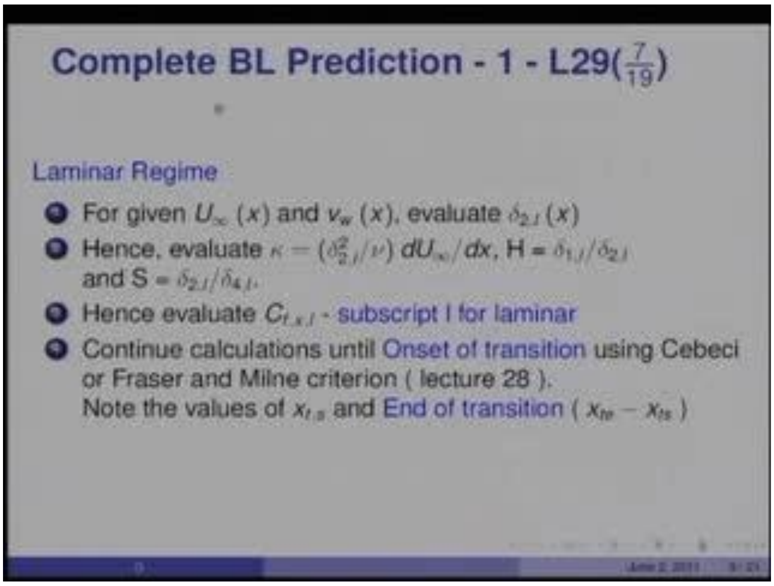
What about the cases when the pressure gradient is highly adverse; what we have done so far is valid only from mildly adverse pressure gradient to highly accelerated. Let us look at the lower limit of the adverse pressure gradient, when the adverse pressure gradient is high, how do we do that? We again recall that the integral momentum equation would be like this (Refer Slide Time: 13:54). We would like also to include the effect of V w should it be present that is suction and blowing are present.

In this case what is done is that H the shift factor is tuned. The manner in which its tuned is quite empirical whatever I am going to say now is very empirical but, it has been validated through a series of computations by earlier workers. It says that the shift factor would be made as function of firstly G, where G is a function of beta, which is a pressure gradient parameter as well as suction and blowing parameter. So - beta is given by that - B is given by that and secondly, it would be function of the skin friction coefficient C f x itself.

Now, how do we evaluate C f x? For example, C f x would be related to C f x for dp dx equal to 0 into 1 plus 0.2 beta raise to minus 1; this is correlation given by Crawford and Kays to account for the effect of pressure gradient the parameter beta. There is also another one given by Ludwig and Tilman, in that C f x itself is made a function of H. So, 0.246 into 10 raise to minus 678 at H into Reynolds delta 2 rises to minus 0.268, both these are meant for smooth surfaces. There is also another recommendation in which C f x here is to be written in this fashions 3.336 into logarithm of 854.6 delta 2 divided by equivalent surface roughness raise to minus 2.

So, depending on the problem in hand one would use either the smooth surface expression or the rough surface expression. These expressions are valid for 1.43 minus 1.43 beta plus B varying from minus 0.143 to plus 12. Now, because H is a function of C f x and C f x is a function of delta 2 as you know, close form solutions cannot be obtained in this case. Therefore, you need to do iterative solution of the integral momentum equation, this equation has to be solved iteratively at each x.

(Refer Slide Time: 16:26)

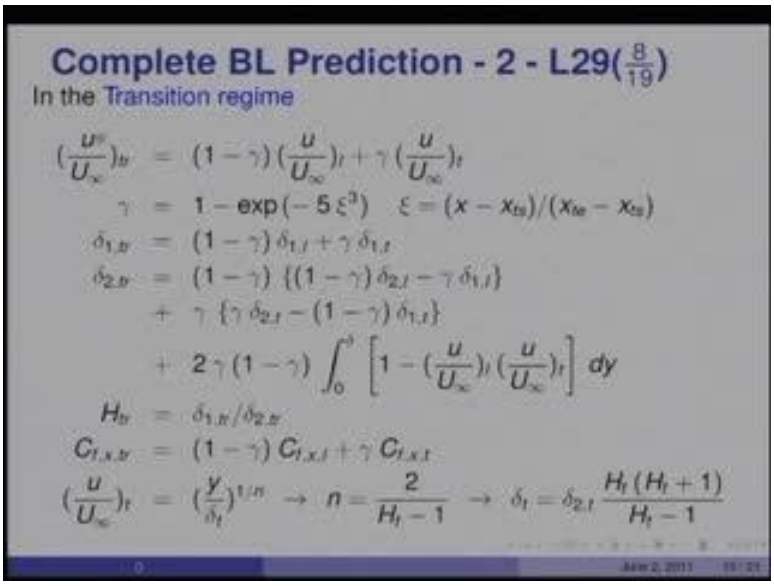


Now, let us turn our attention to complete boundary layer prediction in which we begin with laminar regime. For laminar regime of course, you know this entire very well, you do not really need computer programming because you already have the solutions. Since, we are going to compute through transition and turbulent layer, we would include the laminar calculations in a computer program based calculation and that is what I am showing now.

So, if you know variation of U infinity and V w x then of course, you would evaluate delta two laminar x suffix l represents laminar. Therefore, evaluate kappa - the pressure gradient parameter based on delta 2 - and evaluate H which is delta 1 by delta 2 and then, the shear thick factor which is delta 2 by delta 4.

Once you got your delta 4 out of this then, readily you can evaluate the C f x for the laminar part of the boundary layer. We continue this calculation until the onset of transition and the onset of transition is recognized either by using Cebeci formula or by the Fraser and Milne criterion, which we discussed in the previous lecture. Now, both these formally give you start of transition as well as the criterion for end of transition. As soon as we come to the end of laminar boundary layer, we already know the total length that would be occupied by the transitional layer x te minus x ts.

(Refer Slide Time: 18:14)



Now, let us go to the transitional part of the calculation. In the transitional part we need a velocity profile which is taken as this, u over U infinity transition is taken as 1 minus intermittency gamma into u over U infinity sub l which is laminar, which as you know was a polynomial plus gamma times u over U infinity of turbulent.

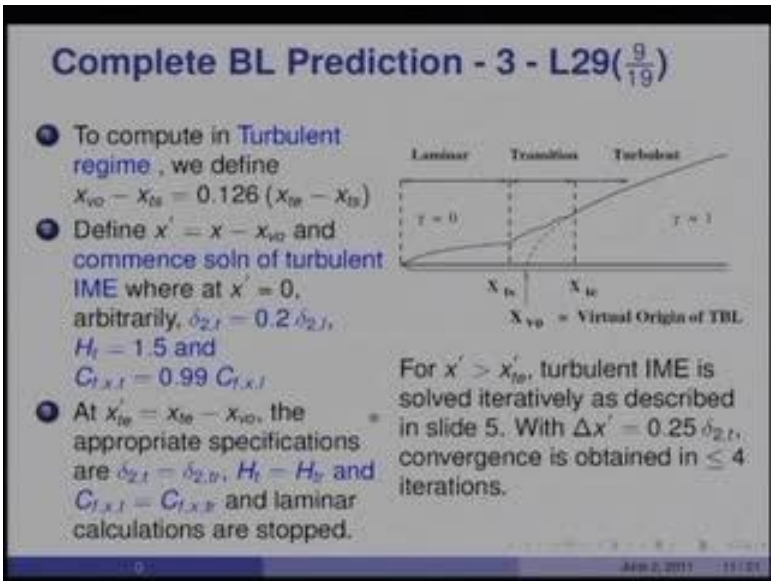
How to tackle the turbulent part? We will see in a short while. Gamma, as you know is a intermittency parameter which varies with x. Since you know x ts and x te already, you can readily evaluate xi and therefore, intermittency distribution over the transitional length is already known. If you substitute this transitional distribution here then, using u over u infinity transition would give you delta 1 transition equal to 1 minus gamma delta 1 laminar plus gamma time delta 1 turbulent. Delta 2 transition again has a similar looking term multiplied by 1 minus gamma plus gamma times gamma of delta 2 t minus 1 minus gamma of delta 1 t.

The delta 2 evaluation is somewhat involved, plus it has a term which requires integration of product of laminar and turbulent boundary layers. As you can see therefore, that close form solutions are again extremely difficult to obtain and computer solutions becomes very valued. Then the H transition for the transitional part; H transition is simply evaluated as delta 1 transition divided by delta 2 transition and that is what I have shown here (Refer Slide Time: 19:56). The C f x in the transition regime would evaluate as 1 minus gamma of C f x due to laminar profile and gamma times C f x due to turbulent profile.

Now, the turbulent profile itself is taken in this manner; u over U infinity t is equal to y over delta t 1 over n which means again a power law profile, but the value of n is taken as 2 over H t minus 1 which is the shift factor of the turbulent part minus 1, which gives you delta t equal to delta 2 t equal to H t plus H t plus 1 divided by H t minus 1. If I have to substitute this then you can get delta 2 t for the turbulent part as a function of delta t or the other way around.

These are the evaluating equations for the transitional part. However, a little trick is required to carry on the computations through laminar transitional and turbulent layers and that is what we shall see on the next slide.

(Refer Slide Time: 21:05)



First of all, in order to compute the turbulent contribution to transitional layer, what is done is that since we know x te already, we somehow back extrapolate the turbulent development and identify what is called as a virtual origin of the turbulent boundary layer x vo. The x vo minus start of transition x ts that is, this length is taken as 12.6 percent of the total transitional length, so that is the first thing done to identify the virtual origin.

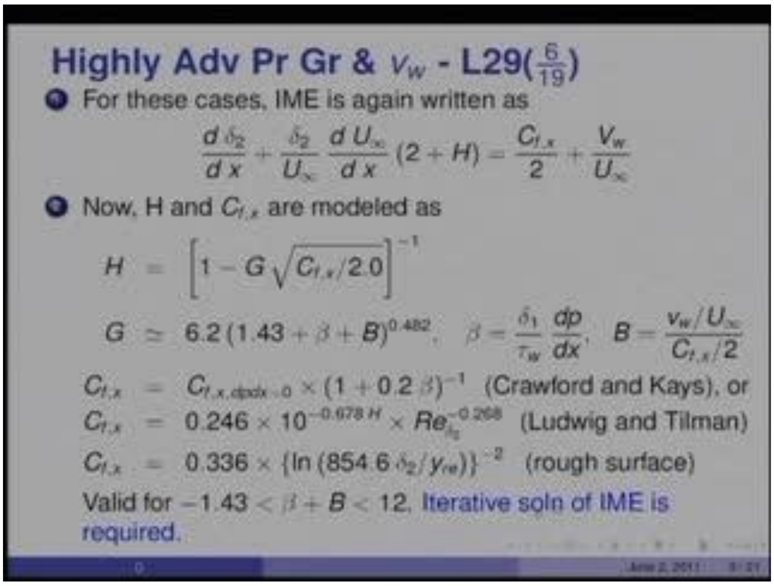
In order to calculate delta 1 t and delta 2 t so on and so forth, we first of all define x dash equal to x minus x vo - the virtual origin a new variable x dash. Commence the turbulent calculations at x dash equal to 0 that means, the laminar boundary layer calculations are continued in fact up to x vo there is no contribution of turbulent at very low intermittency levels.

We simply have no contribution of the turbulent boundary layer but, the turbulent boundary layer contributions begin from x vo onwards. Now, this would require some value of delta 1 and delta 2 t at these points x vo and that is taken as delta 2 t at this quite empirically is taken as 20 percent of the delta 2 as computed from laminar velocity profile. H t is taken as 1.5 and C f x of the turbulent part in this region at this point rather is taken as 0.99 of C f x of laminar. So, these are all empirical settings but, that have been found to be quite good in predicting flows through the transitional regime.

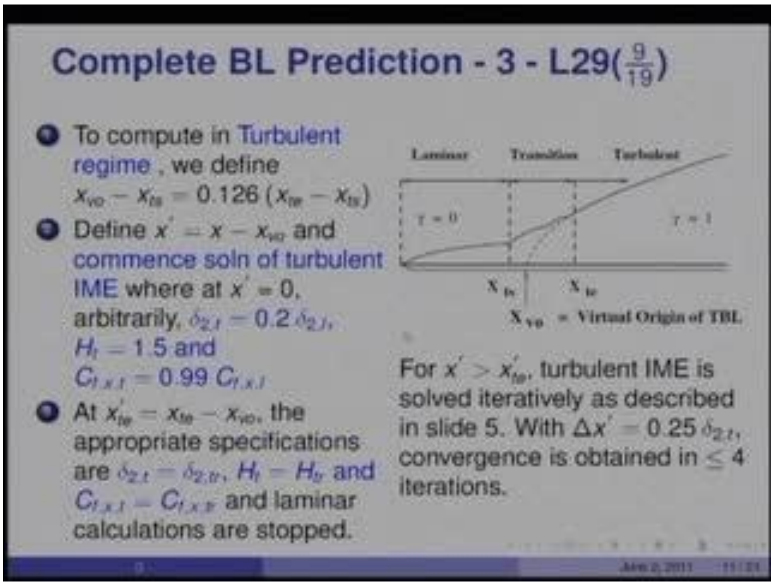
Now both the turbulent and laminar velocity profiles are used as shown in the previous slide. We calculate C f x of transition, delta 1 of transition, delta 2 of transition and so on so forth. Therefore, the H of transition by calculating using both laminar and turbulent velocity profiles, so that is how we continue up to x te, because x te is already known before we started the transitional calculations.

At x dash te equal to x te minus x vo appropriate specifications that are start of fully turbulent calculation delta 2 t would be equal to delta 2 t transition then, H t would be equal to H transition at this point and C f x t would be equal to C f x transition at this point. Therefore, laminar flow calculations are completely stopped and we only integrate the turbulent part. In the turbulent part we use again the power law as set on the previous slide.

(Refer Slide Time: 24:32)



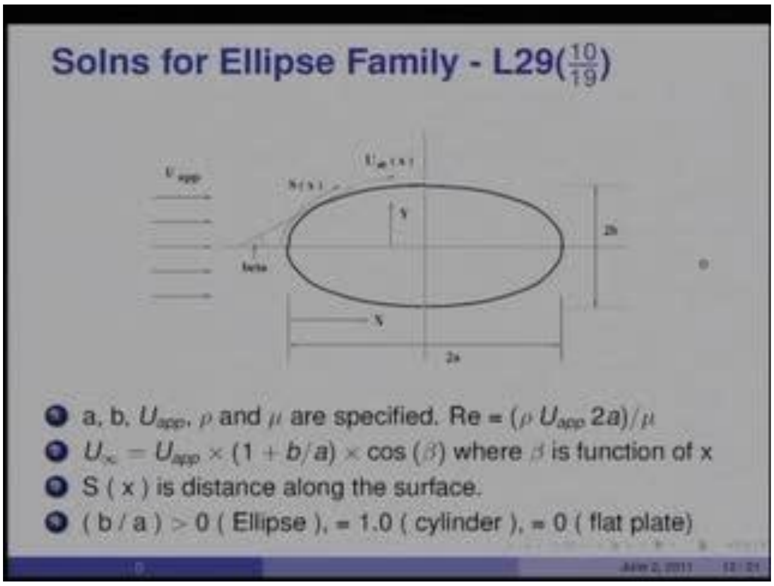
(Refer Slide Time: 24:43)



For x dash greater than x dash te turbulent Integral Momentum and Energy equation is solved iteratively as described why the iterations are required particularly when H is a function of pressure gradient and the suction and blowing parameter as well. We just use this iterative method to compute the flow through the turbulent layer.

Now, in order that the number of iterations required, small typical advice for the step size is that delta x here should be taken as 1 quarter of the momentum thickness at the previous step in the boundary layer. It is taken as one-fourth of the momentum thickness and usually converges are obtained in less than four iterations per step. So, very allegiance computation and very fast computation can be performed using this method from laminar through transition to turbulent boundary layers.

(Refer Slide Time: 25:28)



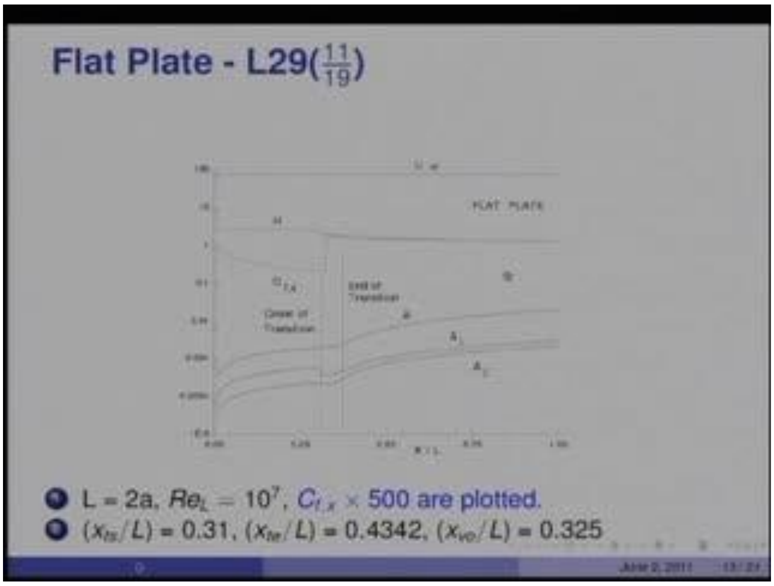
Now, let us see some example, what I am going to do is to consider flow over a family of ellipses. Ellipse is a wonderful family because it has the minor axis 2b and a major axis is 2a. If b by a is greater than 0 then you get an ellipse, if it is exactly equal to 1 you will get a cylinder and if it is equal to 0 that is the b is equal to 0 you simply get the flat plate.

In this case you specify a b approach velocity, U approach the density and viscosity. Therefore, the Reynolds number based on U approach and the major axis length 2a divided by mu would be the Reynolds number. In case of a flat plate 2a would be the length of the flat plate, so 2a would be simply equal to l.

Now, for family of ellipse a U infinity that is, the variation of U infinity along the periphery of the ellipse would be U approach into 1 plus b by a cos beta where cos beta is evaluated like this. At each point on the surface you draw a tangent, wherever it intersects the x axis you note the angle beta and you take the cosine of that to get the free stream velocity U infinity variation with respect to x (Refer Slide Time: 26:38).

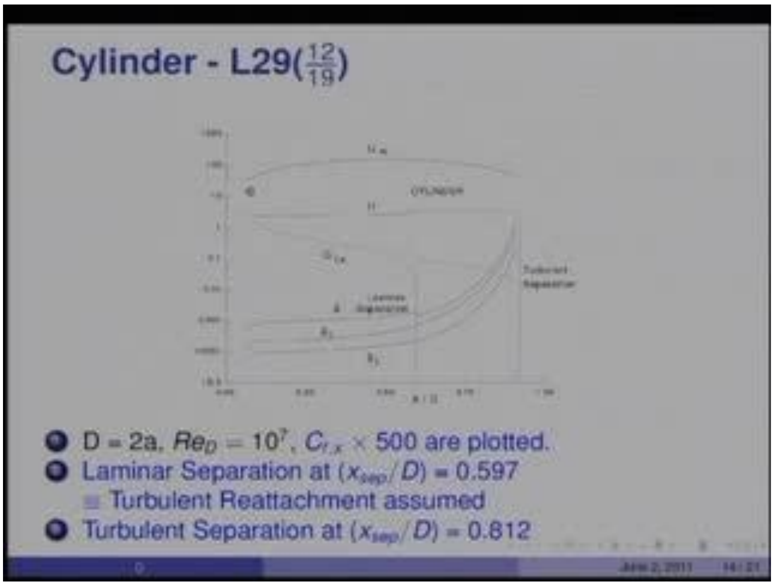
S x is the distance along the surface of the ellipse whereas, x itself is the distance along the axis as shown here, x is a is the distance along axis whereas, S x is measured from the forward stagnation point and along the surface of the ellipse.

(Refer Slide Time: 27:30)



We shall see all the 3 cases 0, 1 and b by a which would be less than 1 and greater than 0 which would be the ellipse. Let us take the first case, I have taken the flat plate case, so U infinity is constant as you can see. The Reynolds number based on length is 10 raised to 7. What do I find? I find that the start of transition is identified at about 0.31 x transition divided by L is identified point transition; end of transition is identified as 0.4342 as you can see here.

(Refer Slide Time: 29:37)



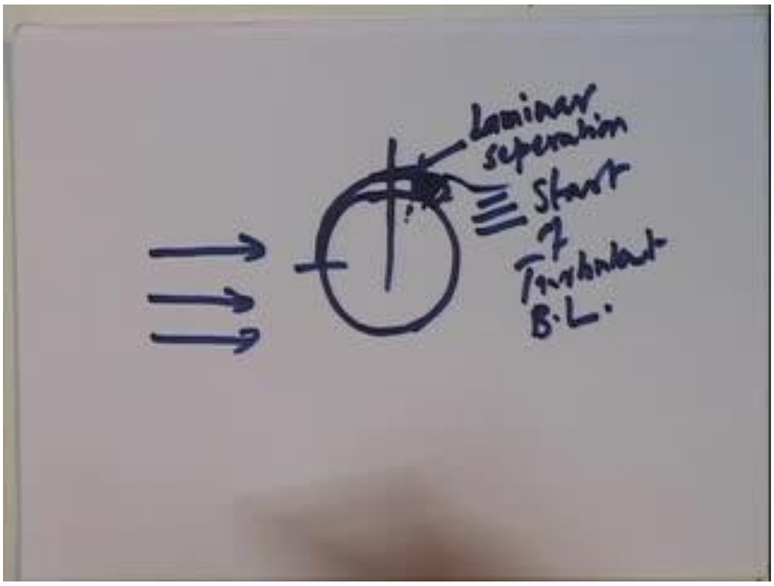
The virtual origin is bit closer to start of transition is 0.325 x vo by L equal to 0.325. You can see that the C f x in fact decreases along the length in the laminar boundary layer up to x vo and then it suddenly raises in the transitional layer, because both turbulent and laminar contributions have been begin. It begins to fall again and meets the end of transition here and then the turbulent part of the skin friction factor continues.

Now, I have multiplied here C f x by 500 to get everything on proper scaling done. The shear factor is of the order 2.5 in the laminar case, then it dips and slowly goes on decreasing and becomes almost equal to 1.29 as we had shown in our calculations earlier.

Here are the rate of growth of delta and we always said that all the integral thicknesses vary as x to the power of 0.5 in laminar case whereas, they vary as x to the power of 0.8 and that is born out very nicely in this figure. So, this is how you can complete the calculation of a flow over a flat plate. Now, let us look at flow over a cylinder. You can see the free stream velocity variation is like this. This is the forward stagnation point to somewhere here where the velocity would be 0 of course, but on log scale you will not see 0 (Refer Slide Time: 29:39).

Then, it accelerates till the top of the cylinder and then decelerates as you go along here. In other words, 2a is the diameter then the Reynolds number based on diameter is 10 raised to 7. Again, I have plotted values of C f x multiplied by 500. Now, very interestingly at this Reynolds number you see the C f x falls in the laminar regime, but it does not turn into transition at all and instead laminar separation occurs somewhat after the midpoint.

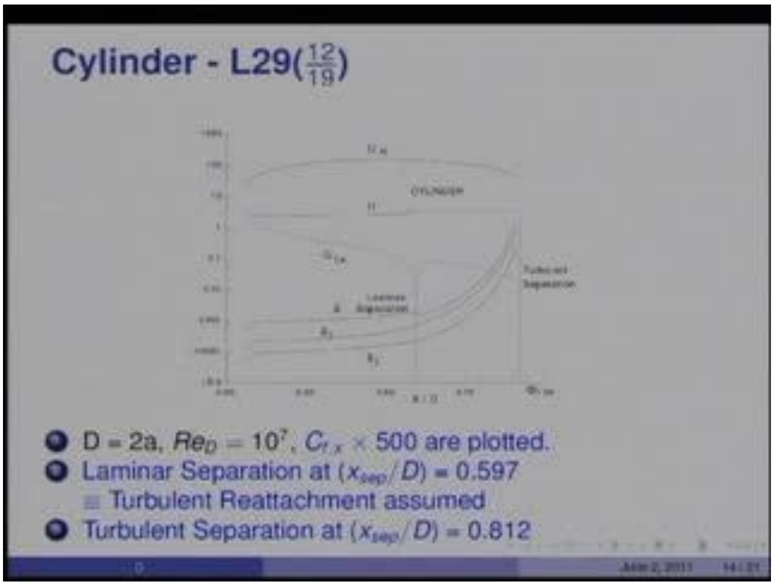
(Refer Slide Time: 30:42)



To show this what I mean is the following that, say if this is the cylinder and this is the approach velocity then, the laminar boundary layer begins to develop here passes the midpoint somewhere here and it separates, so you have separation taking place and it is very close to the wall that separation (Refer Slide Time: 30:44).

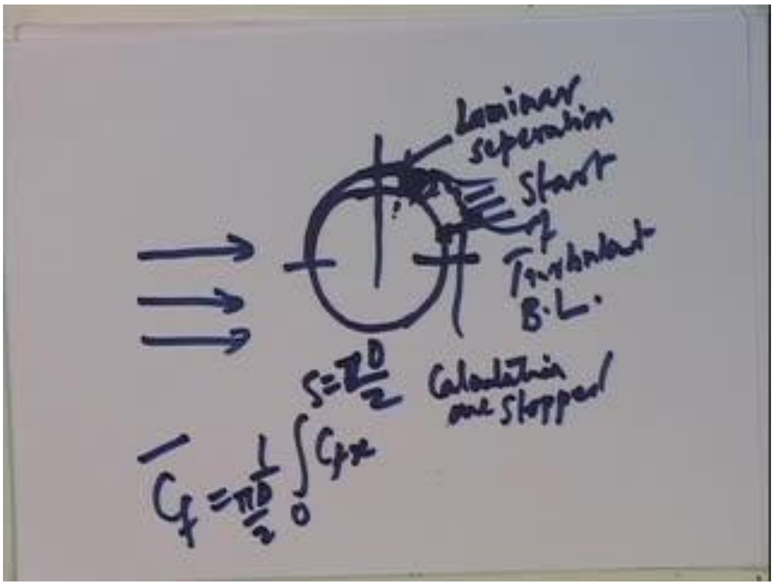
What we do now is to cheat the flow and say laminar separation is equal to start of turbulent boundary layer. We simply start assuming that the boundary layer will turn turbulent after this point of separation and continue the calculations, so that is what I have shown here (Refer Slide Time: 31:45).

(Refer Slide Time: 31:45)



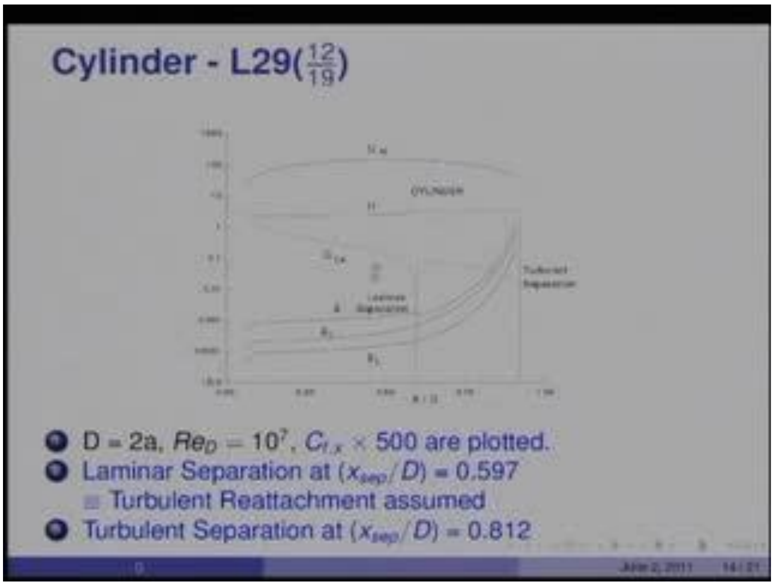
You can see there is an abrupt shift from a very low C f x, which goes down because of separation to 0 and then, very quickly raises up again to the turbulent part of the calculation, with that the C f x again decreases even in the decelerating part of the flow. Now, what happens is at x by d equal to 0.812 the boundary layers separates in the turbulent path, because of the deceleration it separates here that is at the back, somewhere at this point the turbulent boundary layer also separates.

(Refer Slide Time: 32:29)



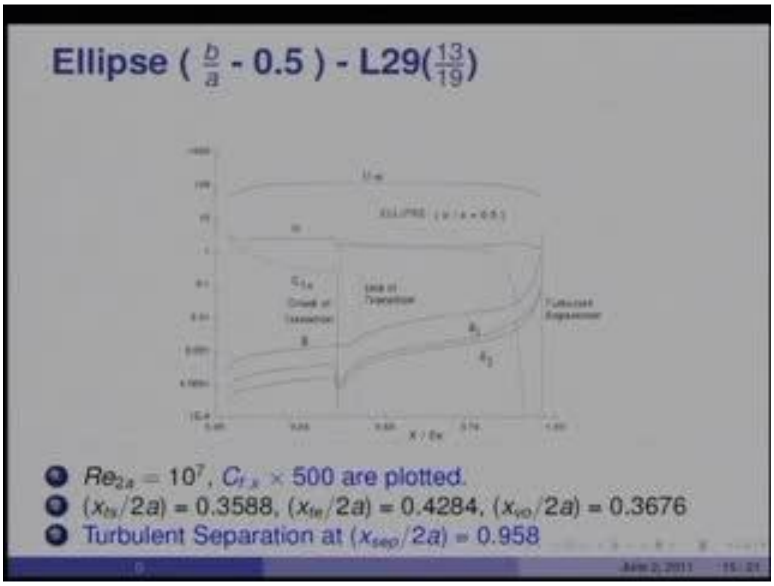
Now, of course, we cannot continue the calculation and the calculations are stopped. Of course, this neglect of this much region say from 0.812 to 1 would not affect the integral quantity of C f x that is our belief that 0 to S, which is equal to pi times d by 2 1 over pi d by 2 will be equal to C f average.

(Refer Slide Time: 33:21)



That value will not be severely affected by neglecting this calculation and therefore, we can still recover the drag of a cylinder from integral method. Laminar separation in this case occurs at 0.597 and it is taken as turbulent reattachment straightaway. In other words, the transitional length is simply absent here and you simply begin the turbulent calculations immediately after laminar calculations have yielded C f x going down to 0, very interesting phenomenon, no transition at all.

(Refer Slide Time: 33:58)

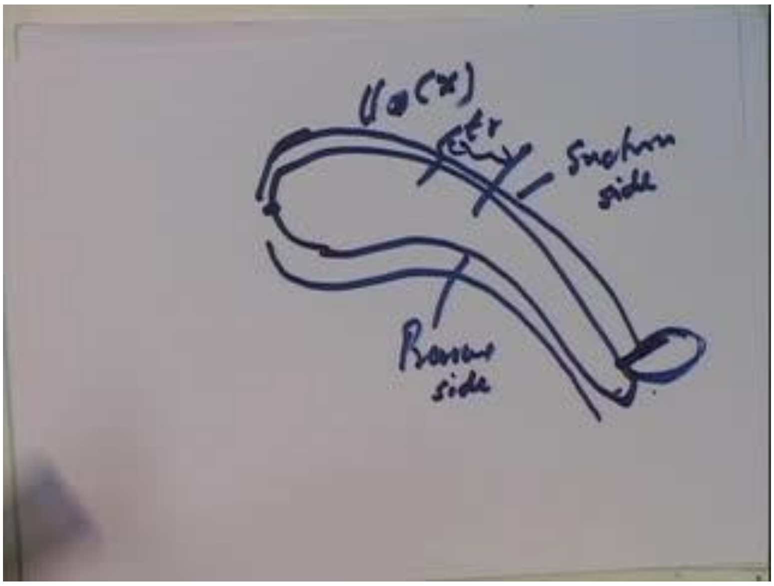


Let us look at ellipse as I have done here and I have chosen b by a equal to 0.5 but, you could choose any other. Again, the flow accelerates till top of the ellipse and then decelerates, but the region here is much more flatter, so there is a considerable region of 0 pressure gradient in case of flow over an ellipse. Again, you will see the C f x value decreases and it does encounter transition at x ts by major axis equal to 0.3588 that is where it encounters transition. The end of transition is predicted at 0.4284 and the virtual origin is 0.3676 or the 12.6 percent of the x te minus x ts length.

There you will see the C f x increases again and then starts declining but then, at 0.958 it again experiences turbulent separation and the computations cannot be taken any forward further because the separation is occurred. You can see C f x behavior is quite different from that for a cylinder but, in both cases the turbulent separation is encountered. Here, finite laminar transition length has been encountered whereas, in the flow over a cylinder we found no transition length. The laminar separation was straight away taken as start of turbulent layer.

The shift factor again is the order of 2.5 in the laminar regime and then, it decreases to around 1.3 or 1.4 so of that order. Computations of this type are very valuable because one can take the case of flow over let us say, any blade or so.

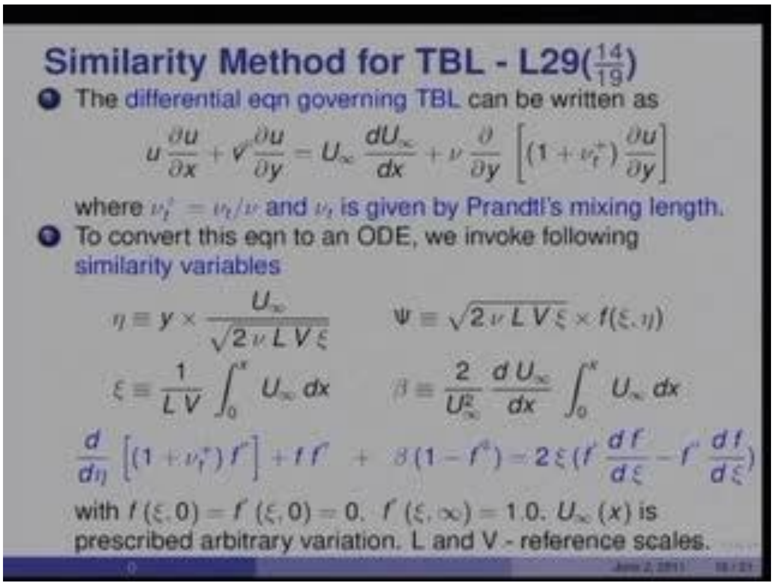
(Refer Slide Time: 36:01)



Flow over a blade and one can begin calculations on the suction side for example, as well as on the pressure side, this is the pressure side, this is the suction side. As long as you know the variation of U infinity as the function of x, you can compute the development of the boundary layer; you can identify where transition occurs and you can identify the length of transition on such blades so on and so forth.

Should any separation occur then of course, that is undesirable and therefore, one would like to adjust the shape of the blade in such a way that no separation is found near the trailing edge. One can do the same way on the pressure side, so calculations of this type have been used in the pre CFD days, these type of calculations where extensively used to design and shape the compressor and turbine blades.

(Refer Slide Time: 37:26)



Now, we turn to the differential method for calculation of C f x through all the three layers and that time average RANS equations for the boundary layer would be like this. These are the convection term, this is the pressure gradient term and this is the diffusion term, where I have taken the new out and that which would make it 1 plus nu plus t, so this is the turbulent part; turbulent viscosity divided by nu (Refer Slide Time: 37:32). The variation of nu t by nu as a function of y is given by Prandtls mixing length for example, that we have already seen.

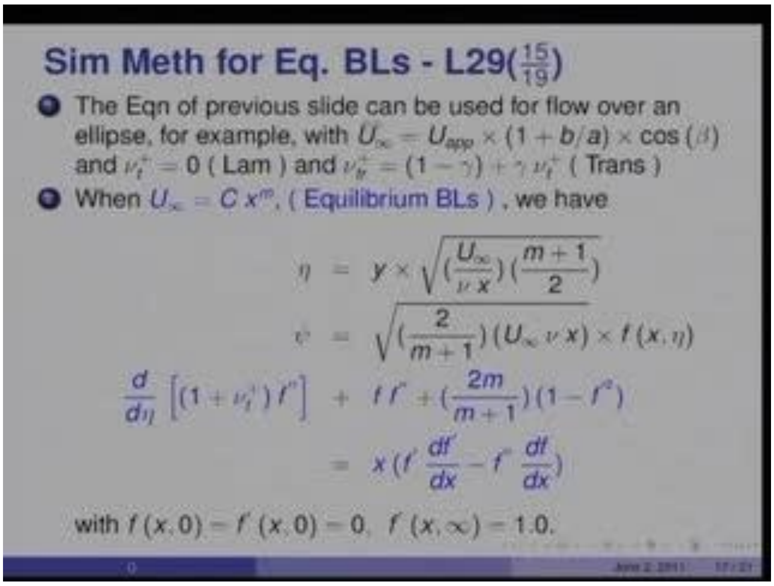
In order to convert these differential equation where this is a function of y to an ordinary differential equation, we invoke these parameters - so called similarity parameters - in which we take L and V as some reference velocity scales. Then, eta is defined as y into U infinity 2 nu L V xi, where xi itself is 1 over L V 0 to x U infinity dx, where U infinity is a function of x. The stream function is defined in that fashion and the pressure gradient parameter beta is defined in this fashion.

If you make use of this definition of xi and substitute here and calculate the gradients so on and so forth, in the usual manner say about two pages of algebra. You can show that this equation would transform to d by d eta of 1 plus nu t plus f double prime plus f f double prime plus beta into 1 minus f dash square equal to not 0 but, 2 xi into f dash d f dash by d xi minus f double prime df by d xi.

These are the inconvenient terms because these are functions of xi and xi itself is a function of x. In other words, our equation in not ordinary differential equation, all we have done is separated out the eta dependent variables on the left hand side, the eta and x dependent variables on the on the right hand side. The boundary conditions of course, will be f xi 0 equal to 0, because we are assuming v equal to 0 at the moment f dash xi 0 because u itself is 0 and f dash infinity would be 1.

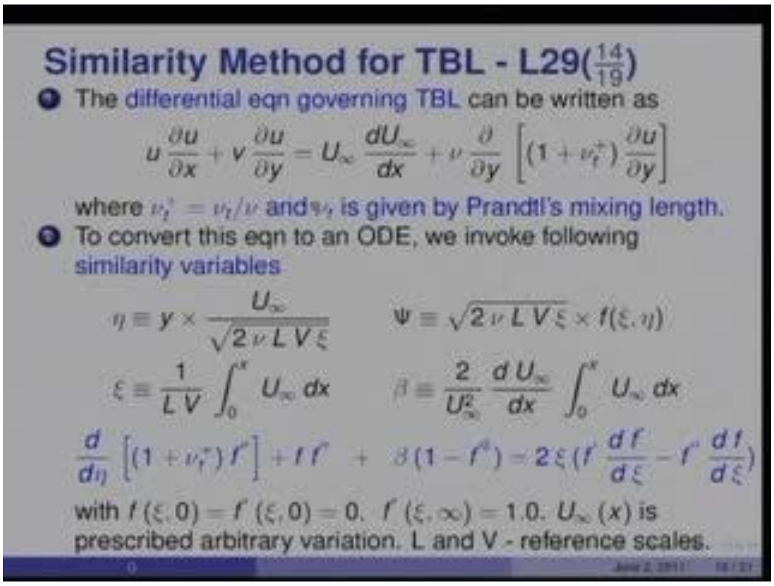
Now, U infinity x is a prescribed arbitrary variation in such a formulation, because it allows you to handle any arbitrary variation of U infinity, where an beta simply takes different values at different actual stations. How do we solve such an equation? In other words, you can see the similarity method has been made amenable to arbitrary variation of U infinity. How do we handle such an equation which has all the left hand side is the function of eta only; whereas, all the right hand side is a function of both x eta as well as, xi or x.

(Refer Slide Time: 40:48)



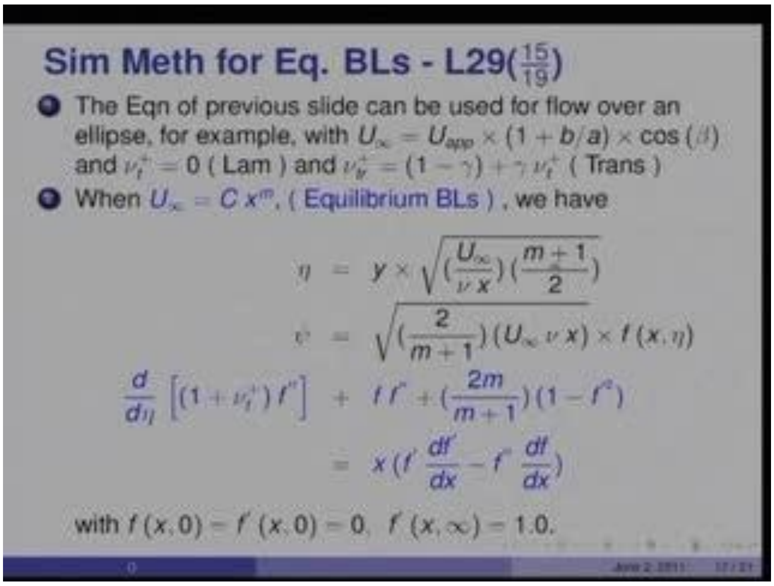
The equation of the previous slide can be used for flow over an ellipse for example, in which U infinity varies arbitrary with x in this manner. It will have nu t plus equal to 0 in laminar range, nu t transition would be 1 minus gamma plus gamma time nu t plus in the transitional regime. Of course, in the turbulent regime gamma would be 1, so that would be nu t plus would be simply nu t plus.

(Refer Slide Time: 41:42)



We can use the previous slide equation to calculate flow over a family of ellipses but, presently what I am going to do is, let us consider what are called equilibrium boundary layers in which U infinity has this special form of variation - the wedge flow variation - U infinity equal to C x to the power of m.

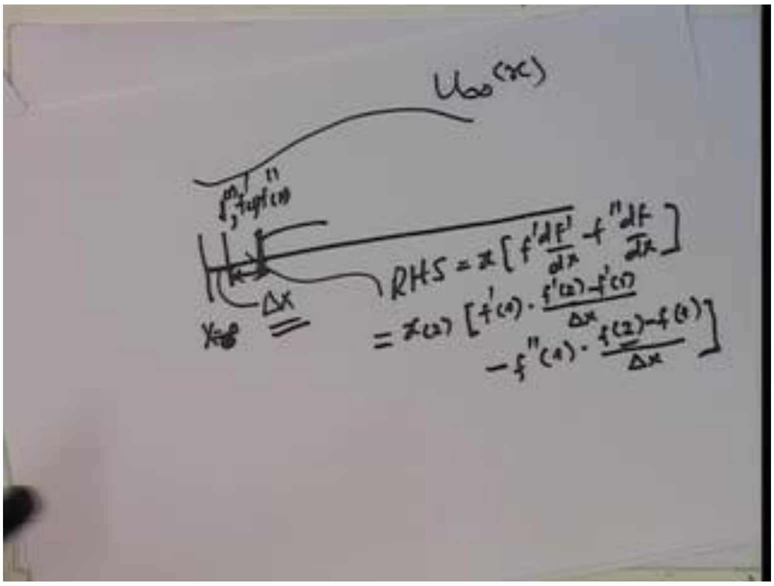
(Refer Slide Time: 41:53)



If I use this expression in all these definition for U infinity everywhere, then you will see the eta variable would become y into under root of U infinity by nu x m plus 1 by 2, psi would become that and the equation itself would be something like this with this as the pressure gradient parameter. Now, you can see x explicitly appearing f dash d f dash by d x minus f double prime f d f by d x and at x equal to 0, sorry, at y eta equal to 0, you have f 0 equal to 0 that is, v equal to 0 as well as u equal to 0 and f x infinity equal to 1.

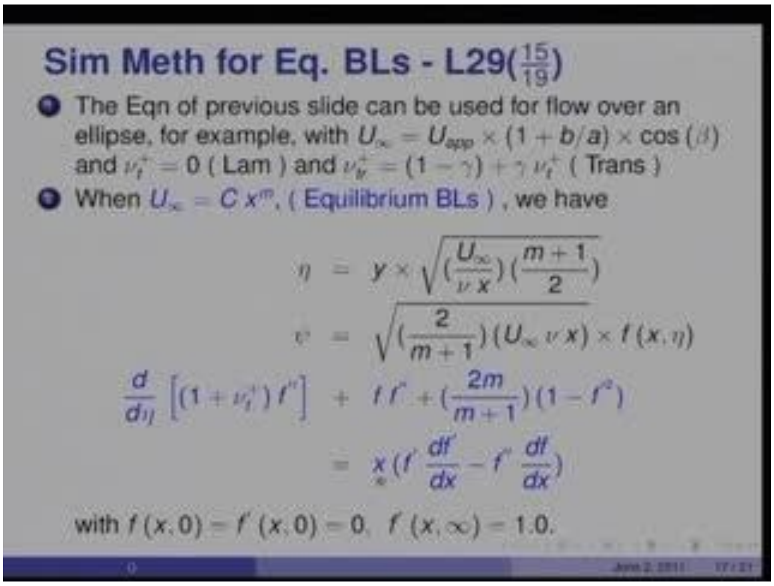
How do we solve such a mixed equation with the left hand side is a function of eta whereas, the right hand side is the function of both eta and x. Let us look how we solve such an equation, it is as follows.

(Refer Slide Time: 42:45)



Let us say, we have a surface then at x equal to 0 that is your starting point and U infinity can vary anyway with respect to x. Let us say, you have chosen the first value of delta x.

(Refer Slide Time: 43:18)

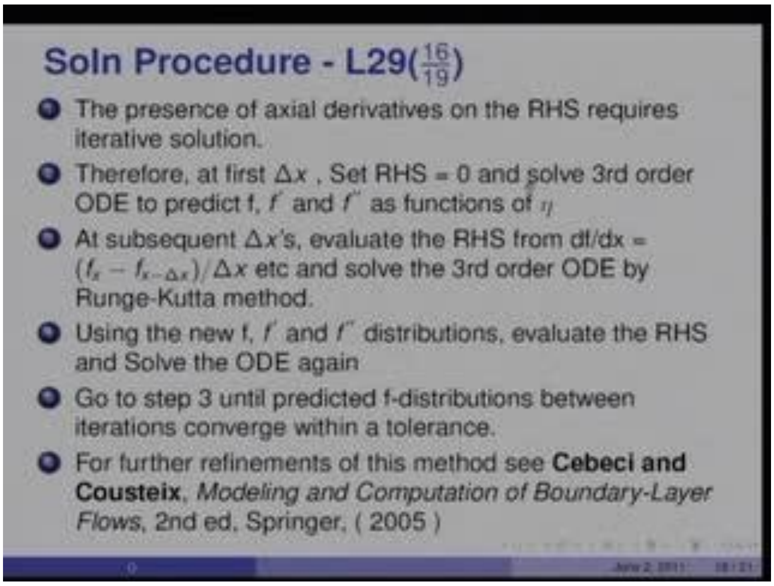


What you do is, you say at the first step which is very small you assume - as this is shown on the slide here - x is very small and therefore, you assume that this is 0 - the right hand side is 0 - in which case of course, this equations can be readily solved because the right hand side is 0 and it is a perfect ODE.

Therefore, you will generate values of f, f dash and f double prime at the end of the first step. Now, you go to the second step here, let us say these are values at the first step. In order to go to the next step, solve the left hand side you need the right hand side. The right hand side which is equal to x times f dash d f dash by d x minus f double prime d f by d x, we write that as x at 2 into f dash at 1 into f dash at 2 minus f dash at 1 divided by delta x minus f double prime at 1 into f at 2 minus f at 1 divided by delta x, so this is how we write (Refer Slide Time: 44:25).

In other words, the right hand side we will have values of f and f dash 2 at location 2 that is a second location. Therefore, since the left hand side is being solved also at location 2 you will have an implicit equation in values of f at location 2 and therefore, you need iterations.

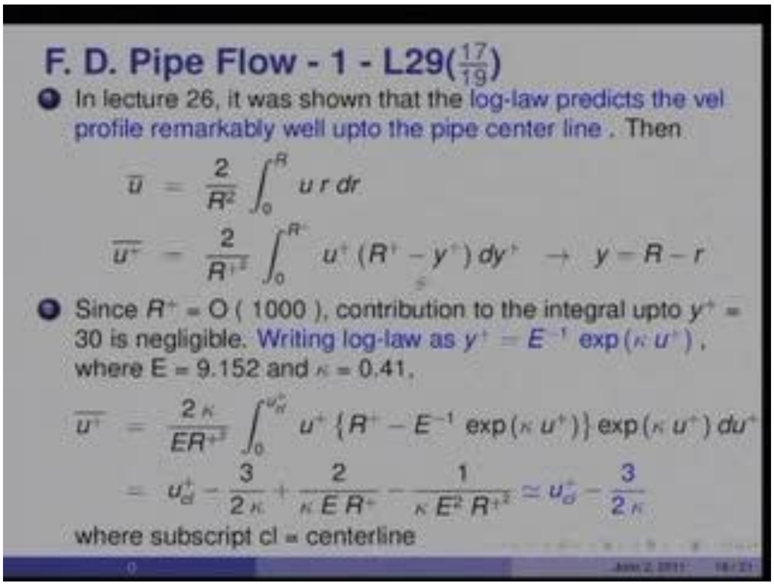
(Refer Slide Time: 45:30)



Once you do the iterations you can get the values, so the procedure as I said here, at the first step simply say right hand side is equal to 0. At subsequent steps the right hand side is evaluated from df by dx equal to f x minus f dash f x minus delta x over delta x and likewise for f dash itself. Solve the third order ODE by Runge-Kutta method and since this is an implicit in f you will get new values of f f dash and f double dash and evaluate the right hand side again and solve the ODE again.

If you find that at this iteration level the values of these quantities were same as the values at the previous iteration level; you say the conversions has been obtained and then you move to the next step.

(Refer Slide Time: 46:22)

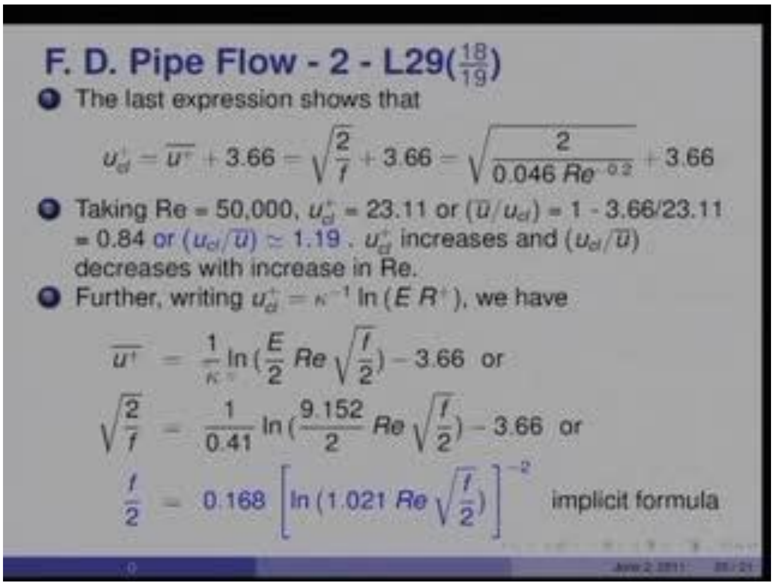


Now, there are many refinements in order to reduce these numbers of iterations so on and so forth; a recent book by Cebeci and Cousteix gives you the details of that. Now, we turn to the internal flow that is, flow in a pipe and we are going to make use of the log-law to assume that it is the velocity profile governed by the log-law then, we evaluate first the mean velocity u bar would be equal to 2 over R square 0 to R u r t r. If I change r to R minus y then, you will see that u bar plus would become that. Now I substitute u bar plus equal to log-law but that integration becomes difficult.

Therefore, you simply make y plus as a function of u plus E rise to minus 1 kappa u plus and simply ignore as before 0 to 30 region that is up to the transitional layer you simply ignore that. Assume this integration would not be affected if you use only the turbulent part of the law. You will see E would be equal to 9.152 kappa equal to 0.41.

Therefore, if I have to substitute that first, convert all dy plus to du plus and then integrate. Then, I will get that expression which is equal to u plus at the central line minus 3 by 2 kappa which would be 3.66 plus 2 over a large quantity, because R plus is of the order of 1000 or so, E is of the order of 9.15 and even bigger quantity here. Therefore, for all practical purposes I could drop both these terms and I would get this approximately equal to central line plus minus 3 by 2 kappa and the cl is the central line.

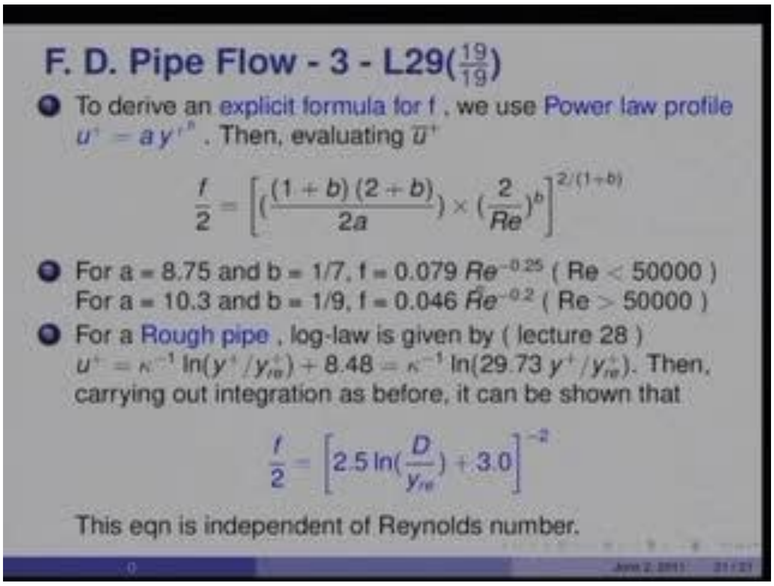
(Refer Slide Time: 48:11)



Then, what is that mean? You get central line plus equal to u bar plus plus 3.66; u bar plus as you know a would be equal to 2 by f plus 3.66. That shows you what the magnitude of U cl by U bar will be in a typical turbulent boundary layer about 1.19 at 50000.

Now, if I substitute in this U cl plus equal to kappa raise to minus 1 ln E R plus, we can organize u bar plus would be equal to that minus 3.66 or that would be equal to root 2 by f, which is u bar plus and the same quantity again. That transforms to this implicit formula for prediction of friction factor as a function of Reynolds number. Now, you have used this formula in your under graduate work, but the origin of that is in the logarithmic law near the wall. So that is very important to recognize that the logarithmic law has been recovered for the friction factor.

(Refer Slide Time: 49:17)

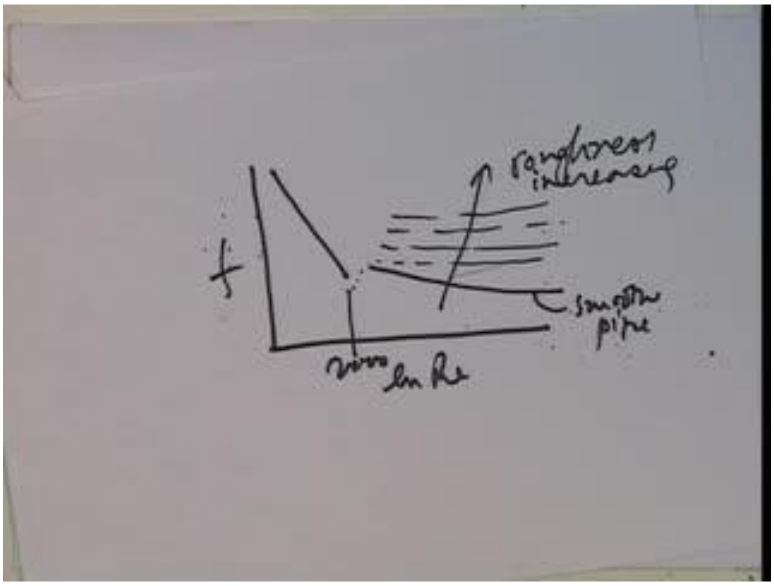


If you wanted to derive the explicit form of a friction factor relationship like for example, this or this, then you use the power law instead of the log law. Then, a again if you evaluate u bar plus then it will integrate like this, what I mean is u plus equal to a y plus raise to b (Refer Slide Time: 49:25).

Then, if I take a equal to 8.75 and b by 7, it predicts this expression which we routinely use for Reynolds number less than 50000. If I use a equal to 10.3 and b equal to 1 by 9 then, it will predict this expression which we routinely use for Reynolds number greater than 50000.

If it is a rough pipe again, we go back to the log-law and use this log-law which can also be written as kappa raise to minus 1 29.73 y plus divided by y re plus. Then, the integration will show that f by 2 is equal to 2.5 ln D by y re plus 3 raise to minus 2. This is a very interesting result that in rough pipes the integration of the logarithmic law of this type shows that the friction factor is no longer function of Reynolds number.

(Refer Slide Time: 50:41)



That is what you had learnt in your under graduate work when you studied Moody’s chart. That, this is the laminar friction factor f versus ln of Reynolds up to about 2000 and then, you have a transitional regime and you have a smooth pipe - this is the smooth pipe regime (Refer Slide Time: 50:57). When the pipes are rough you get friction factors varying like that where roughness increasing. These lines where almost horizontal meaning friction factor over no longer function of Reynolds number and that is something we have recovered in our derivation.

So, everything that you have used in your under graduate work for internal force has also been recovered from our universal laws of the wall. In the next lecture, I will use universal temperature law to predict Nusselt number.