Convective Heat and Mass Transfer

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Module No. # 01
Lecture No. # 03
Laws of Convection

I begin with the third lecture called laws of convection.

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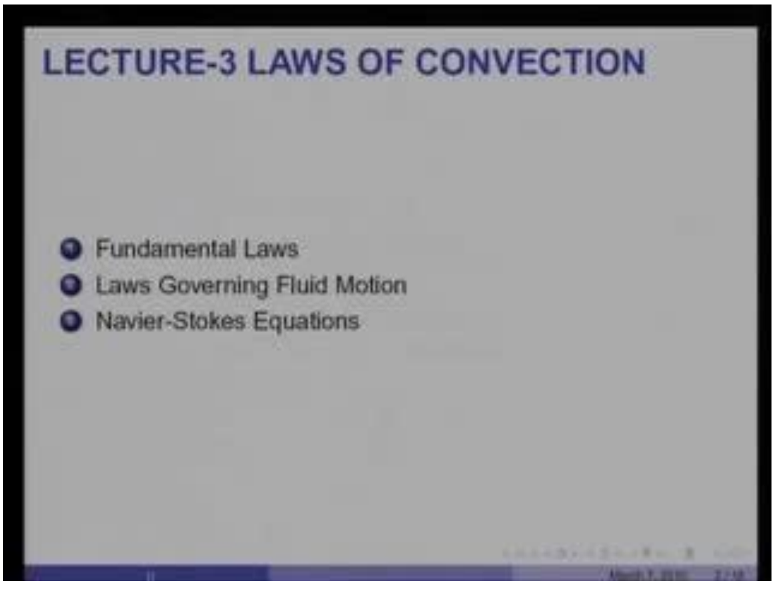

In a way, this is the most fundamental lecture along with the next one, because I will first be stating the fundamental laws that govern our subject of convective heat and mass transfer. Then I will go on essentially, to enunciate the laws governing fluid motion, which are also the tractable forms of governing equations, are also called Navier-Stokes equation.

I will end with complete derivation of the Navier-Stokes equations. In a way, this lecture is very fundamental because fluid motion is so very important in convective heat transfer.

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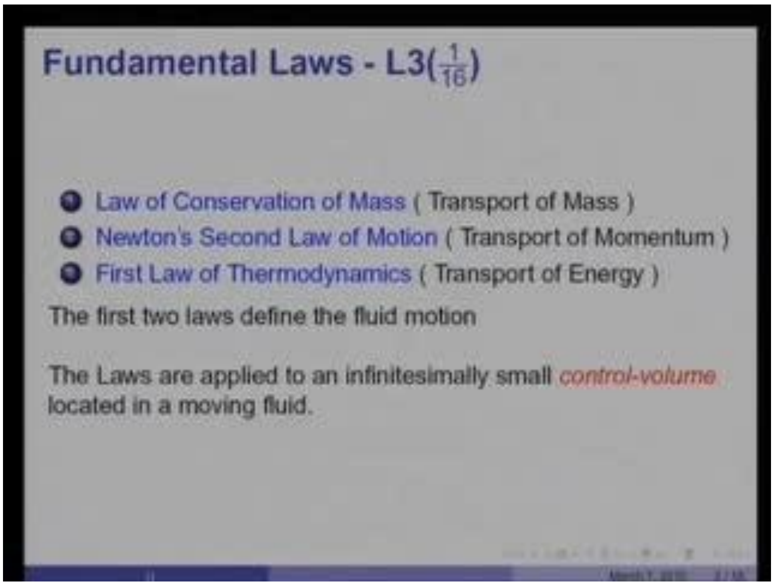

What are the fundamental laws? The first is the law of conservation of mass, which is responsible for transport of mass. Thus, Newton's second law of motion: force is equal to mass into acceleration, is essentially responsible for transport of momentum. The first law of thermodynamics is responsible for transport of energy. The first two laws, namely the law of conservation of mass and Newton's second law of motion, define fluid motion completely and the third one defines the transport of energy.

Today, I am going to consider only first two laws that define fluid motion completely by applying these laws to infinitesimally small control volume located in a moving fluid.

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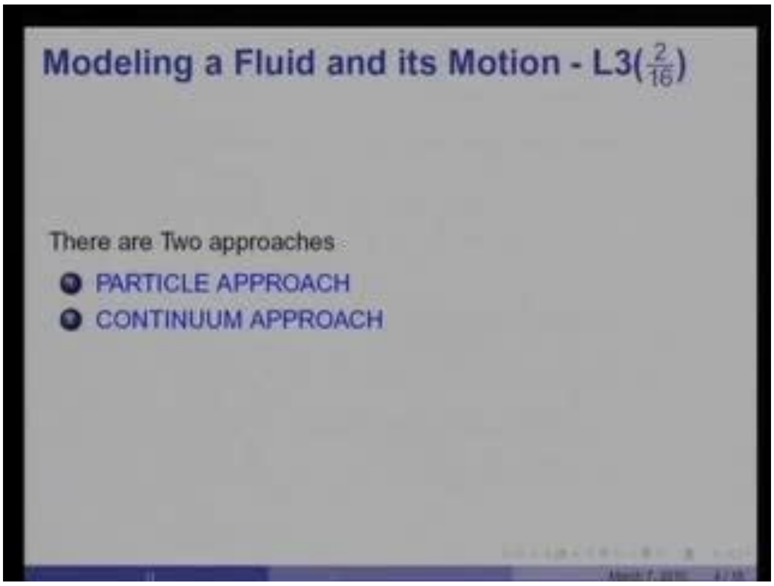

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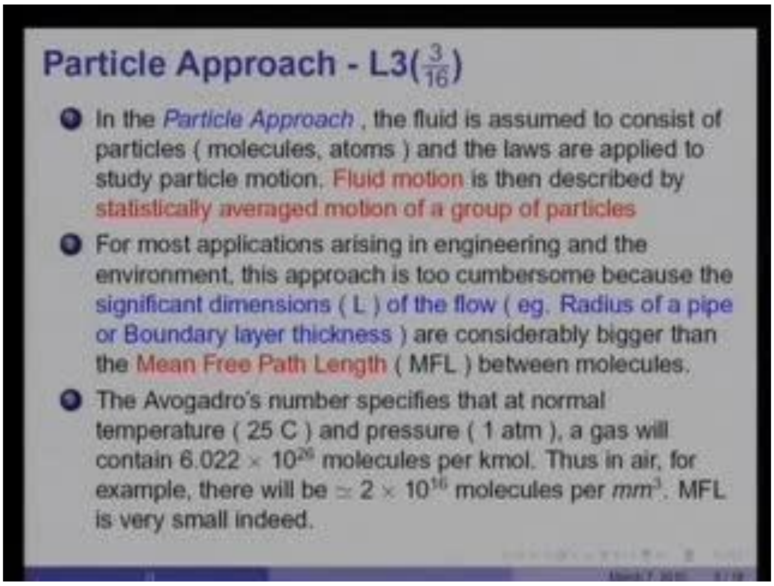

The first question that arises is that, how should we look at fluid and its motion? Though the fluid itself is viewed in two ways – the first one is called the particle approach, and the second one is called the continuum approach.

In the particle approach, the fluid is assumed to consist of particles – molecules, atoms, and the laws are applied to study motion of each particle. The fluid motion that we see and feel is then described by statistically averaged motion of a group of particles.

In this approach, the motion of each particle is studied, but the fluid motion that we speak of is described by statistically average motion of a group of particles.

For most engineering applications, and also environmental applications, this approach is too cumbersome. Simply because the significant dimensions of our flows, say, radius of a pipe, could be anywhere from, say, five millimeters to anything up to two meters diameters is considerable, or a boundary layer thickness.

So, these are the significant dimensions of the flow and these tend to be considerably bigger than the mean free path length between molecules. It means, in a given pipe, there will be simply billions and billions and trillions of particles to track even in a simple one centimeter diameter pipe.

To appreciate this, just consider something that you already know. For example, the Avogadro's number specifies that at normal temperature of twenty five degree centigrade and pressure of one atmosphere, a gas will contain six into ten is[raise to the power] to twenty six molecules per kilo mole.

For example, air, which has a molecular weight of say about twenty nine and a density of about one, it can easily be deduced that there will be two into ten raise to sixteen molecules per millimeter cube. You can very well imagine, therefore, that the mean free path length between molecules must be very small. There will be simply far too many molecules to track, even in a simple case of a flow in one centimeter diameter pipe.

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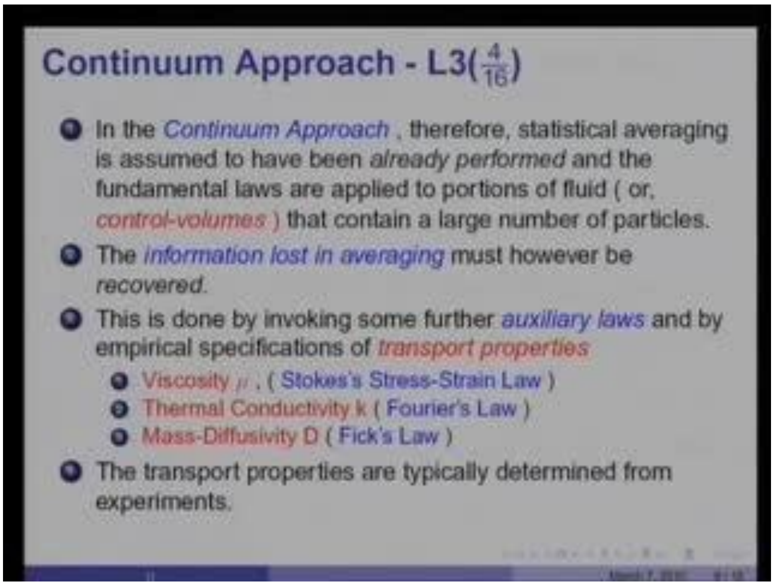

Continuum approach was devised essentially to overcome this difficulty. In this approach, we assume that the statistical averaging is already perform and we consider elements of fluids to which fundamental laws are applied. The elements of fluid are also sometimes called control volumes. These control volumes, although infinitesimally small, actually contain a large number of particles. They are simply very tiny or infinitesimally small. Since, we have already assumed that the fluid is viewed such that averaging has took place, obviously, some information is always lost in any averaging, and therefore, that information must be recovered. This information recovery is done by invoking some additional auxiliary laws, along with the fundamental laws. The three fundamental laws are normally invoked are – firstly, the Stokes's stress and rate of strain law, which defines the fluid viscosity mu; the Fourier's law of heat conduction, which is familiar to you, defines the thermal conductive k; and Fick's law of mass diffusion, which defines the mass diffusivity. If you recall my first lecture, I have said all these are molecular phenomena.

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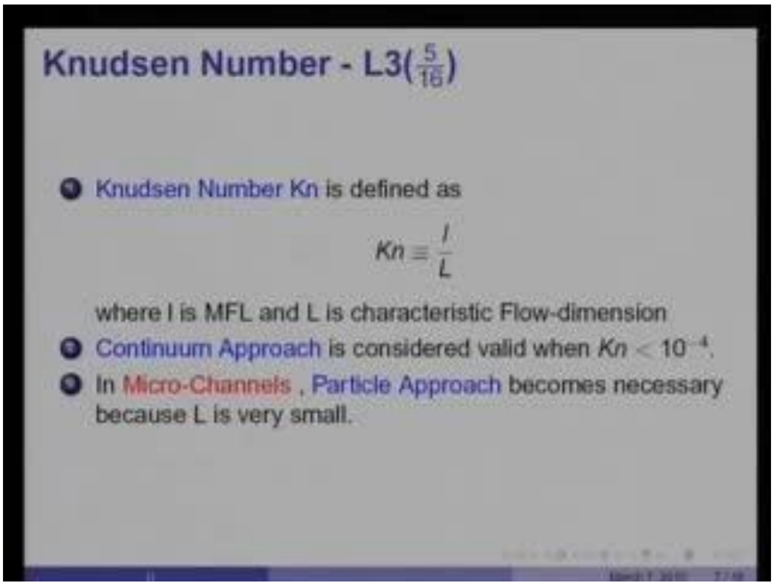

Essentially they are molecular, simply because they represent the last information as a result of averaging at the molecular levels. The transport property is mu k and diffusivity are typically determined from experiments. Although theory, such as kinetic theory of gases, etc., are available to be able to determine their magnitudes, as we move our applications involving very small dimensions, called microscale heat transfer or nanoscale heat transfer, a quantity call Knudsen number is very important one. Knudsen number is simply the ratio of mean free path length between molecules, divided by the characteristic flow dimension – l divided by L. We say that the continuum approach is valid and is also experimentally verified, when Knudsen number is less than 10^{-4}. Today, we have micro channels whose dimensions could be of the order of microns, or tenth or hundredth of microns, and clearly the dimension l itself would be comparable to the mean free path length L. Therefore, in such a case, the particle approach would become necessary.

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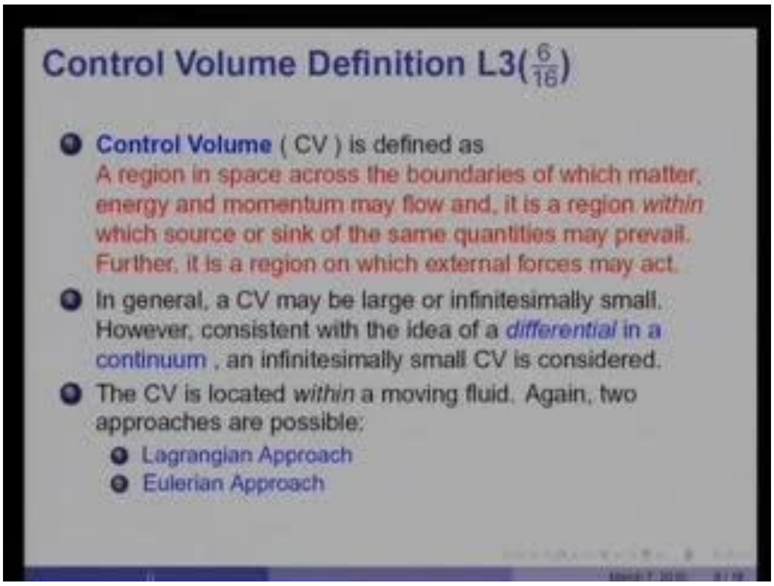

In this course, of course, we are always going to assume continuum approach, by saying that we would be considering situations in which L is much much greater than the mean free path length.

As I said in the continuum approach, the fundamental laws are applied to a fluid element, we call it control volume. Let us define the control volume. The control volume is a region in space across the boundaries of which matter energy and momentum may flow, and it is a region within which source or sink of the same quantities may prevail. Further, it is a region on which external forces may act. So imagine, a fluid inside a flowing flow, very tiny fluid element. Then, we are saying that in this fluid element energy will flow in and flow out, mass will flow in and flow out, momentum will flow in and flow out, and it is an element on which external forces may act.

In general, a CV can be large or infinitesimally small; however, consistent with the idea of a differential in a continuum, an infinitesimally small CV is considered. Remember, the idea of continuum is also invoked in mathematics.

In fact, when we define a derivative, we say, difference in the values of a variable at two distinct points is written as d y by d x, when d x goes to zero. In other words, when d x becomes very small or the two points are brought very close to each other, a derivative is defined.

The notion here is very similar – we want to represent our laws in terms of differential equations. So, in order to be consistent with the idea of a differential in a continuum, we we shall consider infinitesimally small control volume.

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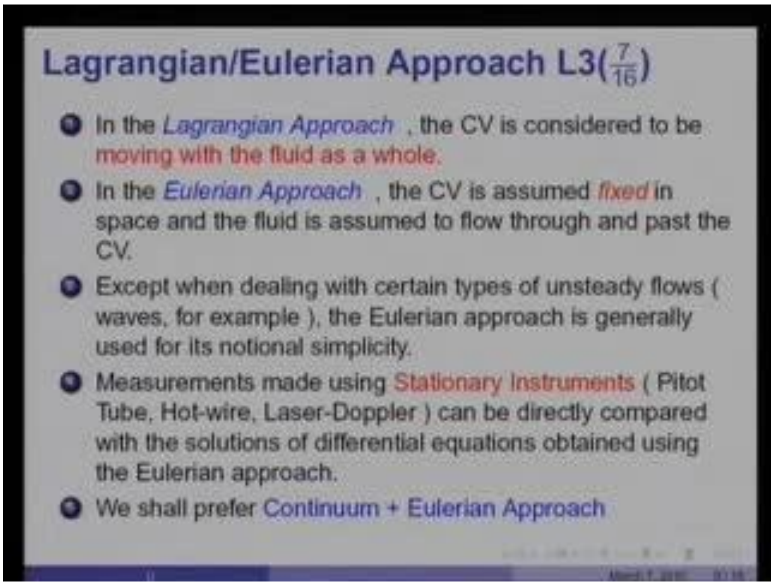

The CV is located within the moving fluid. Again, there are two approaches – first of them is called the Lagrangian approach and the second one is called the Eulerian approach. In the Lagrangian approach, the CV is consider to be moving with the fluid as a whole. In the Eulerian approach on the other hand, if I have a moving fluid, the element that I consider is fixed in space.

In a way, the material of the element is changing continuously because some fluid is coming in and going out, and fluid element contains different materials at different times. But the location of the element is fixed in space.

In the Lagrangian approach, on the other hand, the material of the element does not change but the element itself moves with the fluid. In the Eulerian approach, the CV is assume to be fixed in space and the fluid is assumed to flow through and past the CV. When is Lagrangian approach invoked? It is invoked in only such study of certain types of unsteady flows, such as free surface waves, for example. But this is not something of interest to us, and therefore, we shall be preferring Eulerian approach to application of fundamental laws to control volumes.

There is also one more advantage when we consider Eulerian approach. As I have said, on the Eulerian approach, the fluid element is fixed in space, although located inside the moving fluid, it is fixed in space.

Now recall, that whenever we do experimental measurements using a pitot tube or hot wire for velocity or laser doppler anemometer, and so on so forth, they are all fixed instruments in space, and the fluid moves past them.

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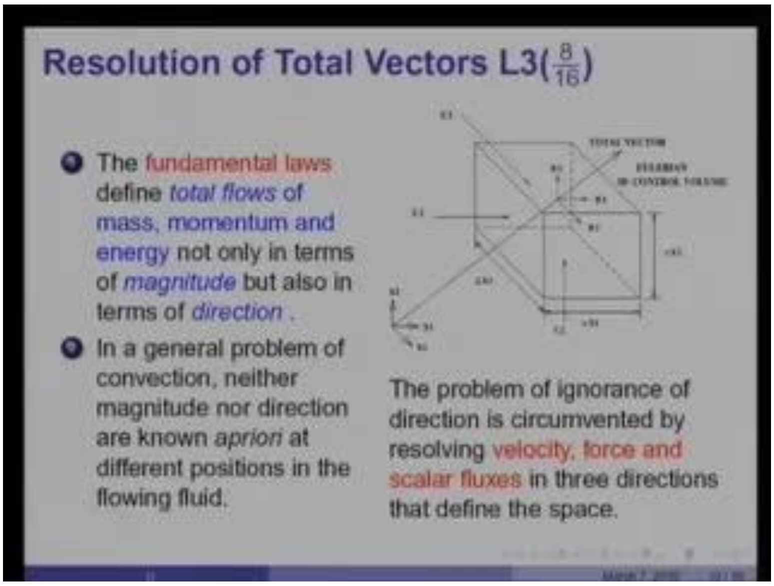

As a result, there is a great advantage in the sense that what is measured by the instruments can now be compared directly with the solution of differential equations that the Eulerian approach defines. As a result, we shall always preferred, throughout this course, to look at the fluid as a continuum and apply fundamental laws to the control volume within the Eulerian approach.

Now, there is still one more matter to be settled. The fundamental laws by themselves are pretty useless, in many ways. Firstly, the fundamental laws apply only to total flows of mass momentum and energy. They define total flows, both in magnitude as well as in direction. So there are two properties involved – magnitude of the total flow and the direction of the total flow. To the extend direction is involved, all these flows are vectors like this, say total vector.

Now, the trouble is that when the fluid flows through a duct or flows over a turbine blade or something like that, at every point, the flow, I simply do not know the direction of the total vector a priory. I am sure you are able to imagine this. In the entrance region of a duct, where the boundary layers grow close to the wall, the direction of the flow will be pointing towards the axis of the tube, inside the boundary layer. But as I move away from the wall, it will be more or less aligned and parallel to the axis of flow. In fact, at the access symmetry, the flow will be absolutely parallel to access symmetry.

But I simply do not know a priory the direction that the total vector will make with the fixed coordinate system x y z, nor do I know the magnitude. So what is done then? In the general problem of convection, since we do not know magnitude and direction a priory, we settle the matter in this way. We say the problem of ignorance of direction is circumvented by resolving the total vector in three directions – x one, x two, and x three, as I have shown here. The total velocity vector u would be resolved in terms of velocity vector u one in the direction x one; u two in the direction x two; and u three in the direction x three. Same thing would apply to forces. Same thing would apply to all other fluxes like heat flux and mass flux. We shall always resolve all the vector quantities in three directions that define the space. Of course, we have increased our work but at least we have made the problem tractable in the sense that now we need to be worrying only about how to determine their magnitudes in three direction, so that the total vector could always be constructed, knowing the three vector, sub vectors in three different directions.

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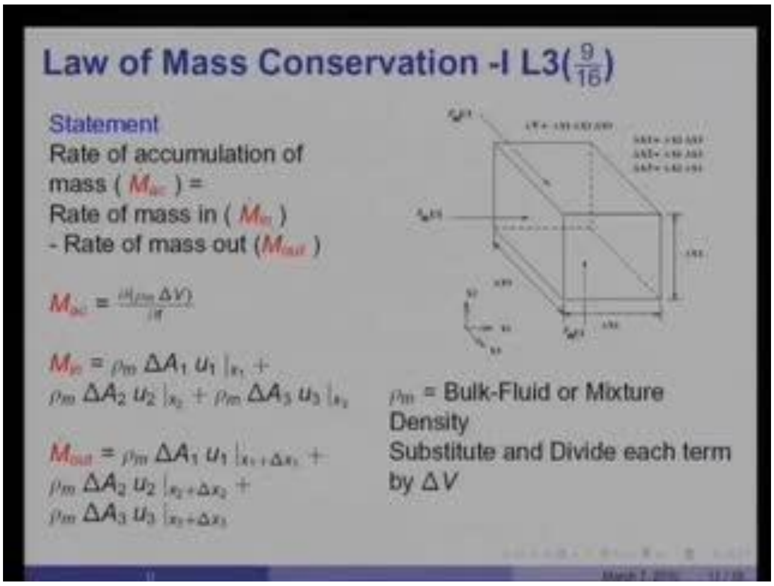

Let us consider the very first law of conservation of mass and here is a control volume. The verbal statement of the law is very simple. It says the rate of accumulation of mass within a control volume would equal the rate of mass in minus rate of mass out. Very simple to understand. There is no difficulty at all.

How do we represent, mathematically, the accumulation of mass – the rate of accumulation of mass – m dot a c? Remember the mass of the element would be simply rho m multiplied by its volume. Rho m is the bulk fluid or mixture density multiplied by the volume, which is delta x one delta x two delta x three, divided by d t. So, partial d by d t rho m delta v would be the rate of accumulation, well, at surface x equal to x one, the mass rate of mass flow in will be rho m into u one, which is the mass flux multiplied by the area delta a one, which is delta x two multiplied by delta x three. So, that is what I have written here. That could be the rate of mass in at surface x equal to x one.

Likewise, there will be rho m u two delta a two, which is the mass coming in from this side, and there will be rho m u three delta a three, which is the mass coming in from the back side at x three equal to constant surface.

M dot out would likewise be rho m u one delta a one going out at x one plus delta x one surface, rho m u two delta a two at x two plus delta x two would be the mass going out of the x two plus delta x two surface, and likewise mass will come out at the front surface in the x two direction.

Incidentally, notice that the directions of x one x two x three satisfy the right-hand screw rule, which means if I start with x one and move towards x two, a right-hand screw would make take me forward in x three direction.

So that is what is implied here. If I turn x one to x two, I will move in x three direction. This is very important to remember in all our subsequent slides that the coordinate directions are so chosen that they obey the right-hand screw rule in cyclic manner.

So, having replace the verbal statement by in its mathematical form, we divide each term here by volume delta v, which is nothing but delta x one delta x two and delta x three. Remember that delta a one is delta x two delta x three delta a two is equal to delta x one delta x three, and delta a three is delta x two delta one.

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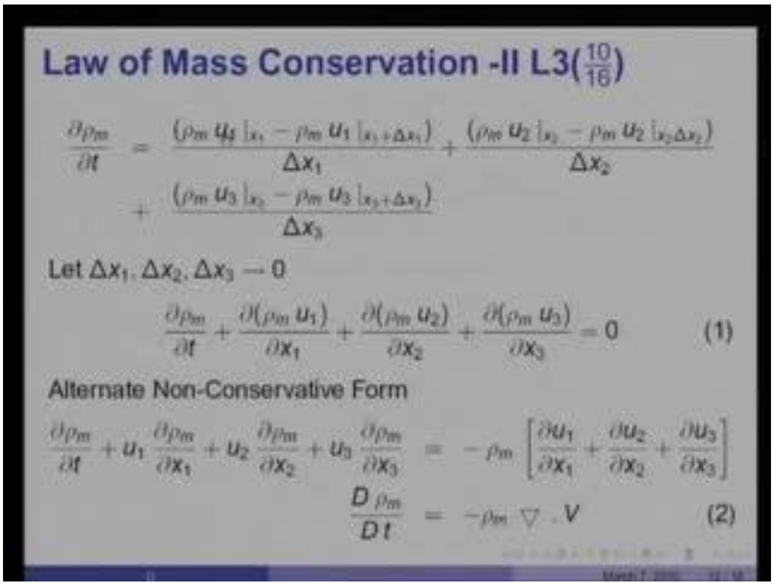

The the law of conversation of mass would read something like this – rho m u one x one, which is the mass flux in, minus rho m u one x one x mass flux out from in the x one direction divided by delta x one. Similarly, in y direction, and similarly in z direction. If I now let each of these delta x one delta x two delta x three go to zero, that means making the control volume extremely small, and infinitesimally small. Then, each of these expressions would simply get converted to a partial derivative. This expression for example, would be minus d rho m u one by d x one, minus d rho m u two by d x two, minus d rho m u three by d x three, which I have transformed on the left-hand side to read as follows. So essentially then d rho m by d t equals plus d rho m by u one d x one plus d rho m u two by d x two plus d rho m by and u three d x three equal to zero, is the statement of the law of conservation of mass in differential form.

It will also call a conservative form of equation, simply because this is how it is derived, and it has conserved all the fluxes in all directions. Non-conservative form can be derived by mathematical manipulation. For example, I can treat this as differentiation of a product, then you will see this will become u one d rho m by d x one and u one d rho m by d x one, which I have written – sorry u one rho m d u one by d x one which is written here on the right hand side.

So, you will see, I will get d rho m by d t plus u one d rho m by d x one plus u two d rho m by d x two plus u three d rho m by d x three equal to minus rho m into d u one by d x one d u two by d x two d u two by d x three. This way of writing is called the non-conservative form of writing the rho of conservation of mass.

You will readily recognize that this left-hand side is nothing but what we call the total derivative – d rho m by d t equal to minus rho m. What is this? This is simply divergence of velocity vector v, and therefore, written as del dot v.

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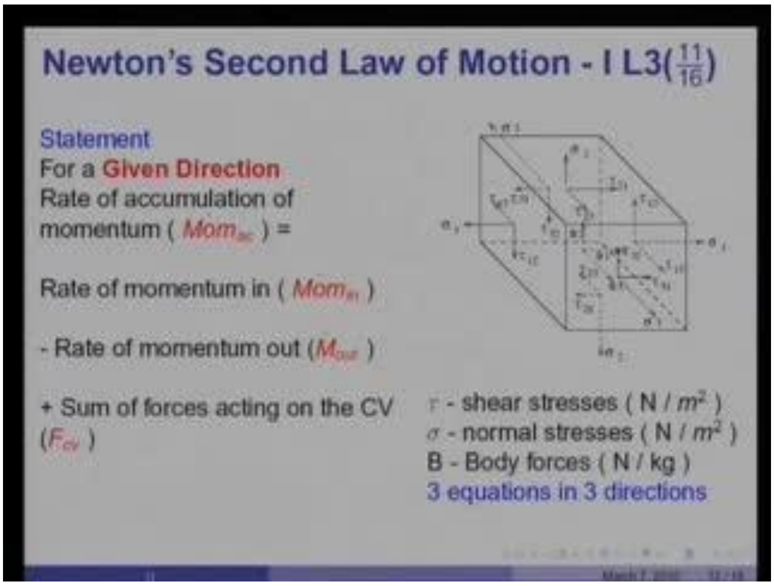

So, d rho m by d t equal to minus rho m del dot v is a non-conservative form of the law of conservation of mass.

I now turn to the second law of motion, which as I said is concerned with the transport of momentum. The statement goes something like this – for a given direction x one x two or x three, the rate of accumulation of momentum equals rate of momentum in minus rate of momentum out, plus some of the forces acting on the c v, in the same direction.

Remember, Newton's second law of motion says that the force equal to mass into acceleration, but implies that the force and acceleration are in the same direction. What are the forces that act on a control volume? Firstly, there are shearing stresses – shearing forces – tau two one is a shearing force acting in x one direction; and likewise tau two one would be at x two equal to constant surface, would act in this direction; tau three one likewise acts at x three plus delta x three surface; and tau three one also acts in the other direction.

The difference in these stresses, provide the shear of the control volume. Sigma is the normal stress, it is tensile, and therefore, points outwards from all surfaces sigma one here, sigma two here, and sigma three in the back; and likewise sigma one here, sigma three here and sigma two there at the other surfaces.

In addition, there could well be body forces due to buoyancy or coriolis force or a centrifugal force or an electromagnetic force. A fluid can experience variety of forces, body forces in particular.

These are all the forces but the important thing is we can consider the Newton's law of motion in one direction at a time. Therefore, since there are three direction, we shall have three equations, as I have mentioned here.

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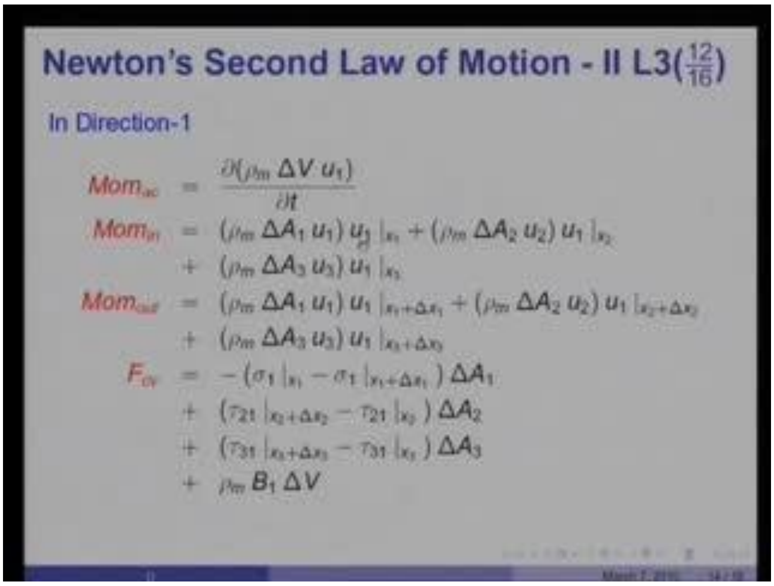

Let us write each term in mathematical form momentum accumulation. Well simply, the mass of the control volume is rho m into delta v. And since I am considering direction one, I must multiply that by u one to get mass into velocity is momentum, and the rate of change of that is the accumulation, so d by d t of rho m delta v u one is the accumulation of momentum.

What about momentum in? Rate of momentum in would be rho m u one into delta a one is the mass coming in from surface x one x two equal to constant. That must be multiplied by velocity u one to get momentum in x direction.

There is also mass coming in from x two equal to constant surface. That also must be multiplied by velocity u one to get momentum in direction one. Likewise, there is also mass is coming from the back at x three equal to constant surface, which must also contribute to momentum in direction one.

I have three terms – mass coming in at surface x one multiplied by u one, mass coming in at surface x two multiplied by u one, mass coming in at surface x three multiplied by u one. Going out fluxes would be – mass coming in at u one multiplied by u one at x one plus d x one, similarly, x two plus d x two and x three

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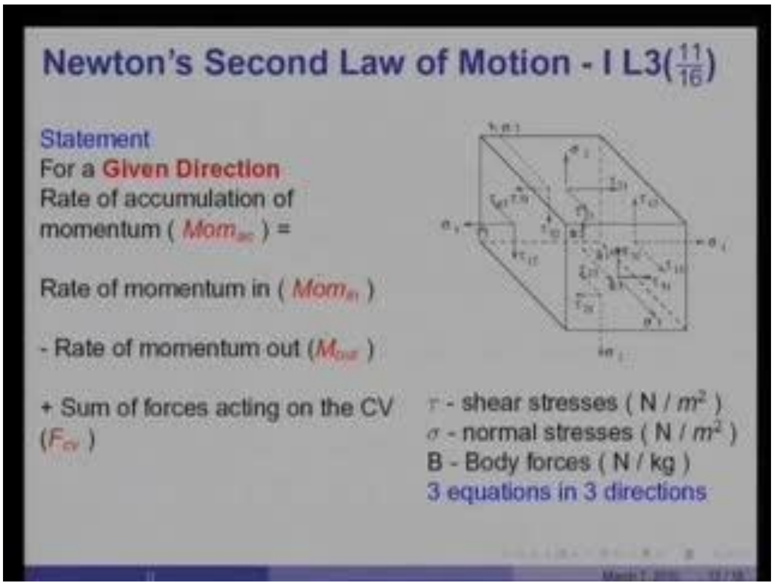

These are the way mass momentum in and momentum out terms. What about the forces? Remember, as I said sigmas are tensile forces, sigma one acts on the area delta a one, sigma two acts on the area delta a two, and sigma three acts on the delta a three.

The first term is indirection x one, positive direction. I shall have minus sigma one at x one minus sigma x one at delta x one multiplied by delta a one. This would be the force in the positive direction one.

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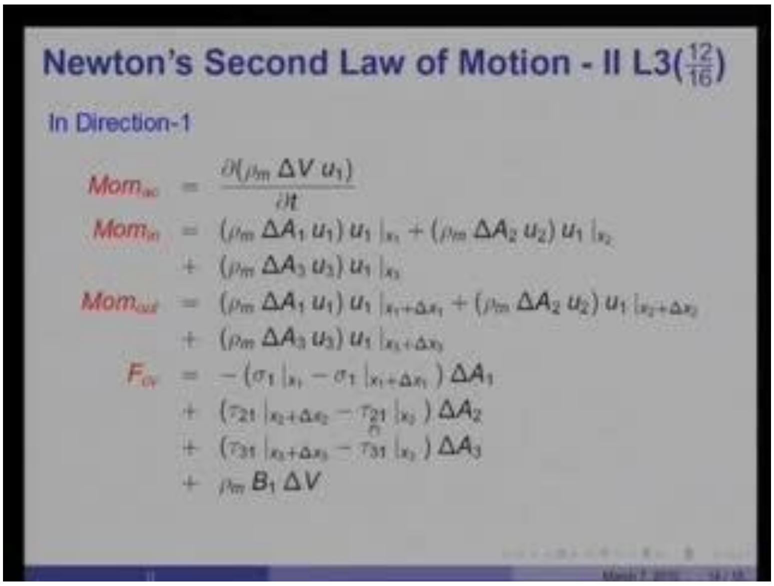

Likewise, tau two one x two plus delta x two surface is acting in positive direction, whereas, the tau two one at x two surface is acting in the negative direction. Therefore, I have tau two one x two plus delta x two minus tau two one x at x two multiplied by delta a two. Likewise, in the surfaces in the z direction, tau three one at x three plus delta x three minus tau three one at x three multiplied by delta x three. Plus, if there is a body force b one in extend direction, which acts on the control volume as the whole, is written as rho m into delta v, which is the mass of the control volume multiplied by b one, meaning thereby that b one has units of Newtons per kilogram. And therefore, it has been multiplied by rho m into delta v.

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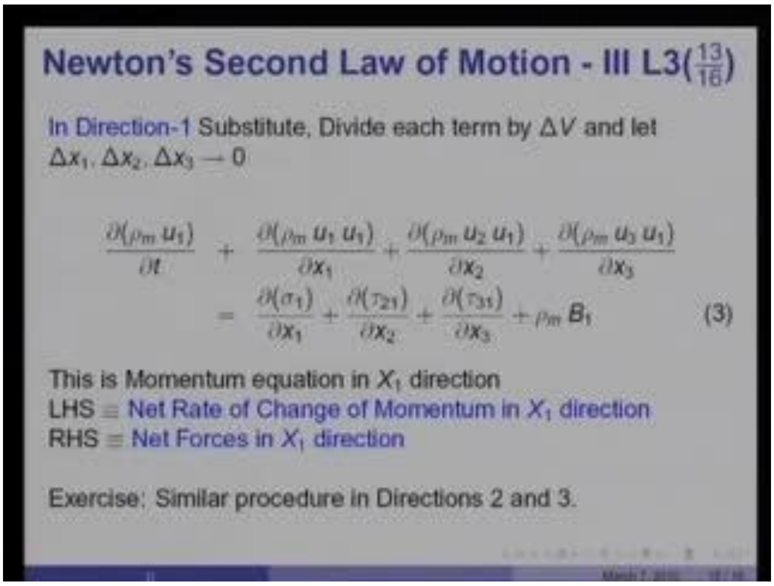

Stresses on the other hand have units of Newtons per meter square and therefore, I have multiplied by area a one a two. So, in other words, the units of each term here is simply Newtons the force.

If you substitute these mathematical terms into the statement of the Newton's second law, and we divide each term by volume delta v, and let delta x one delta x two delta x three go to zero, then it is not very difficult to show that d rho m u one by d t plus d rho m u one u one by d x one plus d rho m u two u one by d x two plus d rho m u three u one by d x three, would simply equal d sigma one by d x one plus tau two one by d x two plus d tau t three one by d x three rho m b one.

This would be the momentum equation in direction x one. The left-hand side, as you will readily appreciate, is the net rate of change of momentum in x one direction. The right-hand side is the net forces in x one direction by fluid stresses and body forces.

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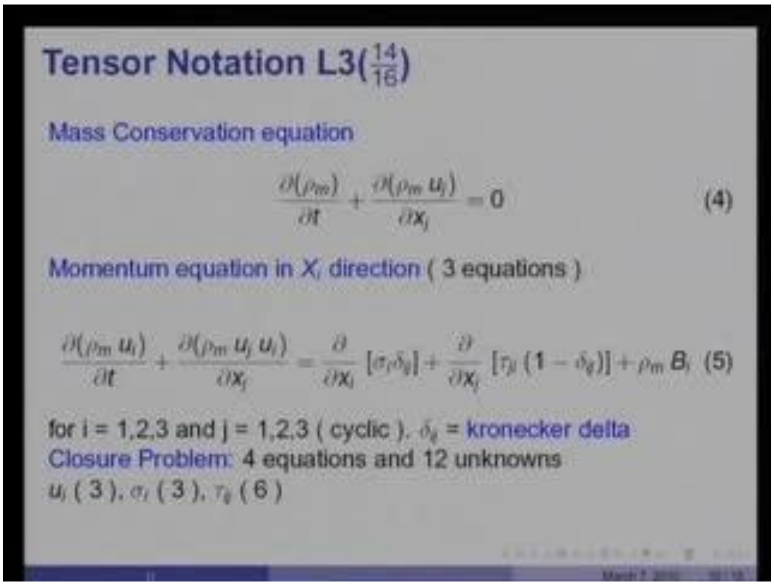

Now, clearly an exercise similar to this can be carried out in direction two and direction three. I leave that as an exercise for you. It is not very difficult at all. You will notice, if you write down three equations together, then they can be represented in a tensor by tensor notation.

For example, law of conservation of mass can be written as d rho m by d t d rho m u j by d x j equal to zero, where j goes from one to three in cyclic order. Momentum equations in direction x i can be written as d rho m u i by d t plus d by d x j of rho m u j u i plus d by d x i of sigma i delta i j, where delta i j is the chronicle delta. It is equal to zero, when i is not equal to j, and equals one, when i is equal to j. These terms will survive only when i is equal to j. d by d x j tau j i one minus delta j i, these terms will survive only when i is not equal to j plus rho m b i and in tensor notation. We write it in this fashion for i equal to one to three and j equal to one to three cyclic. But now, we have a little closure problems, as you can see. These represent three equations, this represents one. So, the fluid motion has been described by four equations, but we have many more unknowns. First of all, we have three velocity components, which are not known. In fact, that is what we wish to determine. Sigma three tensile stresses, which we do not know, and six shear stresses tau i j or tau j i. So, we have essentially six plus three, nine, and plus three, twelve. Twelve unknowns and only four equations, so essentially this is a not a solvable set. In the early days, say around 1960, Euler simply assumed that the tensile stresses and tau j i, the shear stresses would be extremely small and simply ignore them.

That was the situation is 1960. But around eighteen twenty five, a man called Navier found that no, those terms cannot be neglected. This fluid is indeed stressed, as fluid flows through the tube, when it flows, and therefore, fluid stresses could be as big as the body forces that it experiences, and therefore, decided to retain them, which gave rise to these nine unknown stresses.

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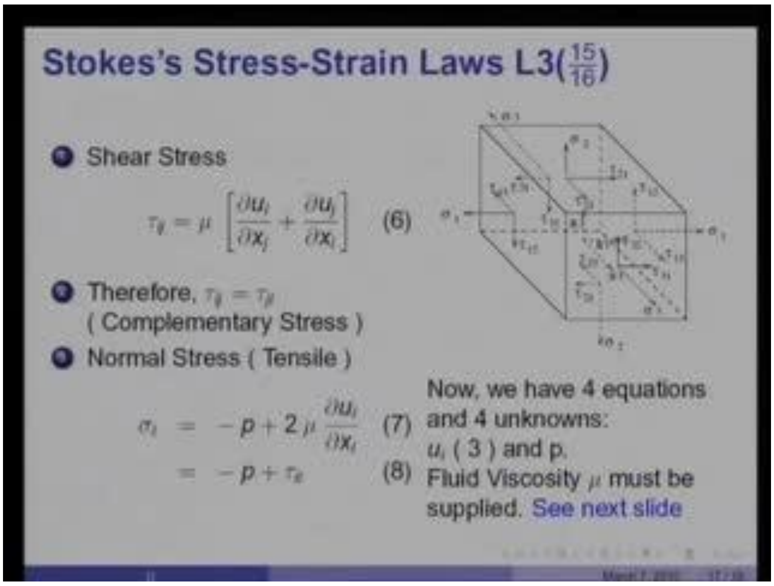

The way forward was found by Stokes, in England. He said that the shear stresses would be related to velocity gradients or the rates of strain, through viscosity, tau i j is equal to mu equal to d u i d x j plus d u j d x i. Remember, this is simply a definition or a model of a stress, related to rate of strain, introducing an entirely new quantity, mu, into our set of equations.

From at the form of the stress strain law, you can readily appreciate that tau i j will indeed be equal to tau j i, because when you change the indices, the expression does not change, when i is not equal to j. And this is precisely what we call complementary stresses.

The tau two one is a stress in that direction, and the stress complimentary to that is tau one two, acting on another face. Likewise, tau two one at x two has a complimentary stress tau one two which acts on x one surface.

The six stresses, which were unknown are now reduced to three unknowns, due to complementarity, and they are now reduced to the velocity components, if you know the value of viscosity.

Normal stresses, which are tensile, are written as minus p plus two mu d u d x i, minus p, because pressure is always compressive, and using Stokes notation, you will see two mu d u i d x i is nothing but tau i i, where i is equal to j.

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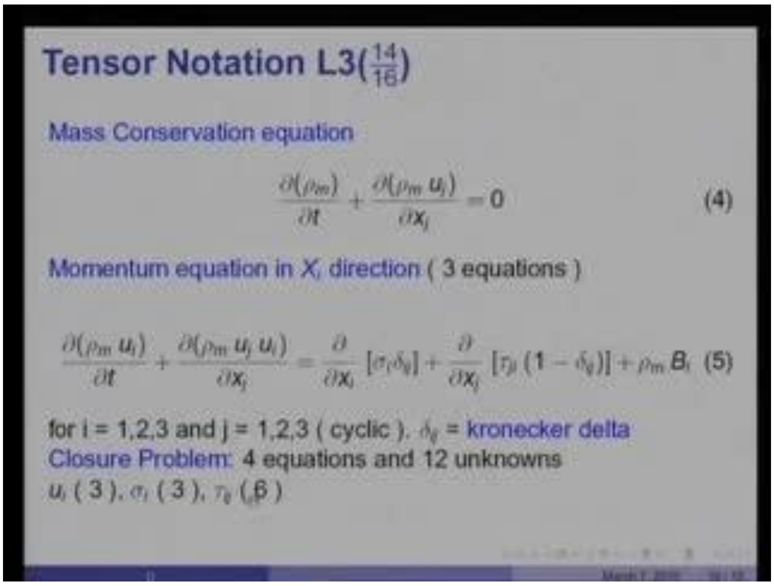

We had four equations and twelve unknowns, but sigma i and tau i j are now replaced by velocity gradient and pressure. Now, we have four equations and four unknowns, three velocity components u i and pressure p, and also one more additional constant called fluid viscosity mu.

Now, this viscosity was simply a constant of proportionality between stress and strain. It is our great fortune that viscosity has turned out to be the property of a fluid, rather than the flow.

Imagine, if I had the stress and strain, which were connected in a such way that viscosity of water, when it flows in a circular tube, is different from when it flows in a square section tube, we would have much bigger problem on our hand. We are very lucky it is an accident of history, if you like, or accident of nature, if you like, that viscosity has turned out to be a property of the fluid.

And not surprisingly, because it is essentially trying to capture the information lost during statistical averaging of molecular motions.

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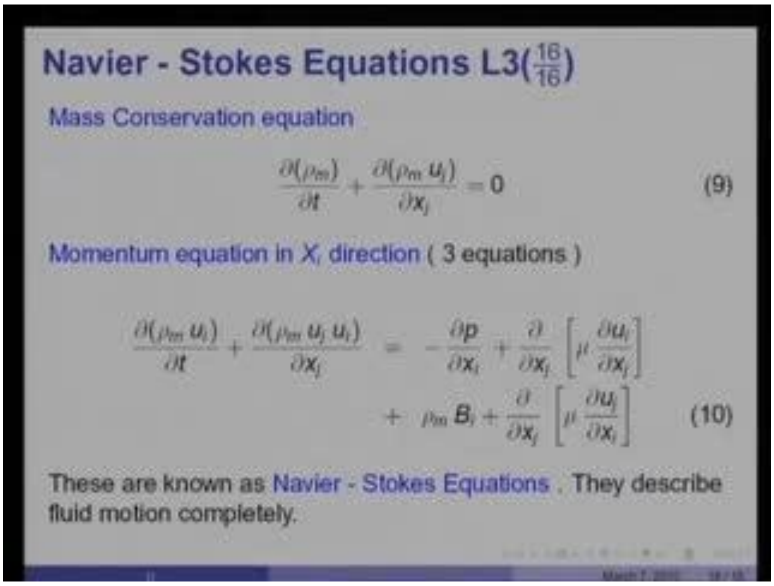

So, we must supply to the equations now, the value of viscosity. As you will see, our equation will now read – d rho m by d t plus d rho m u j by d x j. And the momentum three momentum equations would read in this fashion – d rho m by d y d t d rho m u j u i d x j equal to d p d x i plus d by d x j mu d u i d x j plus rho m b i, which are the body forces and this is the remaining part of the stress stoke stress.

These equations written in this form are known as Navier Stokes equations. Navier was a french scientist engineer and Stokes was the English scientist engineer. Both are credited with formulating these set of equations, whereby including the stress terms, which where ignored by Euler in 1760.

When of course, these tensors are ignore essentially we are saying viscosity is assume to be zero. And therefore, when these terms are zero, we say the momentum equations applied to inviscid fluid; inviscid meaning fluid - having zero viscosity or an ideal fluid. Such an ideal fluid can explain quite a few things in fluid mechanics but not others. Principally, it cannot explain the drag offered by a body when fluid flows past it.

And as I said, this drag is a paramount importance to a convective heat transfer engineers because he must design its surfaces such that the drag is reduced or the pressure drop cause by the drag is reduced. And therefore, these terms are very important to a convective heat and mass transfer analyst.

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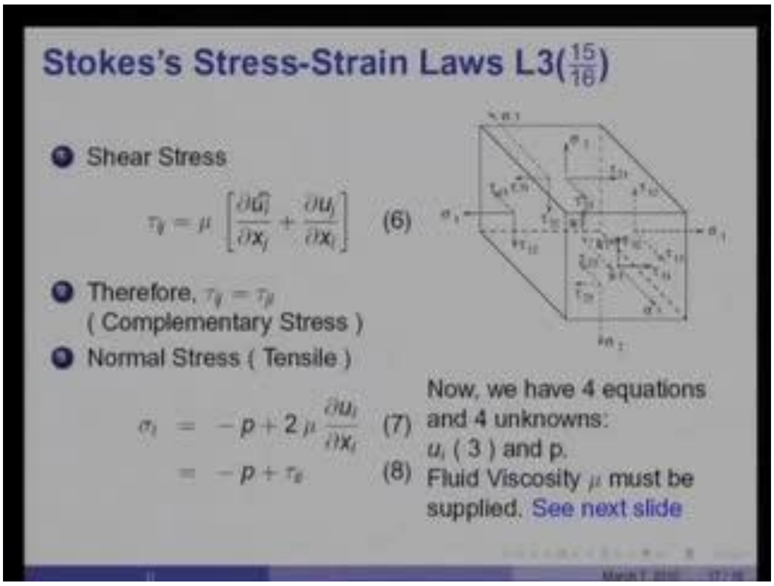

These equations then describe the fluid motion completely. Incidentally, I may mention that when mu happens to be an absolute constant, which is only a property of the fluid, then we say the fluid is Newtonian because stress and rate of strain are then linearly related. But there are fluids like blood or polymers, and so on so forth, in which the viscosity or the magnitude of viscosity itself depends on the rate of strain and sometimes in the manner in which the fluid was strained through time. Therefore, viscosity also happens to be function of time, function of the flow, in which the fluid is situated, and so on so forth.

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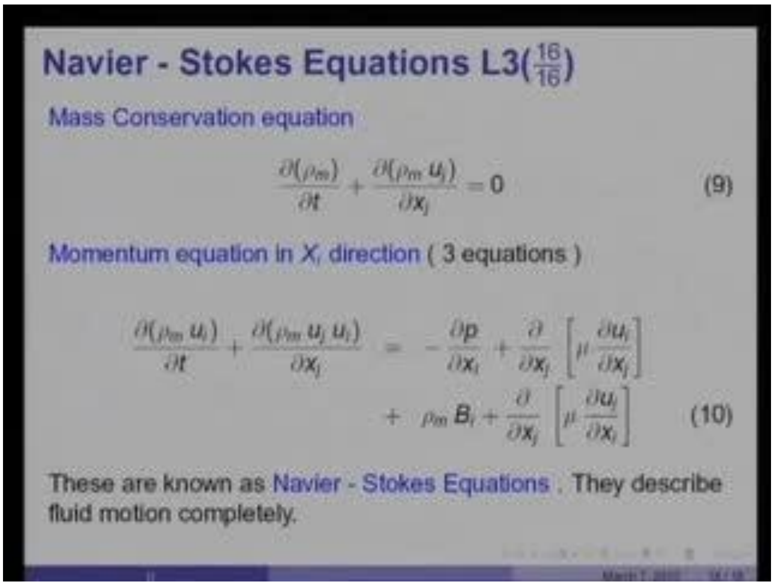

So, my remarks about viscosity apply only to Newtonian fluids, such as air and water at pressures, that we are interested pressures and temperatures in which we are interested in mechanical engineering.

We are leaving out exceptions over applications like blood flows and other things, where the flows are non-Newtonian. In the next class, I will take up the fundamental law of energy, which is the first law of thermodynamics.