Convective Heat and Mass Transfer

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Module No. # 01

Lecture No. # 26

Turbulence Models- 1

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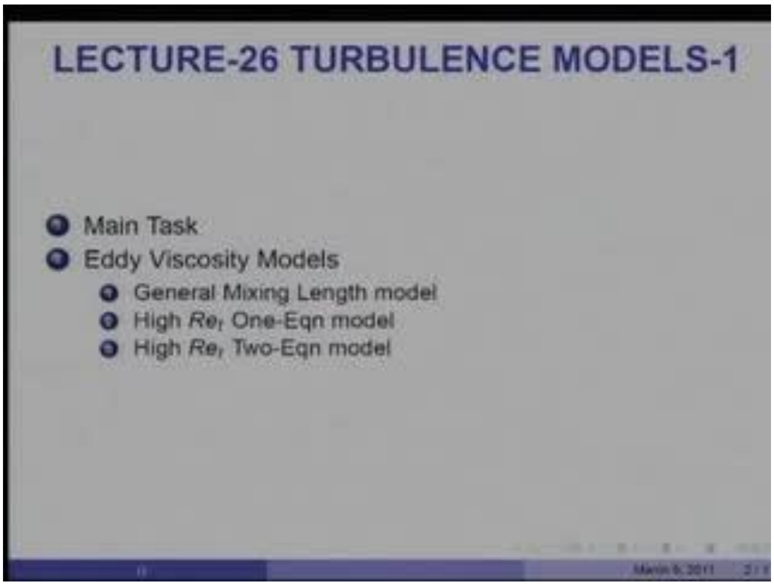

In the previous lecture, we appreciated that in a turbulent layer, the inner layer can be nearly universal at least when pressure gradients are moderate, but the outer layer universality is very difficult to establish and it is here that we principally need the turbulence models and that is the topic of discussion today.

So, I will explain what the main task of modeling. I will take up the so called Eddy Viscosity Models. In that, there are three variants: one is called the general mixing length model, the second one is high Reynolds number one-equation model and high Reynolds number two-equation model.

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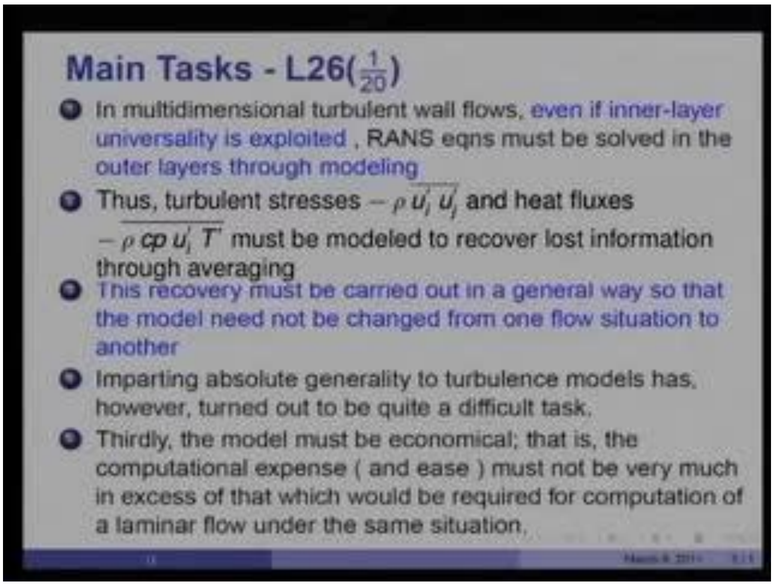

You will understand the meanings of this as we go along. So, in multidimensional turbulent flows, even if inner layer universality is exploited, RANS equations must be solved in the outer layers through modeling and that means the turbulent stresses u i prime u j prime and heat fluxes rho cp u prime i prime T prime must be modeled to recover lost information through averaging.

Remember these terms arise because of Reynolds averaging. This recovery, however, must be carried out in a general way, so that the model need not be changed from one flow situation to another.

Now, it is this imparting absolute generality to a turbulence model has, however, been found to be quite difficult task, but none the less we will see, as we go along what progress has been made. Thirdly, the model must be economical; that is, the computational expense or the ease must not be very much in excess of that which would be required for computation of say a laminar flow under the same situation.

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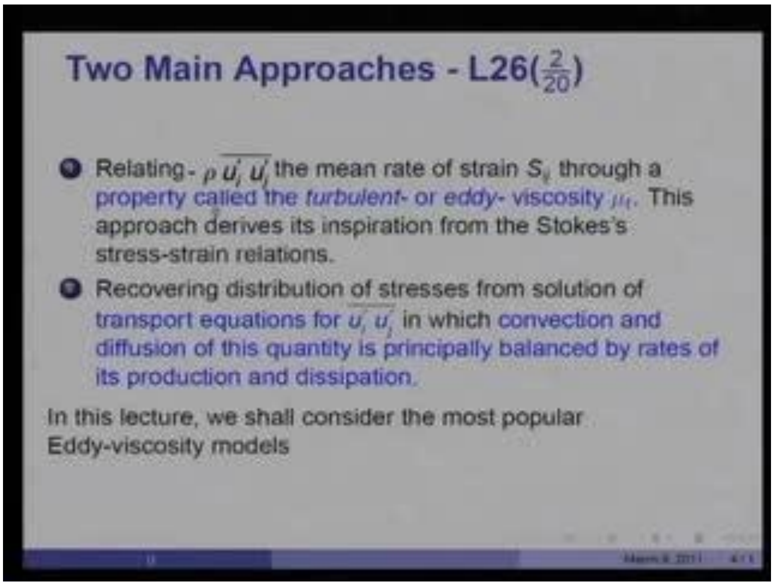

So, with this premises we begin to look at, what are the possible turbulence models. There are two main approaches to modeling a stress: one is to say that the stress is related to the mean strain rate through a property called turbulent or eddy viscosity; this is very analogous to stokes stress - strain relationships.

Alternatively, the stresses can be recovered from solution of transport equations for the quantity u_i prime u_j prime in which, the convection and diffusion of this quantity is principally balanced by the rates of its production and dissipation.

In this lecture, I am going to concentrate on the Eddy viscosity turbulence models, the first type; the second one we will take up in the next lecture.

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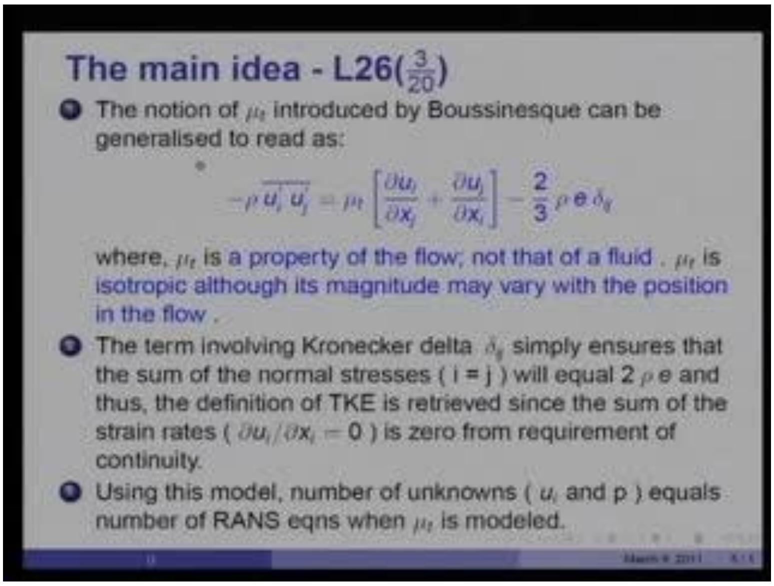

So, the main idea is behind turbulent viscosity mu t is to model a turbulent stress minus rho u prime u i prime uj prime exactly analogous to the laminar stress is except that mu is replaced now by a turbulent viscosity mu t, this is the strain rate and then minus 2 3 rho kinetic energy e into delta i j.

Delta i j is the Kronecker delta; its value 0, when i is not equal to j, but when i is equal to j, it is equal to 1. Remember, mu t is the property of the flow and not of that of the fluid; mu t is also isotropic, that is mu t in all directions xi identical at a point, but at different points, in the flow, its magnitude may vary with position in the flow.

The term involving Kronecker delta, this one is necessary in turbulent modeling because the some of the normal stresses when i is equal to j, remember, d ui, this will be du i dx i and this will be also du i dx i.

So, this will become two mu du i dx i, but that is equal to 0 in a incompressible flow. Therefore, we are left with minus rho ui prime square equal to that quantity, which means it should equal to rho e and that is why that quantity is included in the equations, just to make sure that the kinetic energy is balanced in this representation of a stress and strain relationship.

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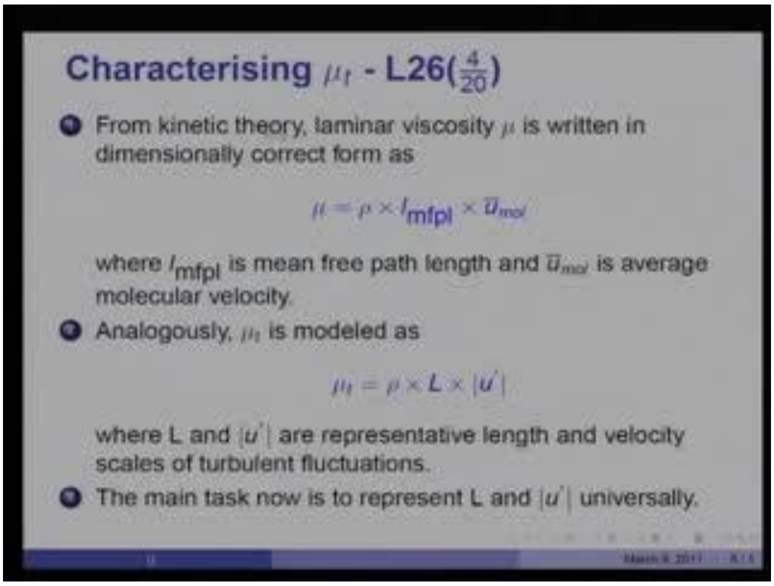

Using this model, the number of unknowns u i and p in the RANS equation equals the number of unknowns, but then the value of mu t now must be provided, somehow, and that is the modeling part characterizing mu t. So, from kinetic theory, laminar viscosity mu is written in dimensionally correct form as mu times rho into the mean free path length l into the average molecular velocity U mole over bar. Analogously, we say mu t would be modeled as rho times some length scale multiplied by a velocity scale ui u dash and these are representative length and velocity scales of turbulent fluctuations.

So then, the main task now is in modeling mu t is to model L and u i prime. Let us see how we can do that in an as universal manner as possible.

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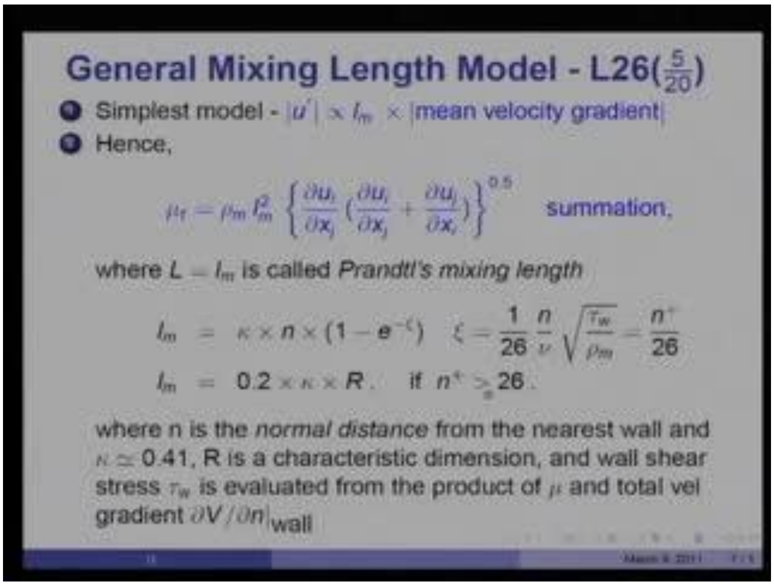

So, for example, the simplest model of ui prime which is called the general mixing length model, is to say it is proportional to some quant length scale l m multiplied by the mean velocity gradient and in a 3 dimensional flow, this would be given as, so that mu t would become rho m times l m multiplied by l m into du i by dx j into du i by dx jdu j by dx I raise to 0.5 with summation. So that, this is really 5 v, if you like or the average mean velocity gradient.

L is equal to l m is called the Prandtl's mixing length and l m is given as you will recall from our lecture on near wall flows kappa times near wall distance y or n in here into 1 minus e raise to minus Xi, where Xi is 1 by 26 n plus, that is n plus by 26 and l m is equal to 0.2 into k into kappa into R or where, R may be the radius of the pipe or it may be the boundary layer thickness, if n plus is greater than 26. Now, the wall shear stress that is required here is calculated simply by the product of mu multiplied by the total velocity gradient at the wall that is the normal velocity gradient at the wall. So, the model is now implementable because the velocity distributions would be available from solution of momentum equations and the stresses in them would be modeled through mu t times this, and therefore, tau wall can always be recovered.

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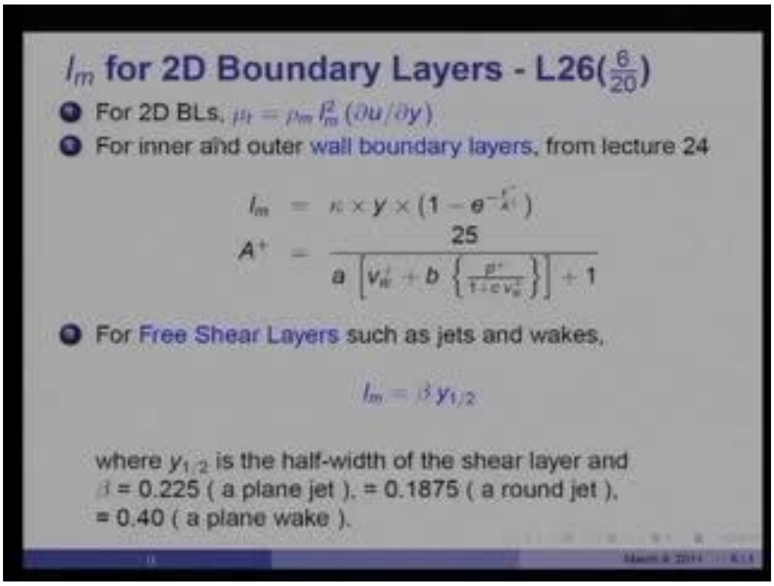

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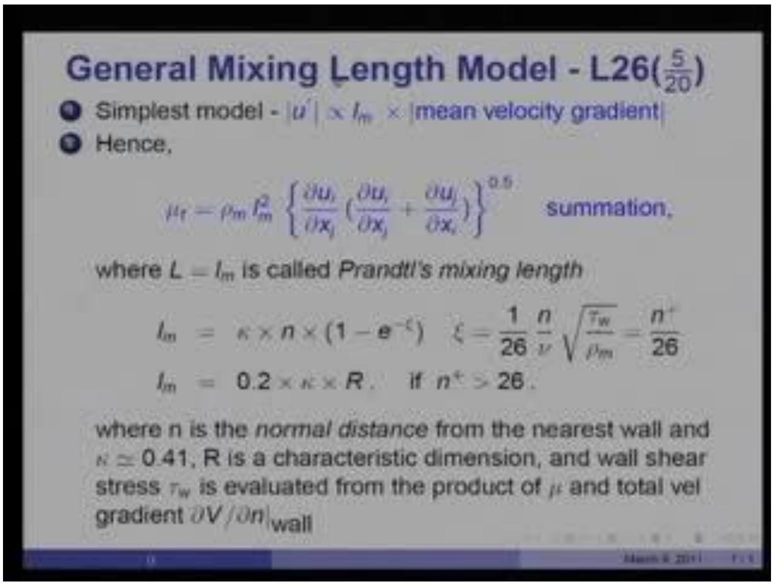

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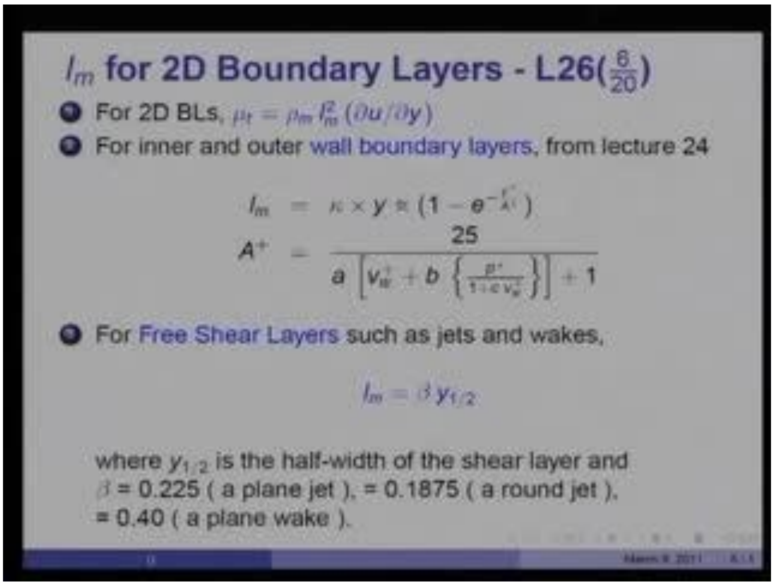

For two dimensional boundary layers, suppose, this mu t which we had represented here for the general three dimensional flows would become simply rho l m square du by dy. Now, for inner and outer layer, boundary layers from lecture 24, we say l m is equal to kappa y 1 minus exponential of minus y plus by a plus and a plus was given by this quantity, where a and b were sensitized to value of the section of blowing parameter or the pressure gradient parameter p plus, and of course, when y plus exceeds a plus, it would simply b equal to what we have shown here, point 2 times kappa times the length dimension of the flow.

When we encounter free shear layers like the turbulent jet or a turbulent wake, then, there is no wall present. In that case, how do we specify mixing length? Well, it is specified as l m time equal to beta times the half width of the jet.

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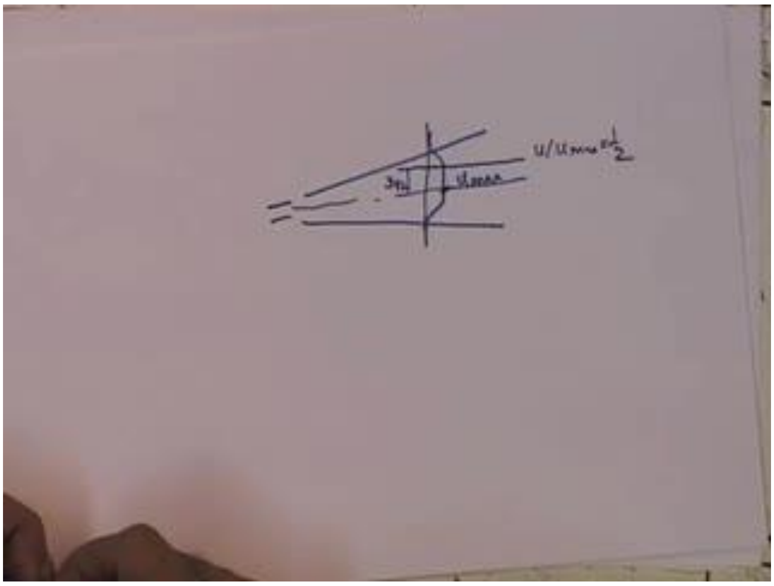

Let us say, the jet emerging, then this would be the velocity profile, and this would be u max, and we say, where ever u divide by u max is equal to half that is the distance y half. This is called the half jet width in a free shear layer, likewise for a wake also.

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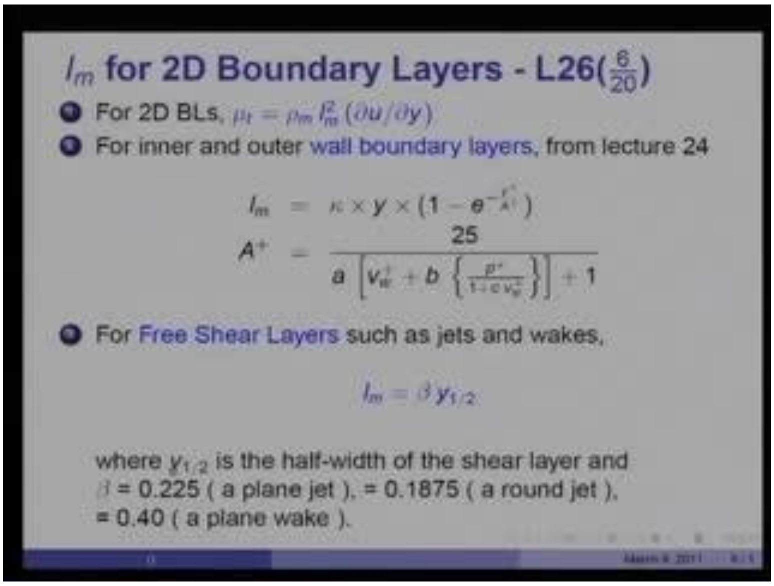

As I said y half is the half width of the shear layer and beta takes different values. This is been observed from experiments as well as has been tuned by numerical experiments. So, beta equal to 0.225, if the jet was a plain jet; if it was a round jet, it would be 0.1875 and if it will a plain wake, it would be 0.4. So, these are the typical values for free shear layers. This is just as an aside, we are not dealing here, in this course, with free shear layers, but none the less, it is useful to document this prescription. Remember, there is no presence of wall in a free shear layer, and therefore, there is no notion of a y plus. So, with this prescription l m is now available, and therefore, the equations can be solved as I said earlier.

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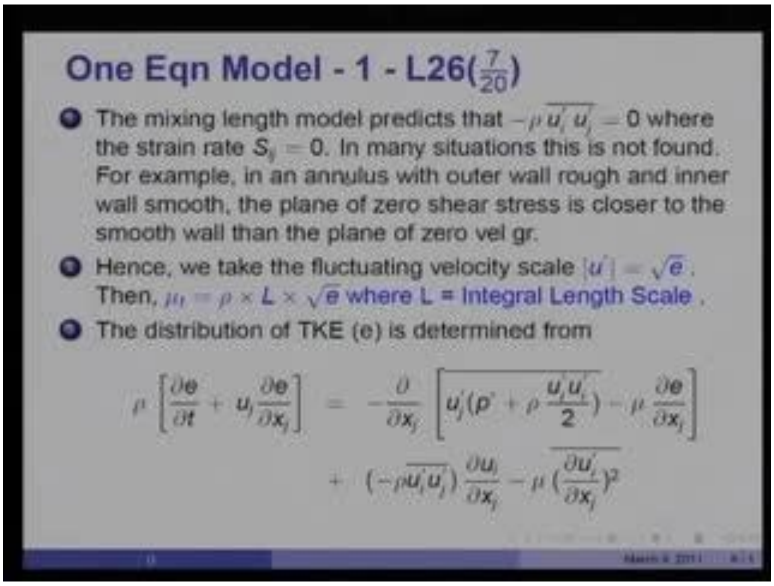

The mixing length model unfortunately predicts that rho u prime u i prime u j prime will be 0, where the strain rate S i j will be 0.

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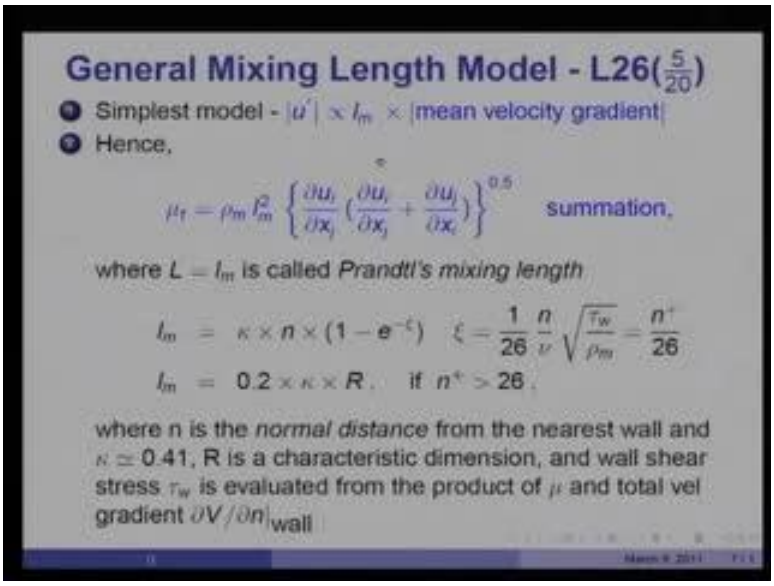

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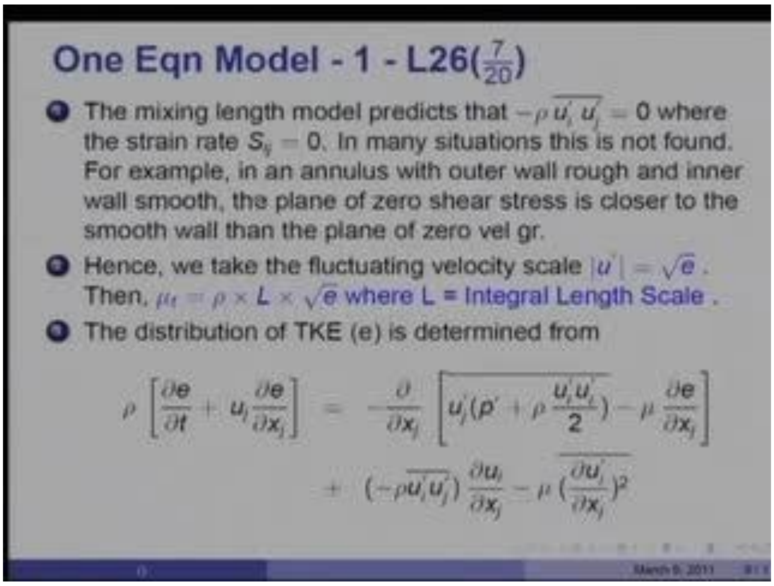

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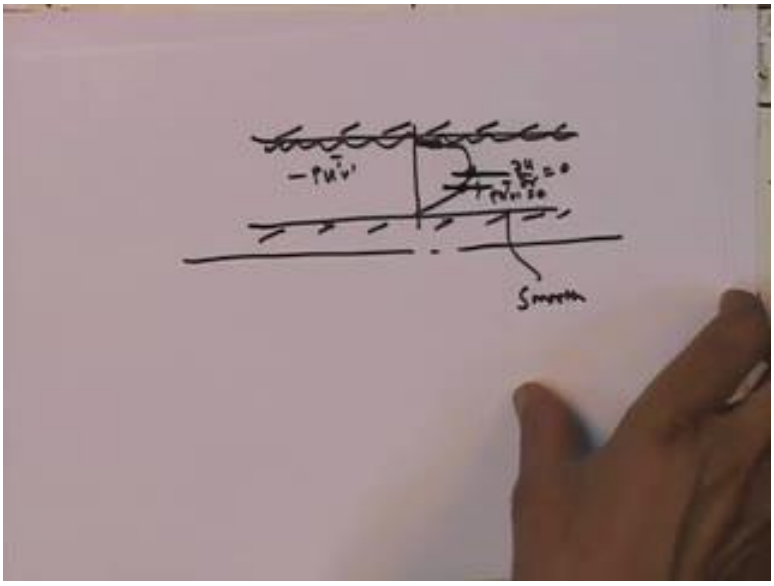

As you can appreciate from this figure, if the mean strain rate was 0 at any point, mu t would automatically be 0. In many situation, this is not found to be the case. For example, if I were to take the case of an annulus, let us say, this is the inner wall and that is the outer wall. Then we would have a velocity profile which will look something like and let us say the outer wall is rough and the inner wall is smooth - this is the smooth wall - and we would expect a velocity profile to be something of this variety. So, the plain of 0 shear, which is here, let us say this is where du by dr will be equal to 0 tends to be closer to and if I were to measure now, in this case, rho u prime v prime then I would find that the plain of 0 shear tends to be closer

This is where rho u prime v prime will be 0. Then, to the smooth wall, then the plain of 0 velocity gradient, so, in fact the shear stress does not really go with the velocity or the velocity gradient or the strain rate.

And there is a separation here, where as the mixing length model would predict 0 shear stress here also, whereas, the true 0 shear turbulence stress is 0.

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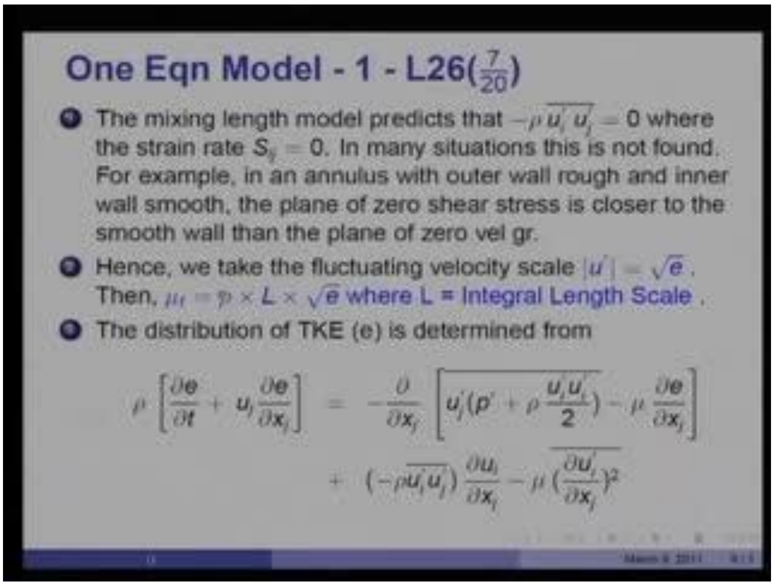

Now, in order to meet with such asymmetric situation, that we need refinements in the models. If we, for example, represent velocity scale u i prime equal to square root of kinetic energy. Then, mu t will recall rho times L square root of e, where L is now has a different meaning and it would be called the integral length scale, which we defined earlier. The turbulent kinetic energy equation e must be obtained from turbulent kinetic equation, which is, as you will recall given by the convection terms here, that, this is the turbulent diffusion is due to pressure fluctuations and velocity and velocity fluctuation Then, there will be the laminar diffusion. This should be the rate at which energy is extracted from the mean motion to produce turbulence and this would the rate at which the energy will be dissipated that we call rho times epsilon.

So, the model of the one-equation type goes like this: it is mu t is equal to rho L square root of e, where e would be obtained from a differential equation, whereas, L would be now specified algebraically, again, like in the mixing length and because we have to now solve this equation, additional equation along with the RANS equation. It is called the one-equation model.

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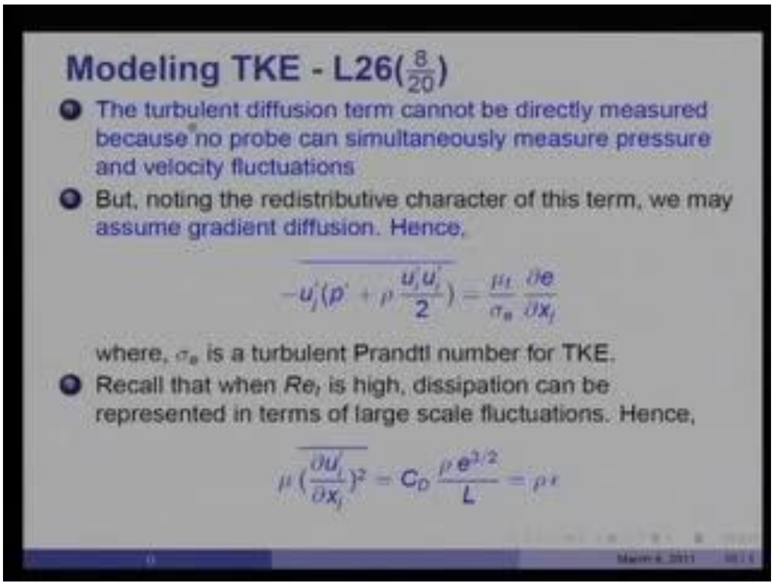

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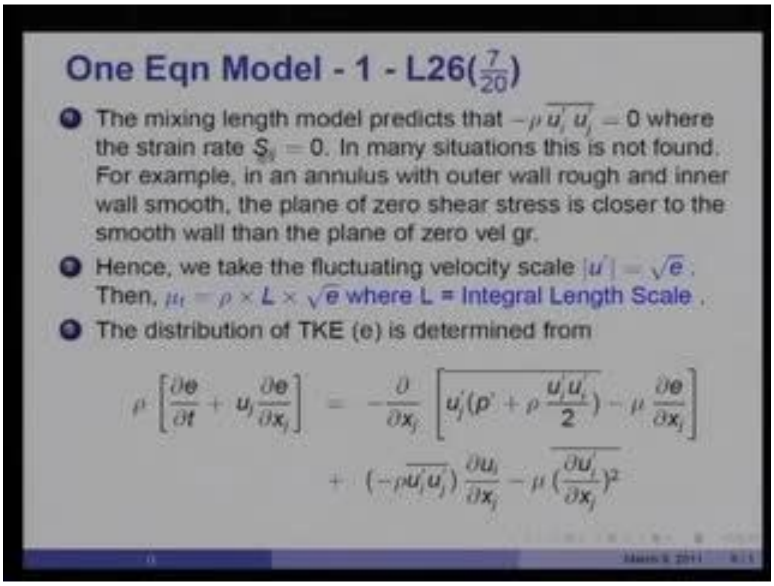

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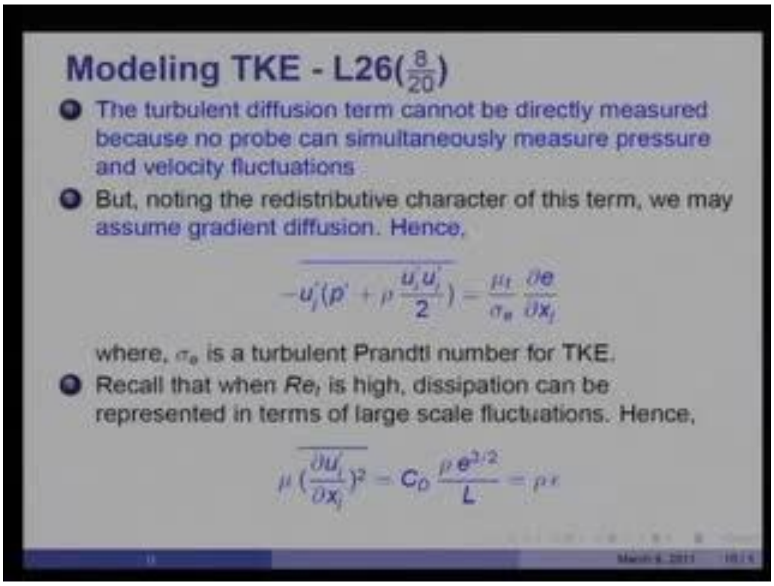

Now, the turbulent term cannot be directly measured. Turbulent diffusion term which contains pressure fluctuations and the triple velocity correlation; these cannot not be measured directly because no probe can simultaneously measure pressure fluctuations and velocity fluctuations. Therefore, we need to model it. So, but noting the redistributive character of this term, you will recall that when we did the energy balance, we noted that this term is redistributive. Therefore, we will assume a gradient diffusion for this. So, minus u j prime p prime rho u i prime u i prime by 2 would be simply mu put as mu t divided by sigma e de by dx j, where sigma e is the turbulent Prandtl number or turbulent kinetic energy.

Now, when Re t is high, dissipation can be represented in terms of large scale fluctuation. This is what we have observed when we looked at the formal aspects of turbulence. Therefore, this mu times du i prime by dx j prime, which we had modeled as rho epsilon can be written as CD times rho times e raise to 3 by 2 divided by L.

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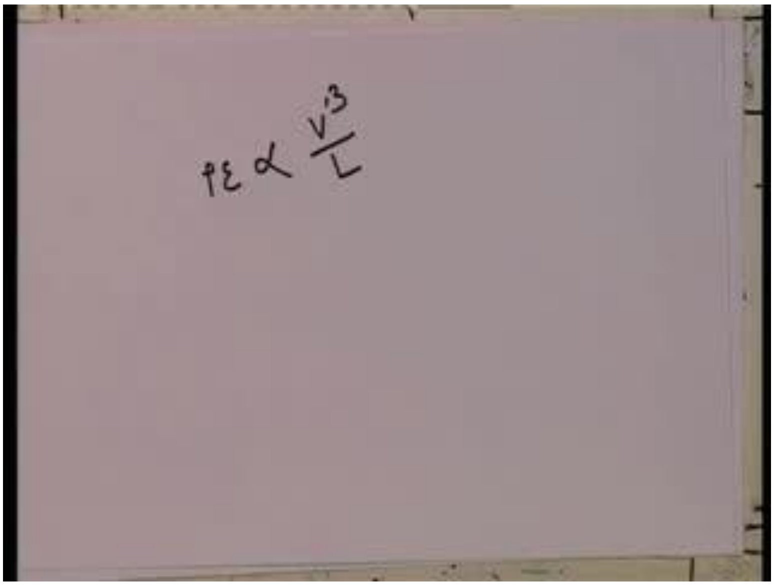

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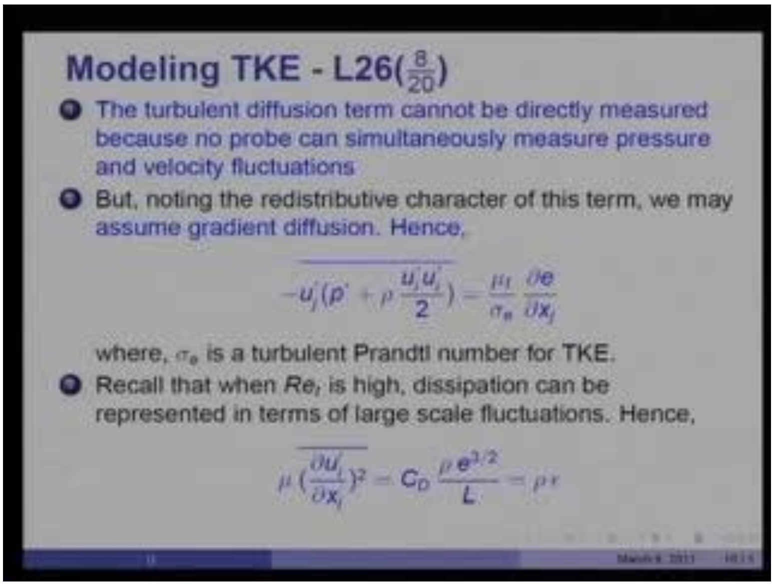

This is now the representative v dash cubed, if you recall. We had said that the rho times epsilon should be proportional to v dash cube divided by Land that is what I am writing here. So, v dash being square root of u, we have said that is equal to rho e raise to 3 by 2 by L.

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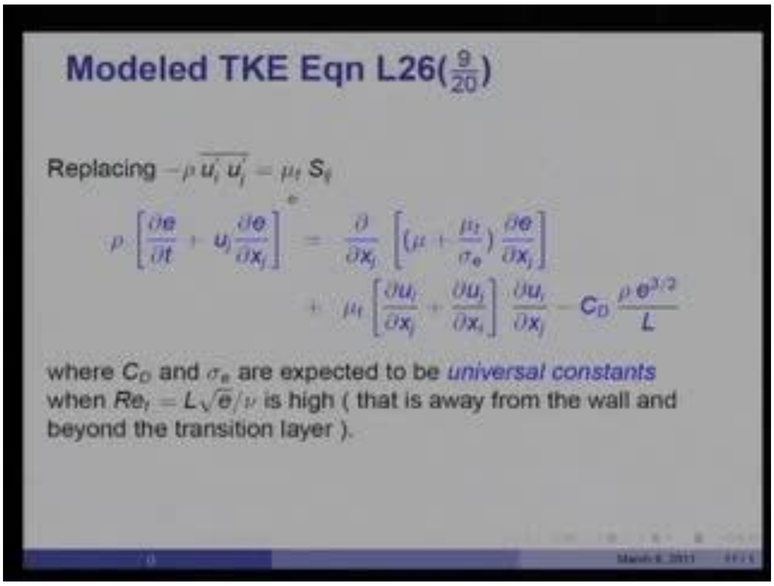

If we now replace minus rho u prime u j prime equal to mu t times S i j, then the model turbulent kinetic energy equation would be simply convection terms here, plus mu plus mu t divided by sigma, which is, which will into this gradient of e which would be the turbulent diffusion. Then, there would be the generation term, energy production term, by replacing this. Then, there would be the dissipation term, where CD and sigma epsilon are expected to be universal constant, when Re t is high - that is away from the wall and beyond the transitional layer.

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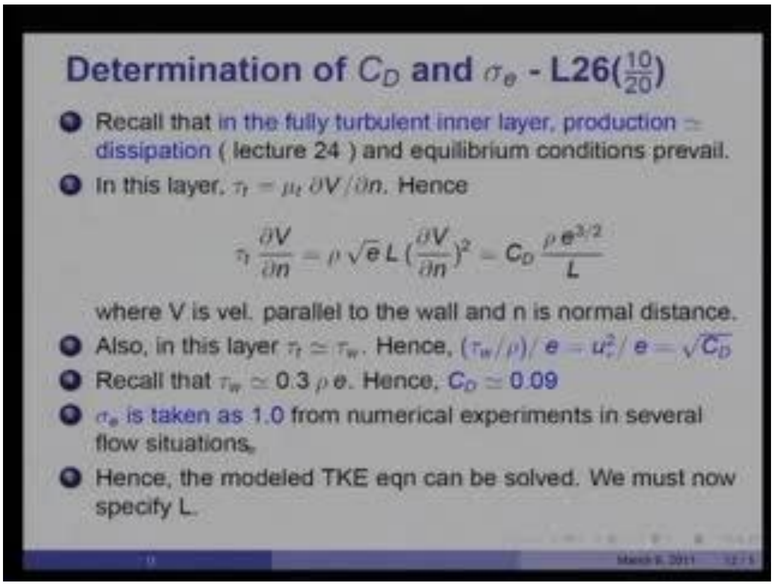

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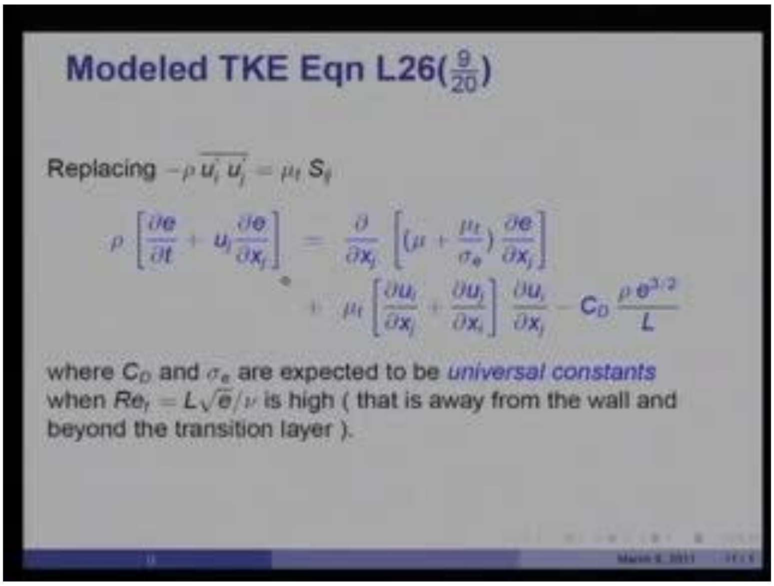

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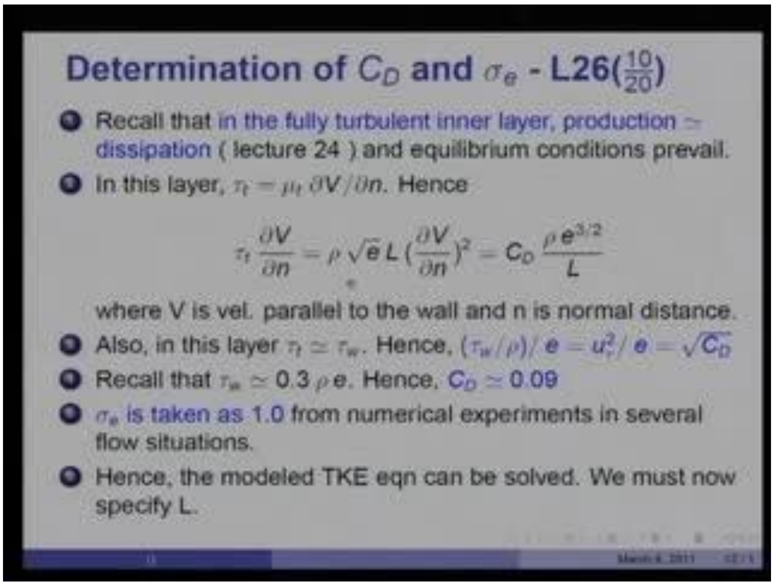

So, how do we get C D and sigma epsilon? Now, recall that in the fully turbulent inner layer, production is very nearly equal to dissipation, as we saw in lecture 24 and the equilibrium conditions prevail and in this layer, tau t is equal to mu t d V by dn. Hence, tau t dV by dn which would be the production term, this term would be the production term in the 2 dimensional boundary layer or very close to the wall minus equal to rho square root of e L d V by d n whole square equal to CD rho e raise to 3 by 2by L which is the dissipation, and the where V is the velocity parallel to the wall and n is the normal distance.

Now, in this layer, we had also said that tau t is very much close to tau w that is the wall shear stress. Hence, tau w divided by rho divided by e would be u tau square by e equal to square root of C D.

Now, you will recall that we had shown from the experimental data that tau w is approximately equal to 0.3 rho times e. Therefore, C D takes a value of about 0.09; sigma e is taken equal to 1 from numerical experiments in several flow situations like a pipe flow or a boundary layer and so on, so forth

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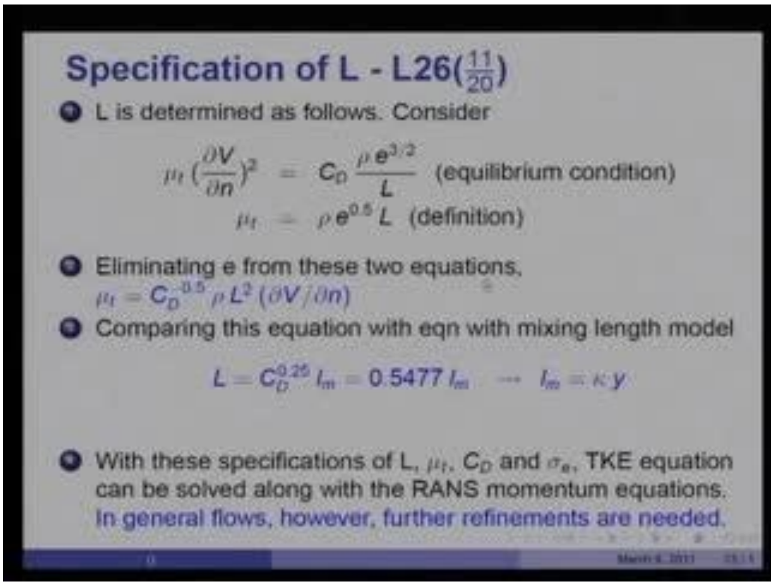

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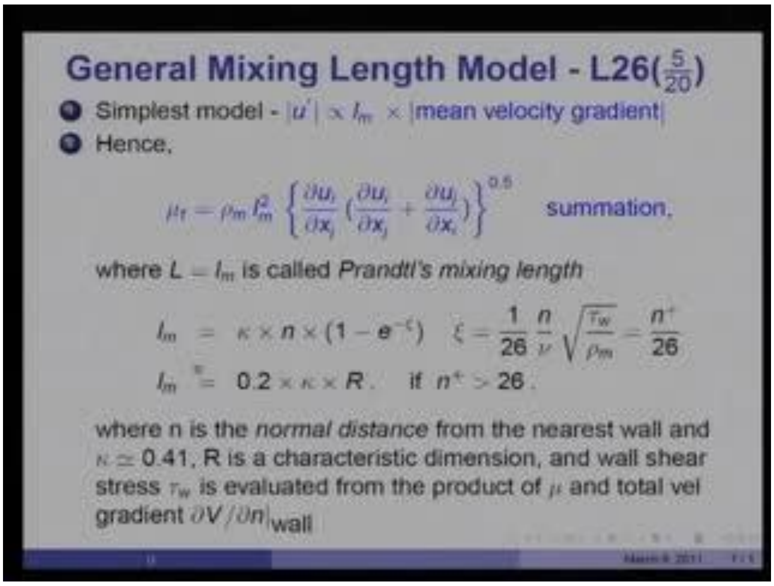

Hence, the model turbulent kinetic energy equation can be solved. We must now specify the length scale L, how do we do that? Well, L is determined as follows. Consider equilibrium condition again; production is equal to dissipation; then mu t will be equal to rho e raise to and the definition of mu t is now rho e raise to half L which is the definition.

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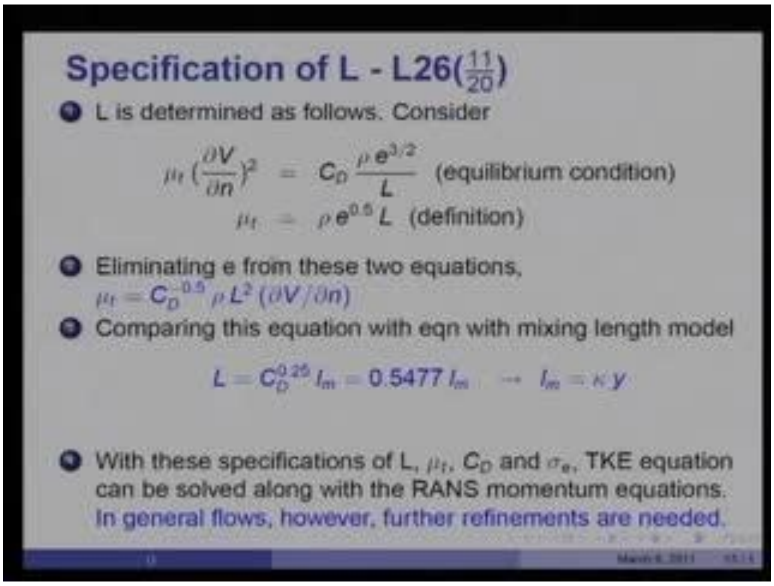

If I combine these 2 equations and eliminate e from them, I would get mu t equal to C D raise to minus half rho L square d V by d m. Now, if I compare this equation with the equation of the mixing length model, then I would see that L would be - you recall the mixing length model - mu t equal to rho l m square into d u d y. So, L would now equal C D raise to 0.25 into l m which is 0.5477 l m and l m equal to kappa y times the function that we normally use.

With this specifications l mu t C D and sigma epsilon turbulent kinetic energy solve along with the RANS equations. In general flows, however, further refinements become necessary, and in the next slide, we will show you why this is so, although kinetic energy can equation can take care of the non coincidence of 0 velocity gradient, and the shear stress, it is not a very good approximation in some other situation as we shall see shortly.

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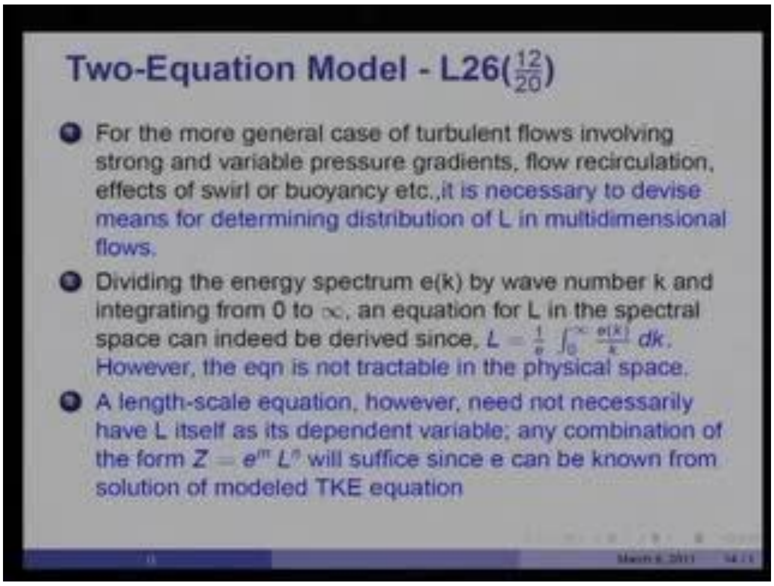

For the more general flows involving strong and variable pressure gradients or even re-circulations or swirl or buoyancy effects, it is necessary to devise means for determining distribution of L in a multidimensional flow.

So, recall the energy spectrum e k in wave number space with wave number k and if we divide this e k by k and integrate from 0 to infinity, an equation for l in the spectral space can indeed be derived since L is equal to 1 over e 0 to infinity e k by k dk.

Unfortunately, this equation is not tractable in physical space. Therefore, we need to find some alternative means of determining the length scale. So, direct equation for a length scale is not possible, which is the need in multidimensional flows with all the effects that I mentioned here.

But then a length scale equation, however, need not necessarily we have L as itself as its dependent variable. Any combination of the form Z equal to e raise to m L raise to n will suffice since e can be known from the solution of the modeled turbulent kinetic energy equation.

There have been some proposals for equation for a Z, which is the composite variable e raise to m L raise to m.

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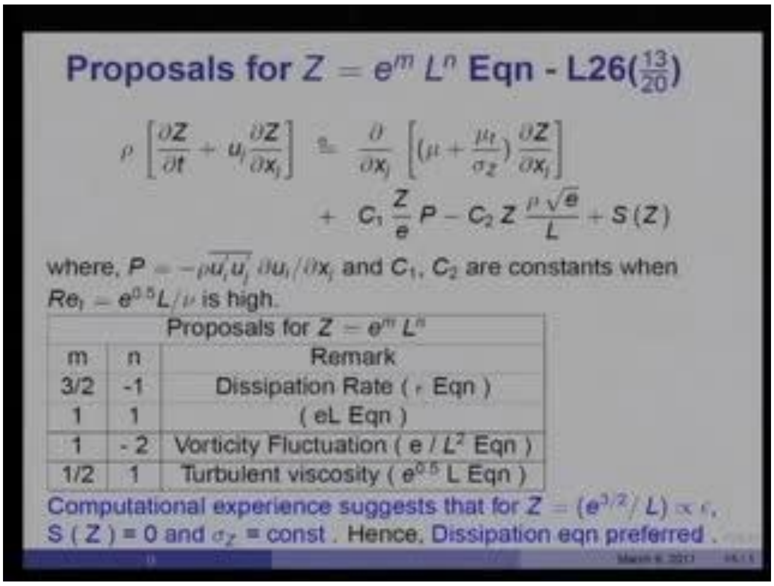

The modeled equation for Z is postulated like this; there would be convection of Z; there would be laminar and turbulent diffusion of Z; there would be appropriately scaled production of Z and there would be dissipation of Z plus a additional source term S Z. So, where P as you recall is the turbulent kinetic energy generation and C1C2 are constants when Re t is high.

Now, there have been some proposals for this. For example, the first one took m equal to 3 by 2 and L equal to minus 1 and that would mean that Z is equal to or proportional to e raise to 3 by 2 divided by L, which is exactly would mean Z would imply dissipation, and therefore, it is called the dissipation equation.

Equations for m equal to 1 and L equal to 1 has also been derived, but then, that equation requires an additional S Z term, which I am not mentioned here. Likewise, you can also have, e raise to 1 divided by L square, that is n equal to minus 2 m 1, which again requires another type of S Z term and you can also have e raise to half multiplied by L, which would be the turbulent viscosity equation itself, but that also requires an S Z term. It is the dissipation rate equation is the only one which does not require S Z and that is equal to 0.

So, we have, if we take Z equal to e raise to 3 by 2 divided by L, we would have a nice convection term, a diffusion term, a production term minus dissipation term with S Z equal to 0. This is the most preferred alternative because this S Z equal to 0 term is an attractive feature. Whereas, in all other models, S Z has to be tuned as it were and has to be modeled separately.

So, and therefore, the most preferred choice is the dissipation equation with e raise to 3 by 2 and L raise to minus 1.

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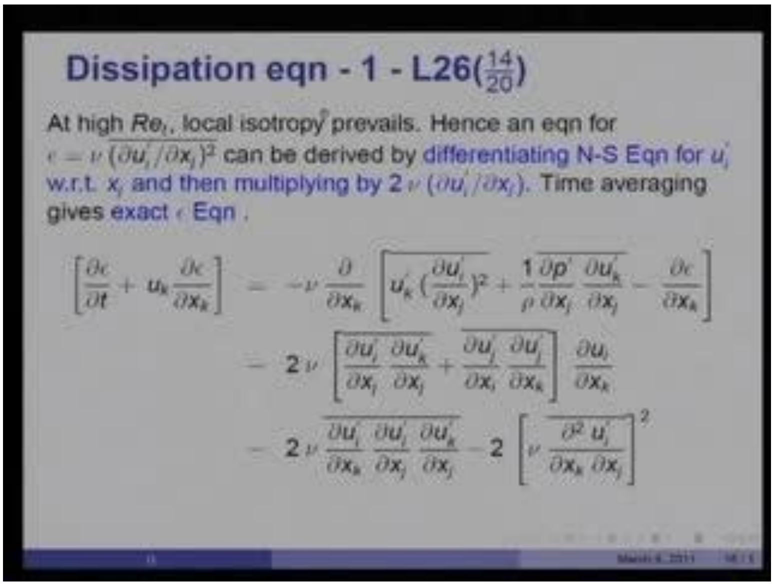

So, one solves this Z equation along with the e equation and simply recovers the L, that we need at each point in the flow so as to form the turbulent viscosity. How do we justify the dissipation equation about which I just mentioned.

Now, at high Re t local isotropy prevails and an equation for new d u i prime by d x j square can be derived by differentiating instantaneous storms on Navier-Stokes equation for u i dash with x j and then multiplying by 2nu d u i dash by d x j. If you then time average this new equation, then you get a exact equation for epsilon the quantity that we have chosen for the second variables Z and that equation looks like this, it is a convection term. Then there is a, these two terms constitute diffusion of epsilon - firstly due to pressure gradient or the pressure fluctuations and secondly due to the velocity fluctuation; this is the laminar diffusion.

This is the production of epsilon; this contributes to the production of kinetic energy and these two are the dissipation of epsilon itself.

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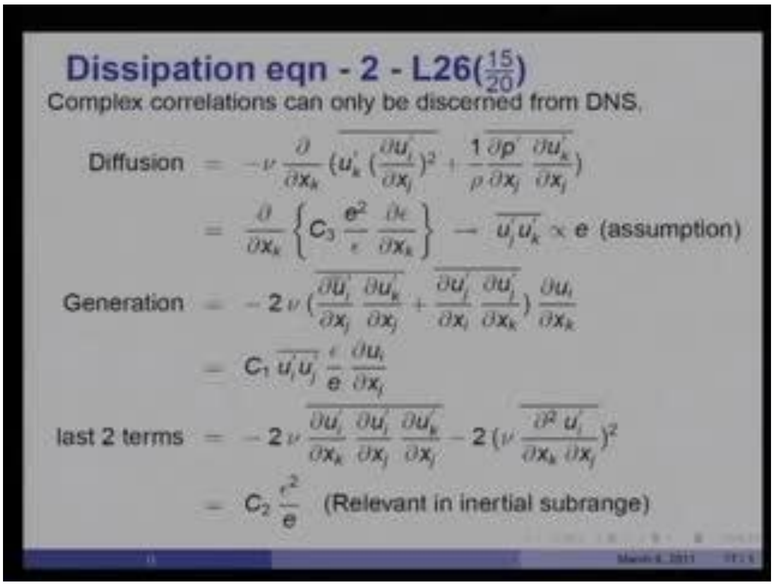

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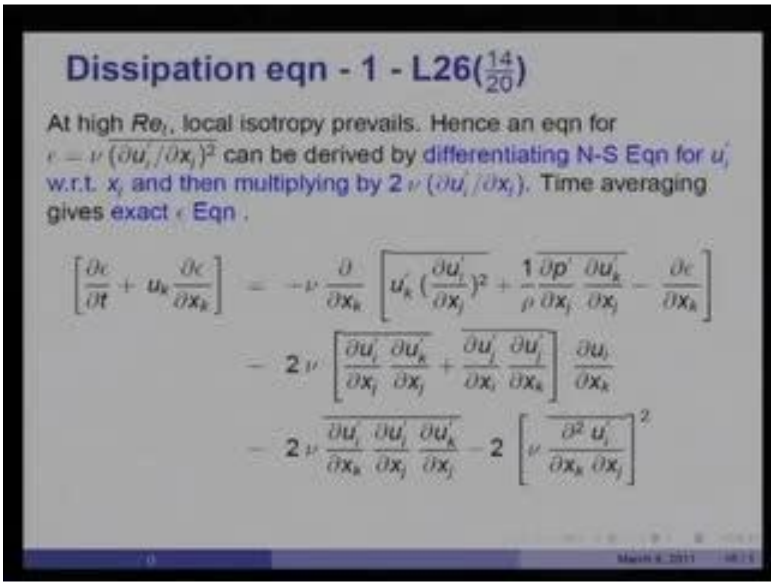

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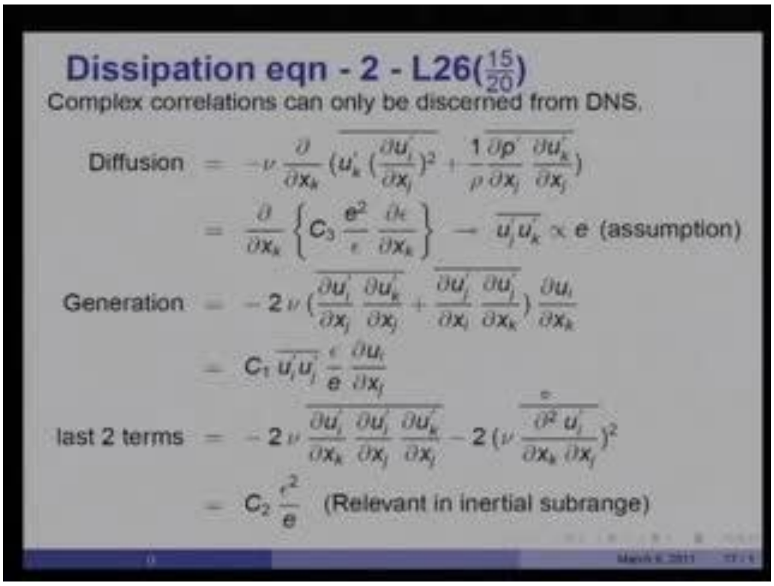

Now, the complex correlations can be discerned from the DNS. So, how can a model these complex terms that you see here. Well, that has been done by looking at computations of direct numerical simulation, and so, the diffusion term apparently can be model as d by d x k into C3 time e square by epsilon d epsilon by d x k in which u j dash u j k dash has been taken proportional to e and this is an assumption in this modeling. The production term about which I mentioned: this is the production term is modeled as simply C1 times the u i dash u j dash multiplied by strain rate multiplied by time scale epsilon by e and then the last two terms are simply modeled as C2 times epsilon square by e. Remember, this term would not be relevant in highest wave numbers space, but in relevant, in the inertial sub range, whereas, this term would be dominant in the highest wave numbers where dissipation really occurs and this two terms can be grouped together and modeled a C2 times epsilon square by e. (Refer Slide Time: 27:47)

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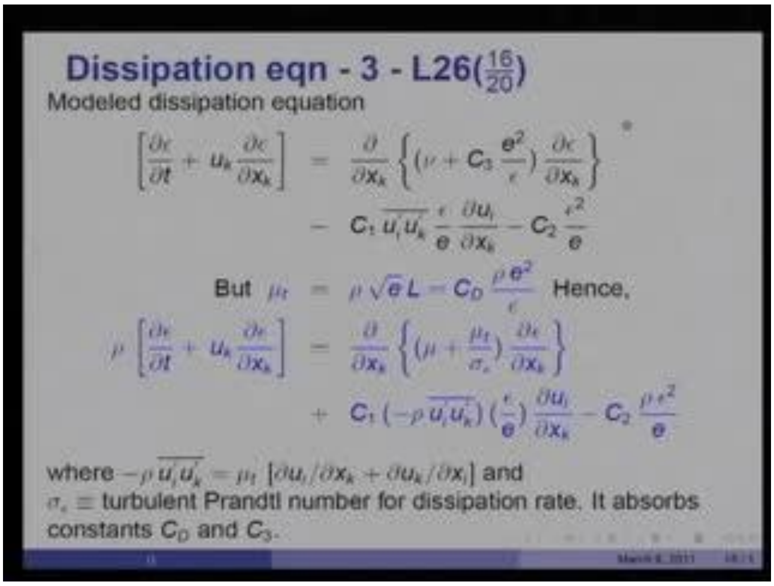

Remember, all this modeling work has been done earlier was made possible from measured turbulence data, but now, it is also possible from direct numerical simulation data. Therefore, we get the dissipation equation which reads like this.

This is the convective term of the dissipation equation; this is the diffusion term; this is the turbulent diffusion; this is laminar diffusion; and then, we get the production term and the dissipation term itself and mu t is rho square root e into L. Therefore, this will simply become C D rho e square by epsilon because epsilon is proportional to e raise to 3 by 2 by L.

Hence, the dissipation equation would be written as d by d x k diffusion gradient of epsilon into C1 times production minus C2 times rho epsilon square by e, and where, u i dash u k the strain rate multiplied by the viscosity this is the model that we choose to model stress which appears here.

Sigma epsilon is the turbulence Prandtl number for dissipation rate; it absorbs constant C D and C3. This C 3 and having model u i dash u j dash by this as C D times. So, the mu t is this whole term is replaced by mu t by sigma epsilon and C3 and C D are absorbed in it.

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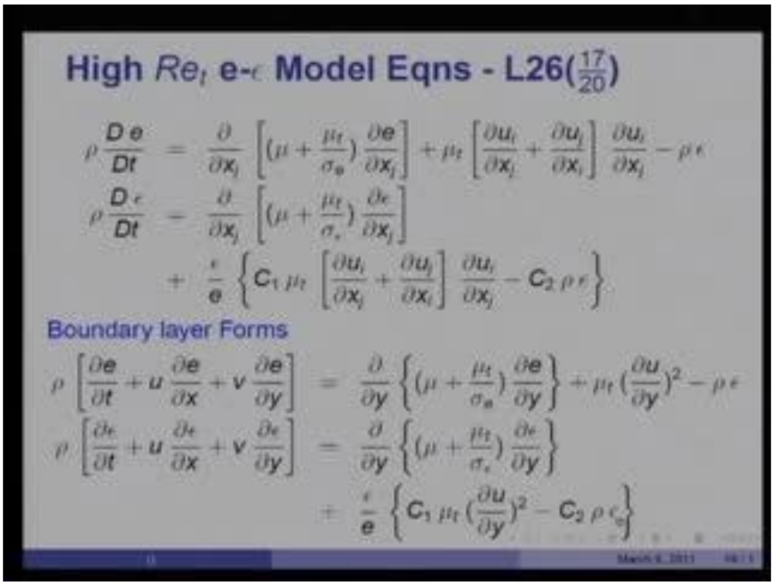

So, this is the final form of the epsilon equation which we shall choose. The so called 2 equation model is essentially this; there is the kinetic energy equation, which you have already appreciated earlier and this would be the dissipation equation. Now, written with u i dash u dash replaced by the strain rate and the boundary layer form of this would be, these are the convection terms of e, diffusion terms, generation term minus dissipation, and for dissipation, this would be the convection term; this will be the diffusion term; and this will simply be mu t du by dy whole square minus C2 rho epsilon.

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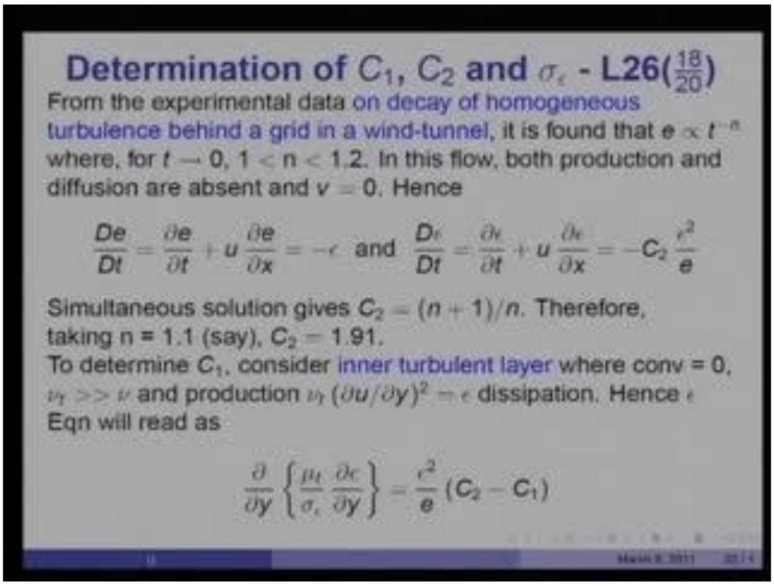

The question now is how do we determine C1 and C2 and sigma epsilon? These are determined from observation of experimental data in decay of homogeneous turbulence.

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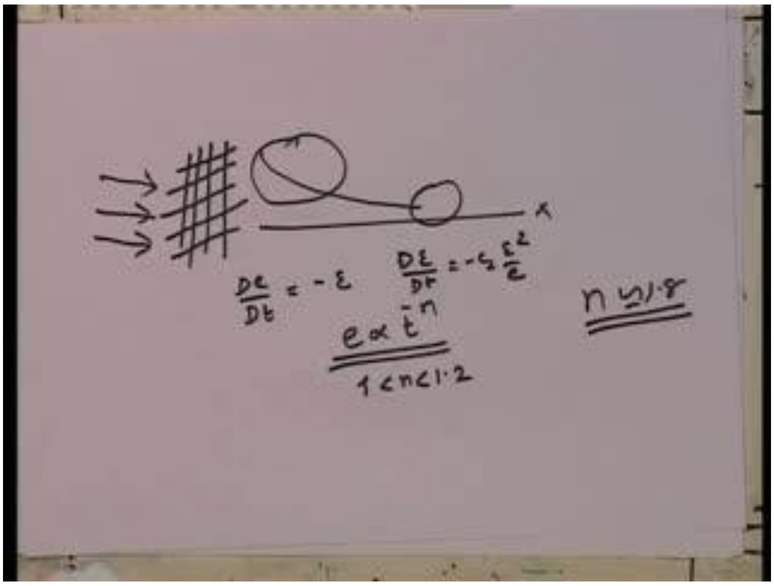

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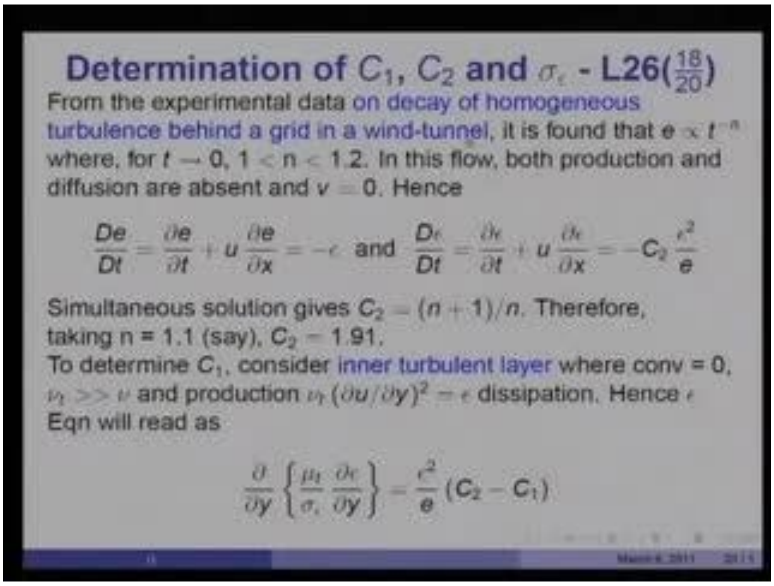

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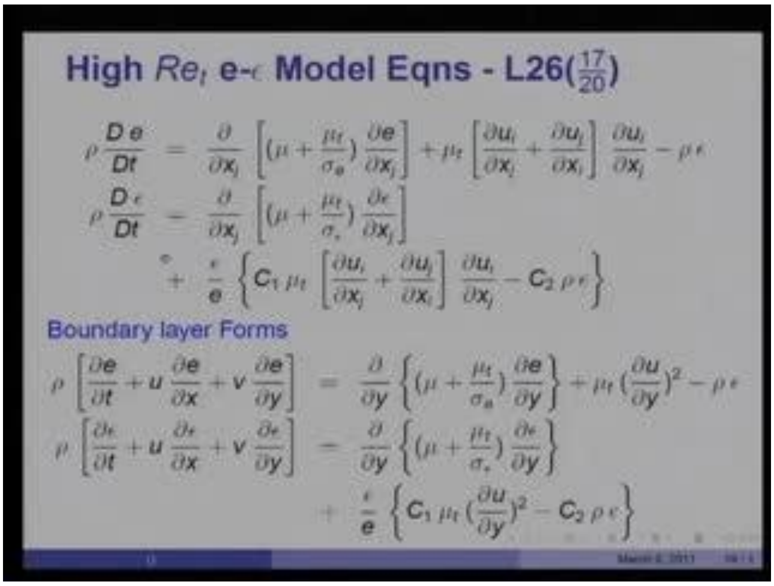

So, what is done is, that, in a wind tunnel, in a wind tunnel, you have, you put a screen like so and the air flows through this screen and because of the screen, high production of energy would take place, and the energy, kinetic energy will go on with x; it will go on declining. If you look at D e by D t in this case, in this flow both production and diffusion are absent and v the normal velocity component is 0. Therefore, D e by D t would be simply you will see, in this case, D e by D t would be simply equal to minus rho epsilon.

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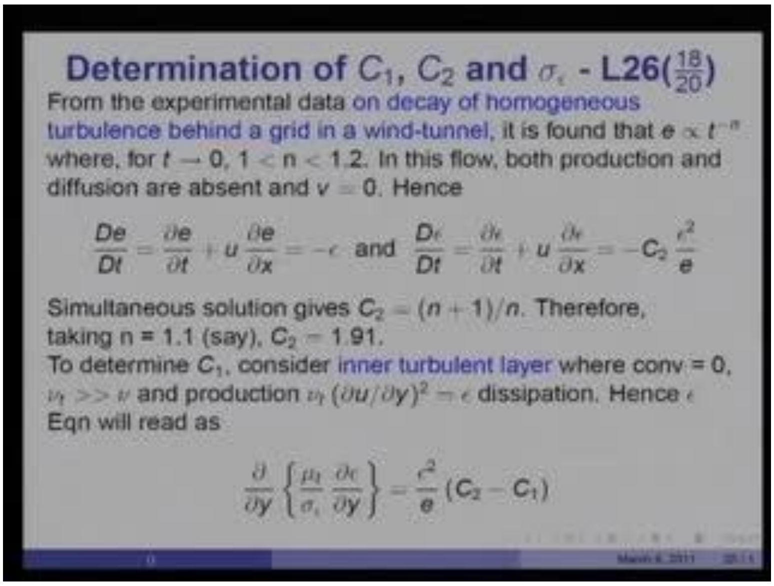

So, rate of change of energy is simply balanced by the dissipation and that is what I have written here and likewise D epsilon by Dt would be equal to minus C2 epsilon square by e.

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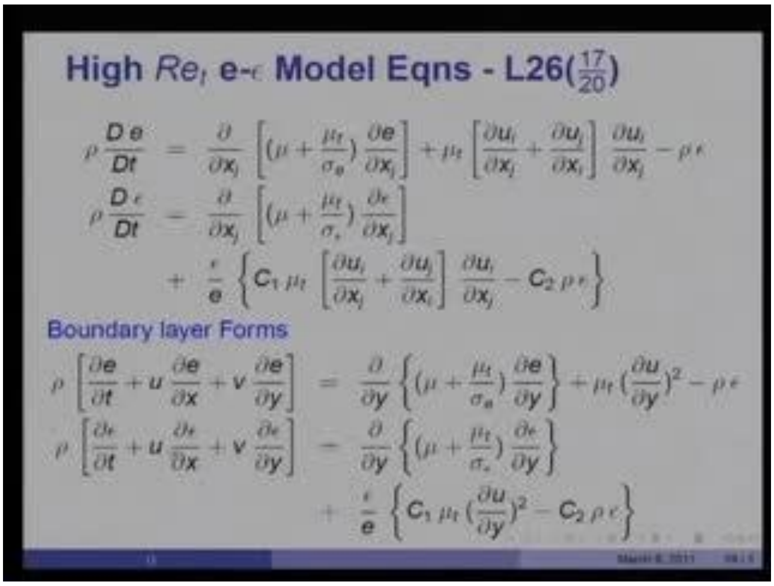

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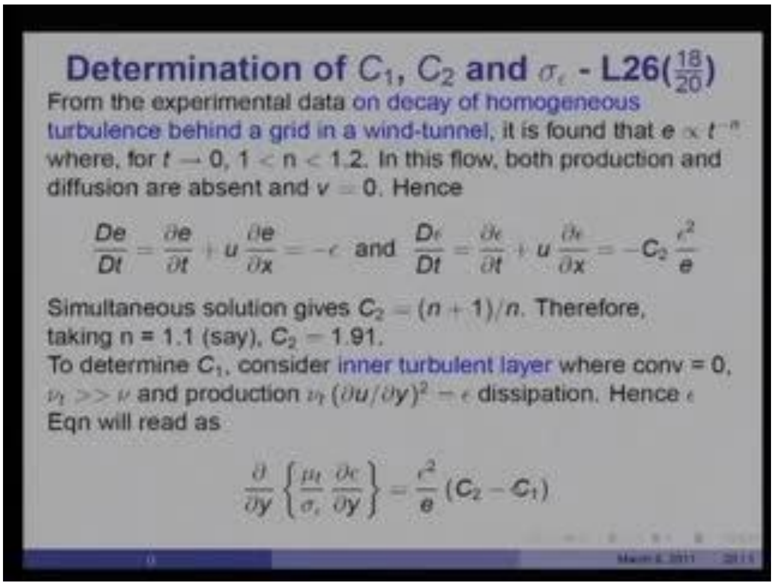

If we solve these two equations Du by Dt with the observed fact that e is proportional to t to the power of minus n, it follows from this De by Dt is equal to minus epsilon and D epsilon by Dt is equal to minus C2 times epsilon square by e. Therefore, inter substitution shows that e should vary as t raise to minus n, and if you look at the experimental data, you find that n varies between1 and into 1.2 at least in the early part for small times and for later times n is of the order of1.8 or so which we shall look at a little later.

So, in the initial times, at least, it is between n equal to 1 and 1.2. Therefore, the solution gives C2 equal to n plus 1 divided by n simultaneous solution. Therefore, taking n equal to 1.1, say, we will get C1 equal to about 1.91. To determine C1, we consider the inner turbulent layer where convection is 0, nu t is very much greater than nu and production nu t du by dy square is equal to dissipation. Hence, epsilon equation, this is 0 almost, and therefore, production is equal to dissipation because that is also 0.Since convection is 0, we will get the total diffusion equal to that term and which we have modeled as epsilon square equal to, remember production would now be nu t du by dy whole square equal to dissipation. Therefore, epsilon square by epsilon e into C2minus C1.

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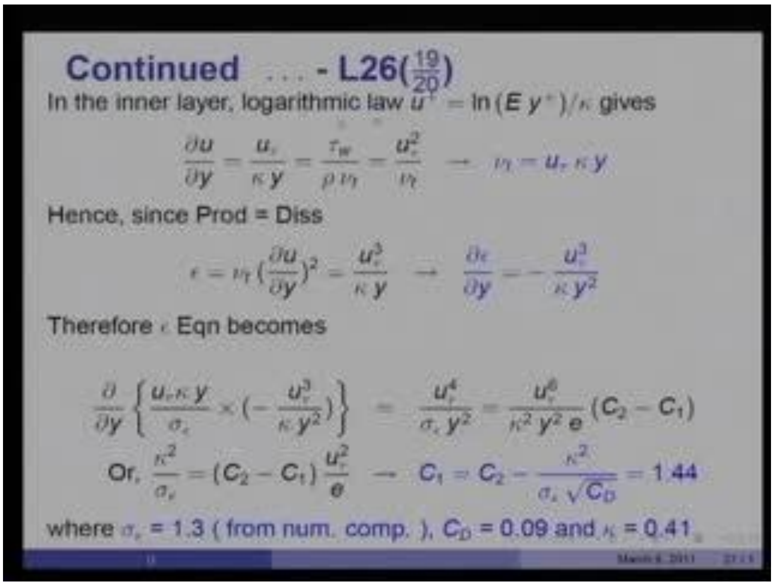

In the inner layer the logarithmic law suggest that u plus equal to l n E y plus by kappa gives du by dy equal to u tau by kappa y and that is equal to tau all over rho nu t and that is also equal to u tau square over nu t which gives nu t equal to u tau kappa y, and hence, since production is equal to dissipation, epsilon would be nu t du by dy whole square equal to u tau by kappa y which gives relationship between epsilon and y. Therefore, d epsilon by d y would be minus u tau cube over k y square.

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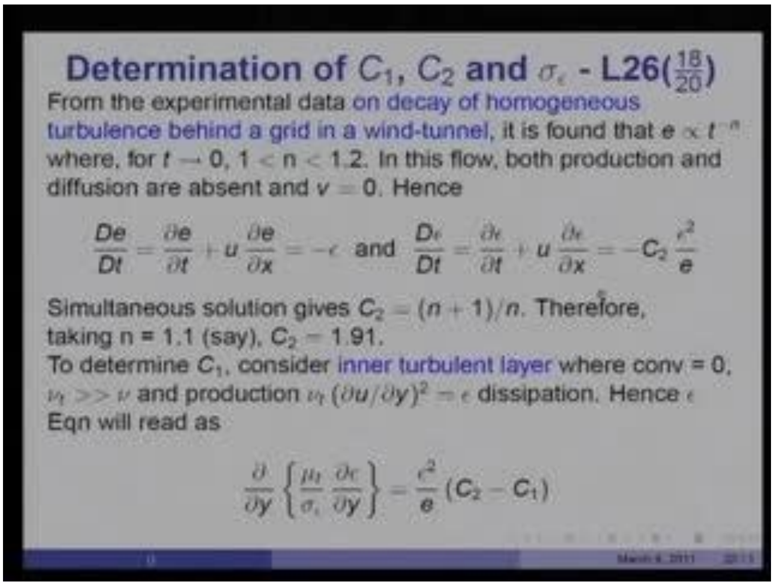

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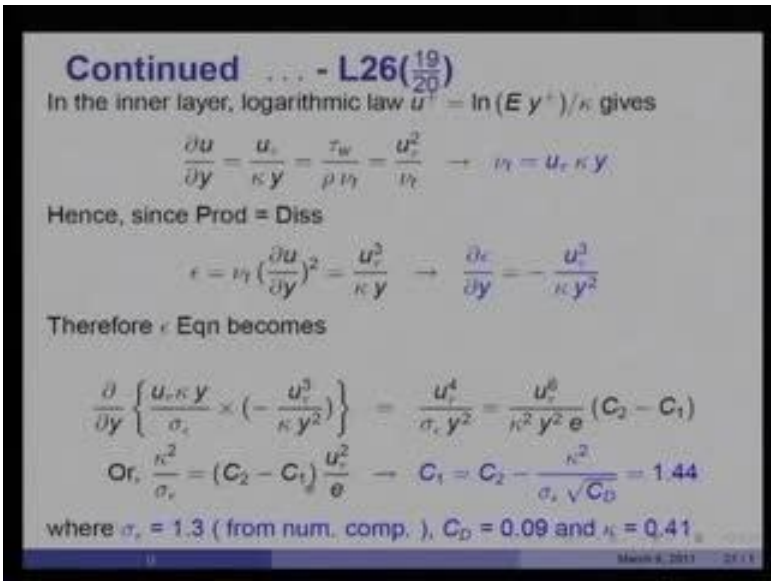

So, if we now substitute d epsilon by d y here, in this equation, and also take mu t appropriately as shown here and d epsilon by d y as shown here.

So, then you will get dy dy over u 2 u tau kappa y by sigma epsilon into this quantity equal to u tau square by sigma epsilon y square equal to all this or kappa square over sigma epsilon would be C2 minus C1 u tau square by e. Therefore, C1 would be equal toC2 minus kappa square over sigma epsilon over C D.

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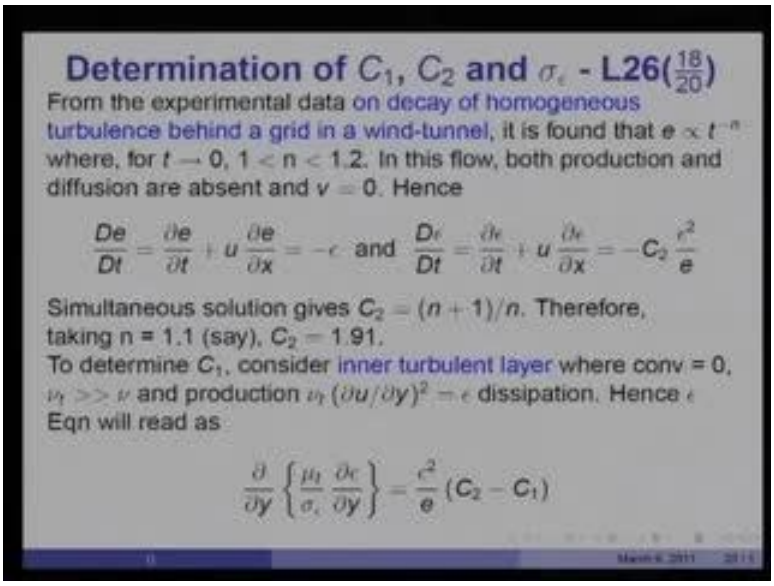

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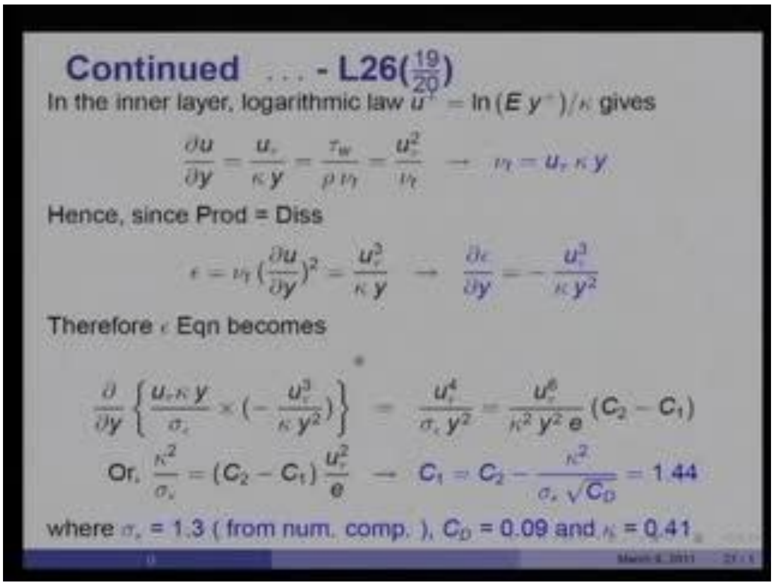

And that would turn out to be equal to 1.44 because C D recall was equal to 0.09, kappa was 0.41, and sigma epsilon has been found from numerical experiments to be approximately 1.3. Therefore, C 1 will be equal to C 2 minus kappa square over this equal to 1.44. So, with this, we conclude how C 1 and C2 are determined. Remember, we had said C 2 was equal to 1.91, and therefore, C 1 can now be determined from this and it turns out to be about 1.44.

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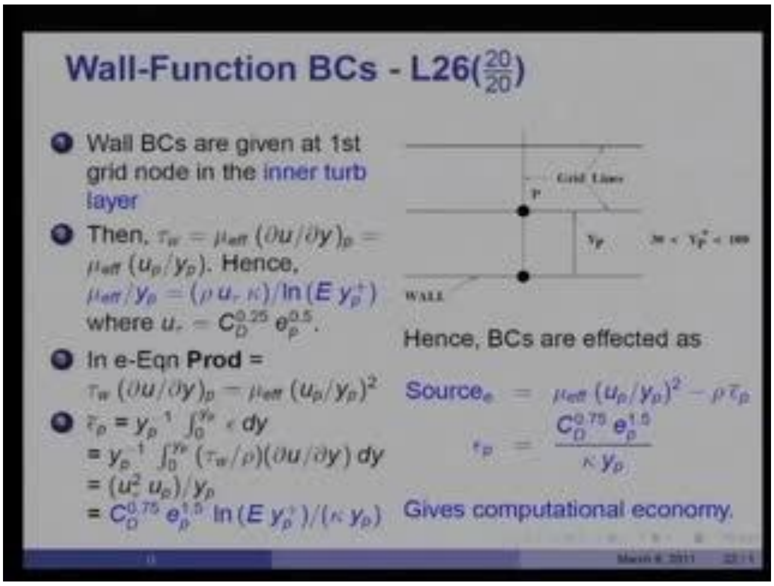

Now, of course, these equations apply as I said to high turbulent Reynolds number region which means beyond the transitional layer. Therefore, in a practical flow, the layer up to the transitional layer must be somehow captured in a computation; that is done as I have shown here.

So, in the equations are solved, let us say by a finite difference procedure. Then, if this is the wall, then you will take, this will be the grid node at the wall but the next grid node would be taken at a distance substantially away from the wall. So that Y p plus would be anywhere between 30 and 100 and you are sure that you are in the inner turbulent layer.

So, the point p must be in the inner turbulent layer, that is the most important part and then you only solve for the outer layer which is beyond point p.

So, all the boundary conditions must be applied at p. Now, of course, we do not know the exact boundary conditions for e and epsilon at p, but they can be derived.

So, for example, if we take tau all equal to mu effective du by dy p equal to mu effective u p by y p, and hence, mu effective by y p becomes simply rho u tau kappa l n E y plus p where, u tau is equal to C D raise to 0.25 e p raise to half. You will recall that we had shown this for the inner layer or a turbulent part of the inner layer. Therefore, u tau can be estimated like this in rather than from mu du dy equal to 0 because we are not computing up to this point; we are only computing from this point onwards. u would be of course obtained from the RANS equations at this point.

So, in the e equation, if we say production equal to that and equate it to an average dissipation over this length, epsilon bar p would be equal to y p raise to minus 10 to y p epsilon d y; substituting for epsilon equal to production, we get that and that would integrate to simply this. Therefore, the source term of e is simply mu effective u p by y p whole square minus rho epsilon bar, which is what I have shown here, what epsilon bar is and epsilon p itself would be simply C D raise to 0.75 e p raise to half 1.5 into kappa y p. So, what is done is the source term at point p is modified in the k n epsilon equations.

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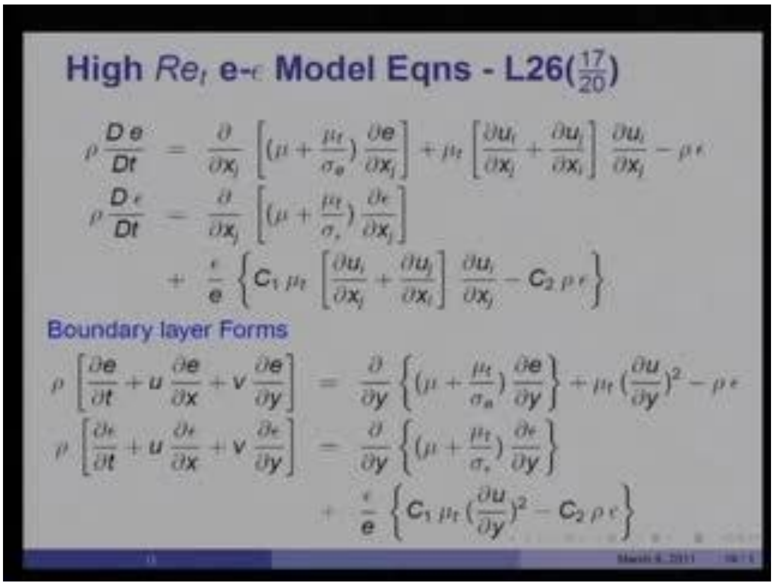

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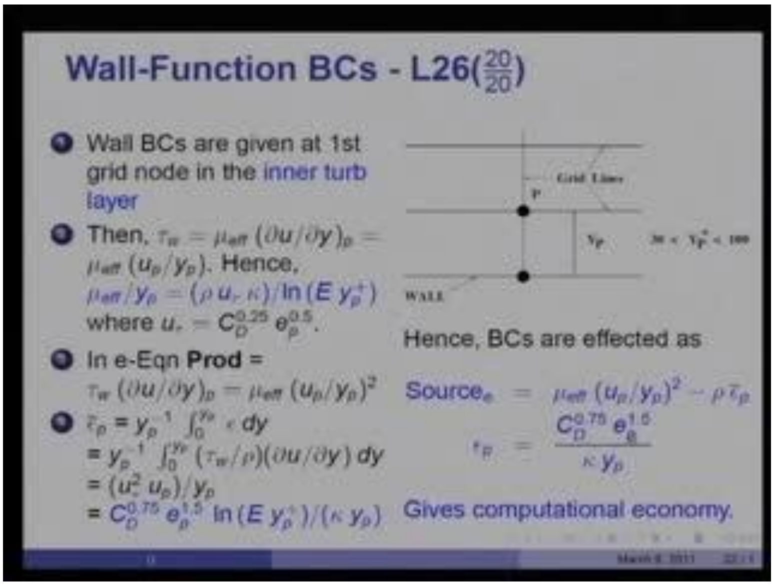

So, these are the source terms in the k n epsilon equation. Then at the point p, these two source terms are simply modified to read as I have shown here.

Now, with this the region where very sharp gradients of velocity kinetic energy and other quantities take place is completely circumvented and everything begins from here; we have essentially exploited the universality of the inner layer in dimensionless quantities. This gives enormous savings in computational time, and therefore, it is routinely used in two equation turbulent model. This is called the wall function approach to solving the two equation turbulence model equations.

In the next lecture, I will consider further refinements for situations in which the AD viscosity models do not quite apply in a precise manner. Therefore, one has to turn to the modeling of the turbulent stresses themselves, or obtaining them turbulent stresses through transport equations for ui prime uj prime.