Convective Heat and Mass Transfer
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Module No. # 01
Lecture No. # 23
Sustaining Mechanism of Turbulence-2

In this lecture, we shall continue with our discussion of the Sustaining Mechanism of Turbulence, but by employing Spectral Analysis and Vorticity Dynamics.

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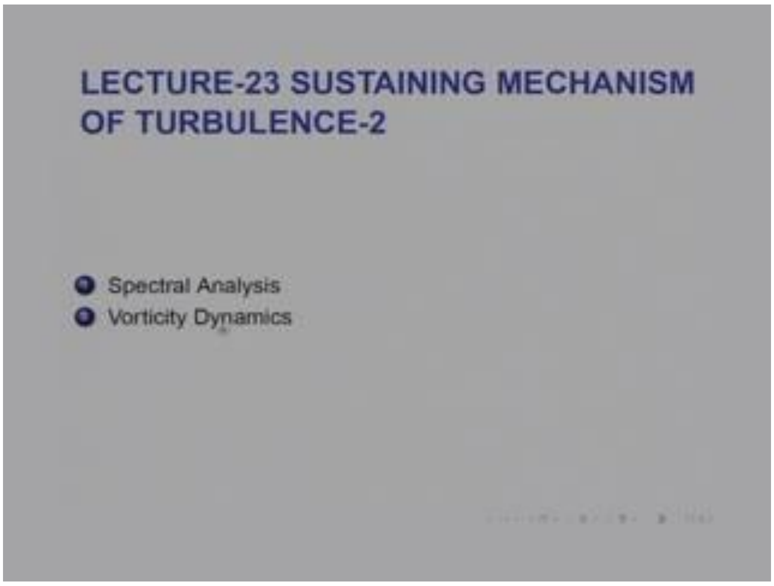

Both these terms need explanation. In the course of this lecture, you will understand what they mean. I miss one that the subject matter is somewhat mathematical. Therefore, a very close attention would be required.

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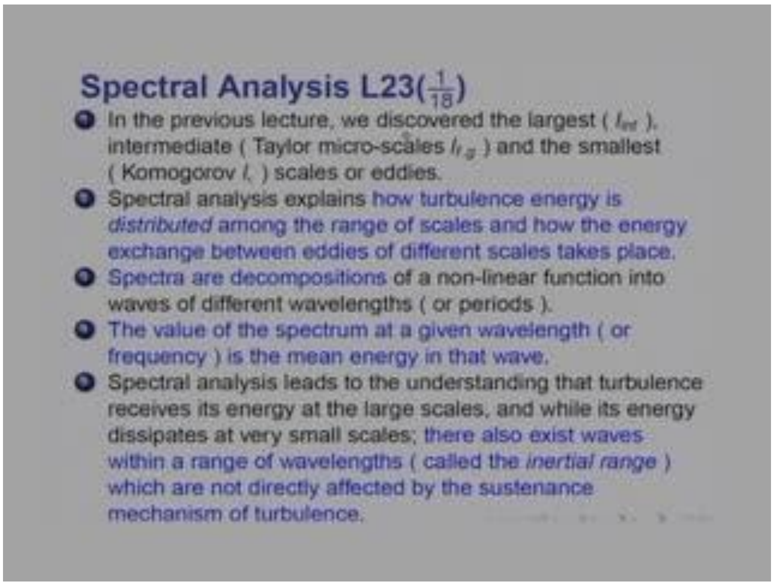

We recall that in the previous lecture, we discovered three characteristic length scales of turbulence. One was the largest - the integral length scale, which was obtained by integrating spatial correlation coefficient l int, intermediate Taylor micro-scale l f g; f being longitudinal and g being transverse length scale. Then, the very smallest scales, which we call Kolmogorov scales; l sub epsilon, where energy dissipation takes place because of the strong influence of viscosity. Spectral analysis goes little further. It explains how turbulence energy is distributed among the range of wave scales and how the energy exchange between eddies of different scales takes place.

In this sense, it is a much more continues description of energy transfer from largest scale to the smallest scale. To do this, we use the notion of spectra. The spectra are decompositions of any non-linear function into waves of different wavelengths; or they can even be periods.

The value of the spectrum at a given wavelength or frequency in case of time-dependent function is the mean energy in that wave. Spectral analysis leads to the understanding that turbulence receives its energy at large scales. While its energy dissipates at very small scales, there also exists waves within a range of wavelengths called the inertial sub range, which are not directly affected by the sustenance mechanism of turbulence. That is the overall story that we wish to narrate by going through somewhat tedious mathematics.

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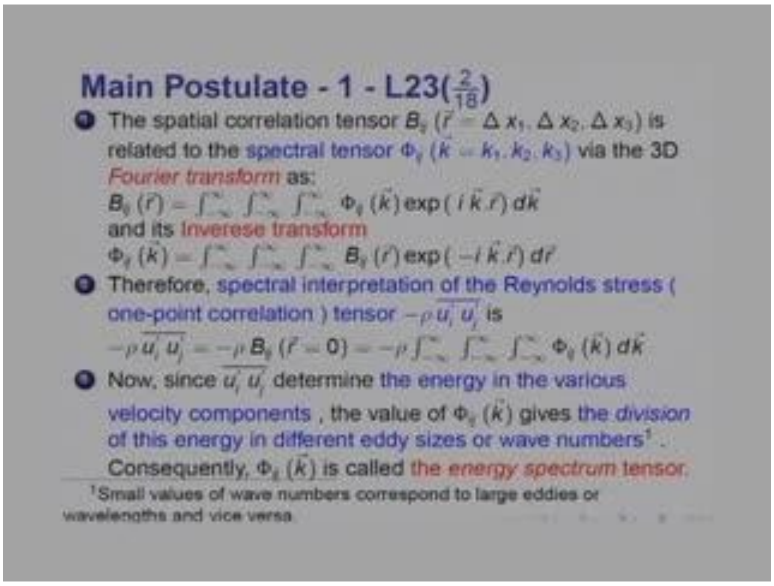

You will recall the spatial correlation tensor B i j at a vector position r equal to del x 1, del x 2, del x 3 is related to the spectral tensor phi i j as a function of wavelength vector k via the 3D Fourier transform. The Fourier transform is defined as B i j at vector r equal to integral from minus infinity to plus infinity 3 times over phi i j vector k exponential of i k dot r dk; i is the complex number under root minus 1. The inverse transform of this is given by phi i j vector k equal to integral from minus infinity to plus infinity 3 times B i j r exponential of minus i k dot r dr.

Therefore, spectral interpretation of the Reynolds stress, which is of the one-point correlation tensor rho u i dash u j dash; both u i dash and u j dash are measured at the same point. Therefore, minus rho u i dash u j dash will become minus rho B i j r equal to 0 equal to minus rho integral from minus infinity to plus infinity 3 times over. Since r is 0, exponential of that would be 1 and you get phi i j k vector d k. Now, since u i dash u i j dash time average determine the energy in the various velocity components of the fluctuations, the value of phi i j k gives the division of this energy in different eddy sizes or wave numbers. The small value of wave number corresponds to a large eddy and vice versa.

A large wave number or wavelength corresponds to very small eddies and very small wavelength corresponds to large eddies. So, what the Fourier transform does is - it gives you the division of the energy of the fluctuations in different eddy sizes. Consequently, phi i j vector k is called the energy spectrum tensor.

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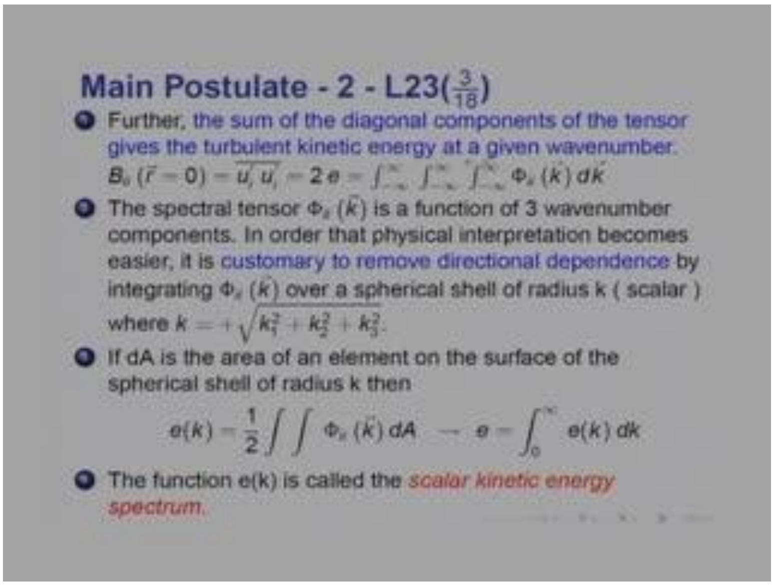

Further, if we take the sum of the diagonal components of the tensor, gives the turbulent kinetic energy spectrum at a given wave number. So, B i i r equal to 0; u i dash u i dash will be 2 times the kinetic energy e equal to integral from minus infinity to plus infinity 3 times over phi i i k dk; here, e is the turbulent kinetic energy.

The spectral tensor phi i i k is a function of three wave number components k 1, k 2 and k 3 because k is a vector. This makes interpretation of this expression (Refer Slide Time: 06:19) somewhat difficult. Therefore, in order that physical interpretation becomes easier, it is customary to remove directional dependence by integrating phi i i k bar over a spherical shell of radius k; where k is square root of k 1 square plus k 2 square plus k 3 square. In other words, the total magnitude of the vector k is taken as the radius.

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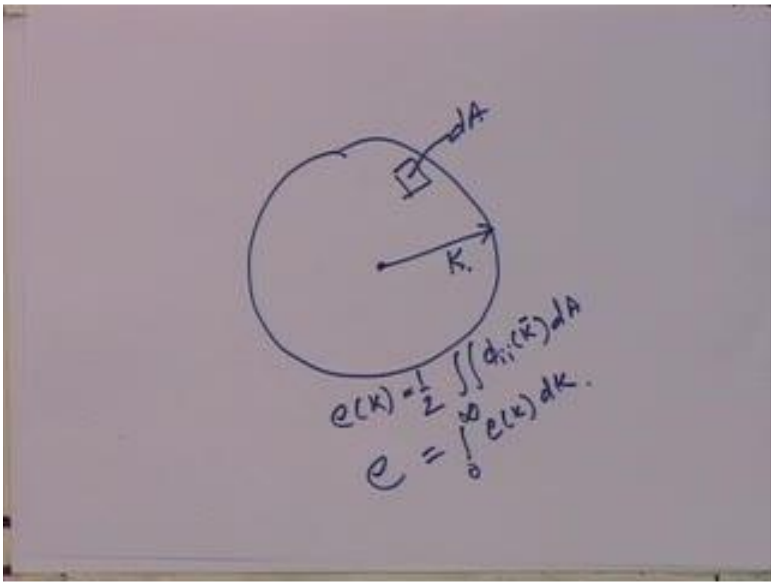

On this spherical shell, we form a sphere of radius k. If we take an element dA on this sphere, then the energy spectrum of e k is half times integral phi i i k dA. e would be equal to integral 0 to infinity e k dk.

This is what I have shown here (Refer Slide Time: 07:37). Energy at wavelength k is half area integral of phi i k bar dA. That gives e equal to integral from 0 to infinity e k dk, which is the kinetic energy e. The function e k is called the scalar kinetic energy spectrum.

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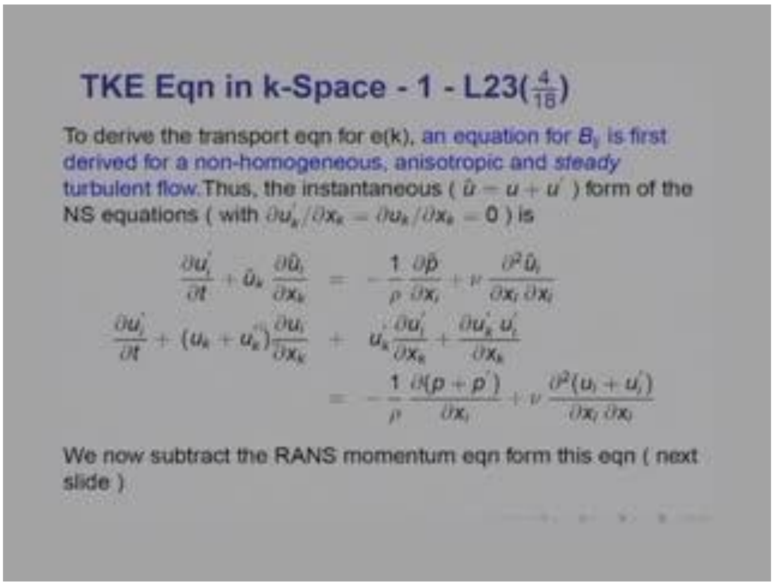

To derive the transport equation for e k, an equation for B i j k is first derived for a non-homogeneous, anisotropic and steady turbulent flow. As you know that in a steady turbulent flow, du dash k dx k plus du k by dx k is 0 from continuity. The instantaneous form of the Navier-Stokes equations would be given by du i dash d by dt plus u cap k du cap i by dx k equal to the pressure gradient term and the diffusion term.

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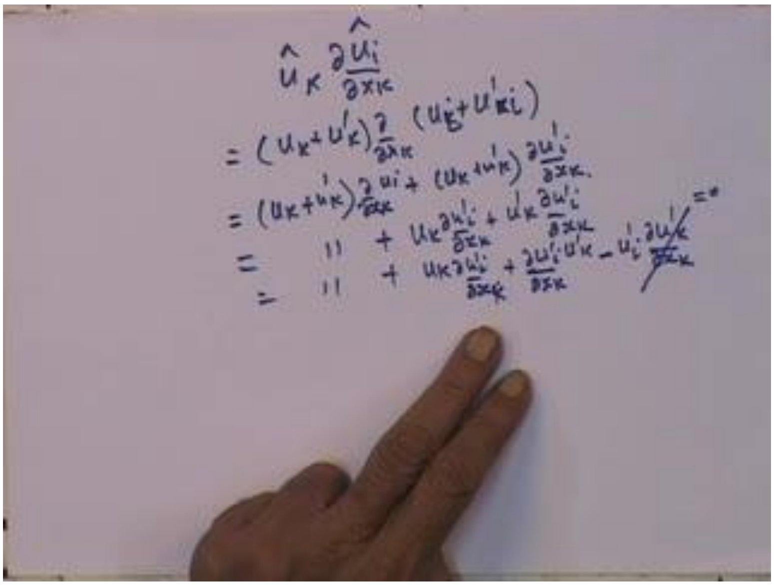

Now, what I am going to do is explain how the second term is modified. u k du i by dx k is written as u k plus u dash k d by dx k of u i divided by u dash i. That gives me the first term as u k plus u dash k d by dx k of u i plus u k plus u dash k of du dash i by dx k. So, this term remains as it is; plus u k du dash i by dx k plus u dash k du dash i by dx k. This equals u k du dash i by dx k plus du dash i u dash k by dx k minus u dash i du dash k by dx k. Now, that term is 0 because of continuity and you will see this is what I have written here. So, you have three terms u k plus u dash k du i by dx k plus u k du dash i by dx k du dash k u dash k u dash i by dx k equal to all that.

If I subtract from this equation (Refer Slide Time: 10:52), the RANS equation. You will recall - what is the RANS equation?

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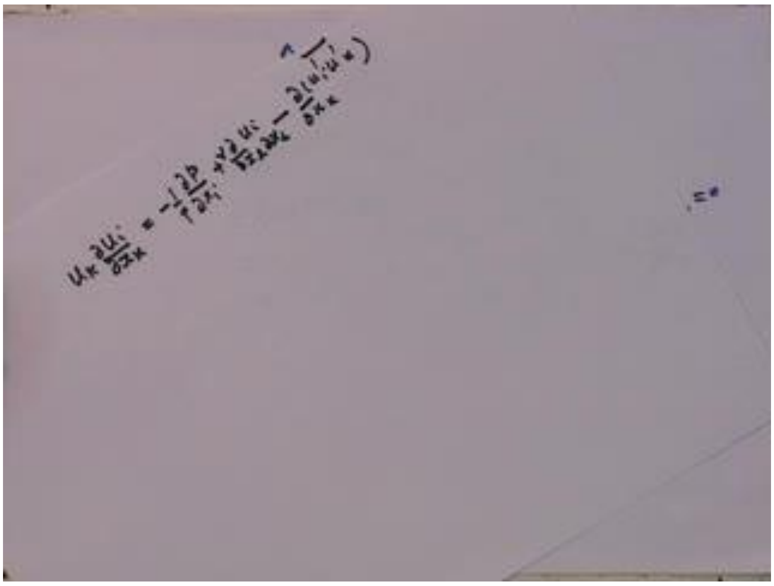

The RANS equation is u k du i by dx k equal to minus 1 over rho dp by dx i plus d by dx k u i into nu minus du dash i u dash k by dx k.

If I subtract this equation from this equation (Refer Slide Time: 11:46) that you see here, then you will notice that I would get the following equation:

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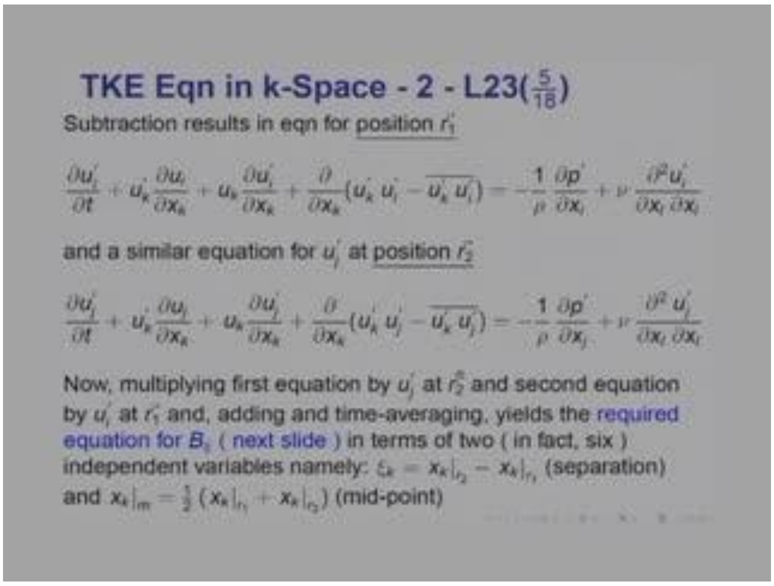

du i dash i by dt plus u dash k du i by dx k plus u k du dash i by dx k plus d by dx k into u dash k u dash i minus u dash k u dash i; time average is equal to minus 1 over rho dp dash by dx i plus nu d 2 u dash i by dx l dx l. So, this is simply subtracting the mean RANS equation from the instantaneous form of the momentum equation. If I say this is the equation for u dash i at position r 1, I could also write a similar equation for u dash j at another position r 2.

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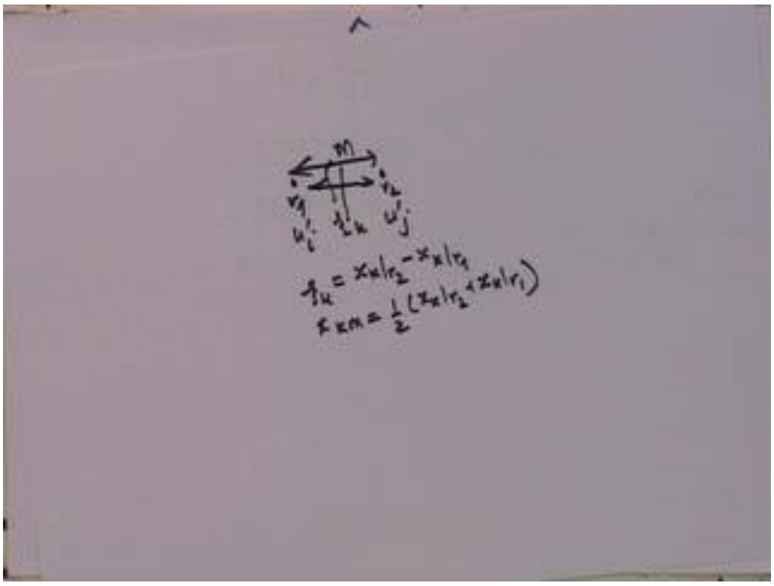

That would read as let say position r 1, where I wrote an equation for u dash i and position r 2, where I wrote an equation for u dash j.

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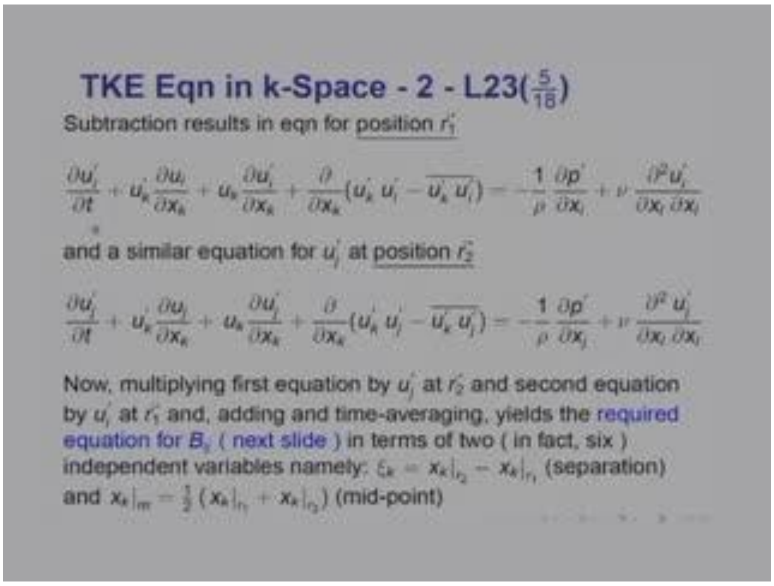

That equation would again read very much like the first one with the dp dash by dx j here and these terms. To do further development, we multiply the first equation; this equation at position r 1 by u dash j at r 2 and the second equation by u dash i at r 1. Then, we add the two equations and time average, which would yield the required equation for B i j, which I will show on the right hand side with 2 independent variables xi k; which is the difference between these (Refer Slide Time: 13:44). So, xi k equal to x k at r 2 minus x k at r 1 and x k mean equal to half of x k at r 2 plus x k at r 1. So, you have the midpoint m and the difference xi. So, we have two independent variables in this equation. This may be little bit unfamiliar to you, but it is a straightforward algebra to show that this is indeed the case.

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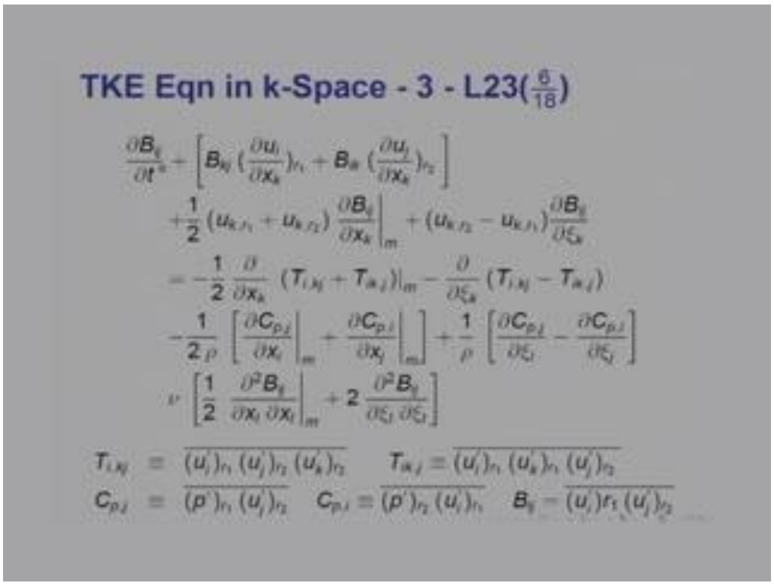

The total equation for non-homogeneous anisotropic turbulence would read like this. dB i j by dt plus B k j du i by dx k at r 1 plus B i k du j by dx k at r 2 plus half of u k r 1 by u k r 2 into d B i j by dx k at mean point plus u k r 2 minus u k r 1 into dB i j by d xi k; equal to minus half of T i k j plus T i k j; T i k j here is defined as u dash i at r 1 multiplied by u dash j at r 2 multiplied by u dash k at r 2; T i k j the second term is defined as u dash i at r 1 multiplied by u dash k at r 1 multiplied by u dash j at r 2. So, you get that transfer term minus d by d xi k based on the difference between the two points. Then, minus 1 over 2 rho dC p j by dx i at m; C p j is the product of pressure fluctuation at r 1 multiplied by u dash j at r 2. That is the C p j. Likewise, C p i is the pressure fluctuation at r 2 multiplied by u dash i at r 1; B i j is u dash i at r 1 into u dash j at r 2; is the tensor B i j plus nu times 1 by 2 B i j d x. These are the diffusion terms.

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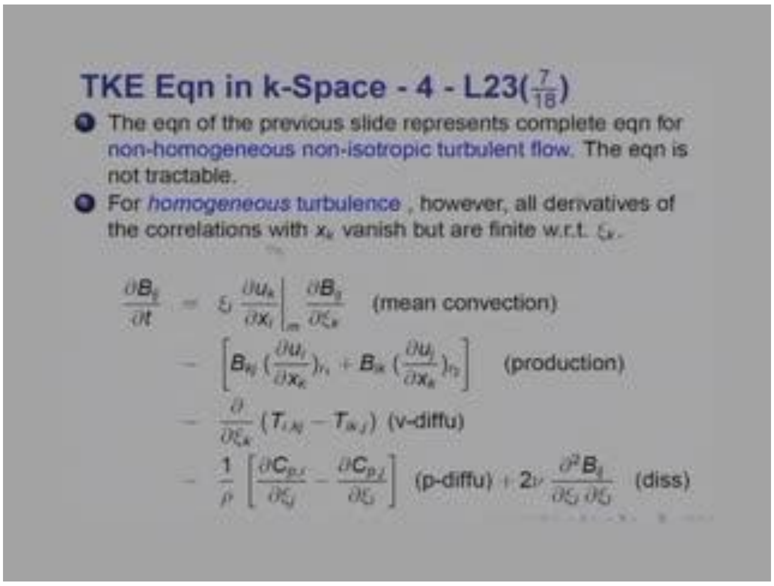

To make further progress, we postulate - the equation of the previous slide represents complete equation for non-homogeneous non-isotropic turbulent flow. The equation is not tractable because it has two types of independent variables; as you can see, xi k and x k (Refer Slide Time: 16:49). So, midpoint and the difference between the two points.

It is not tractable, (Refer Slide Time: 16:58) but we can make it tractable somewhat by postulating homogeneous turbulence. As we will recall, in homogeneous turbulence, derivatives of correlations with x k vanish, but are finite with respect to xi k, the difference.

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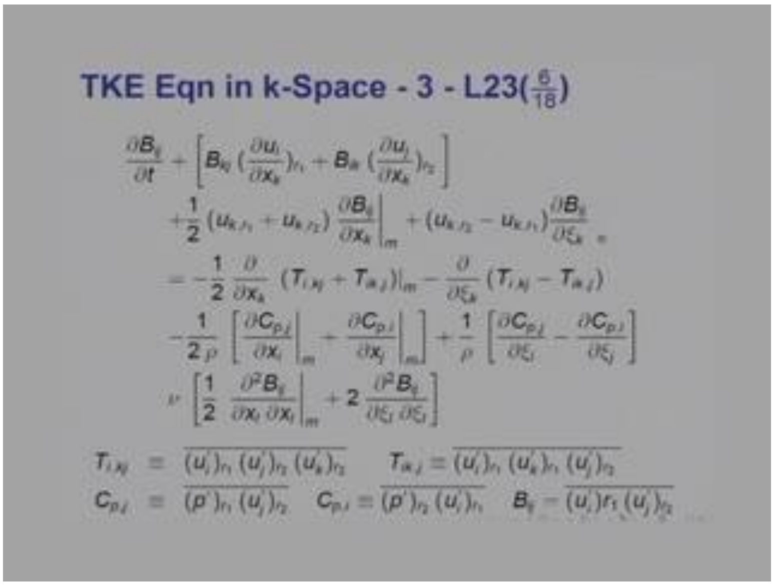

Therefore, you will get all these terms. So, this would vanish and you would get this equation (Refer Slide Time: 17:32) - dB i j by dt equal to xi l times du k by dx l into dB i j by d xi k. That is a mean convection term; this term (Refer Slide Time: 17:46). Because this is already 0, it is this term u k r 2 minus u k r 1 into dB i j by d xi k. That mean convection (Refer Slide Time: 17:56) plus production. Remember: The product of stress multiplied by rate of strain means strain; gives you the production term. So, that becomes the production term.

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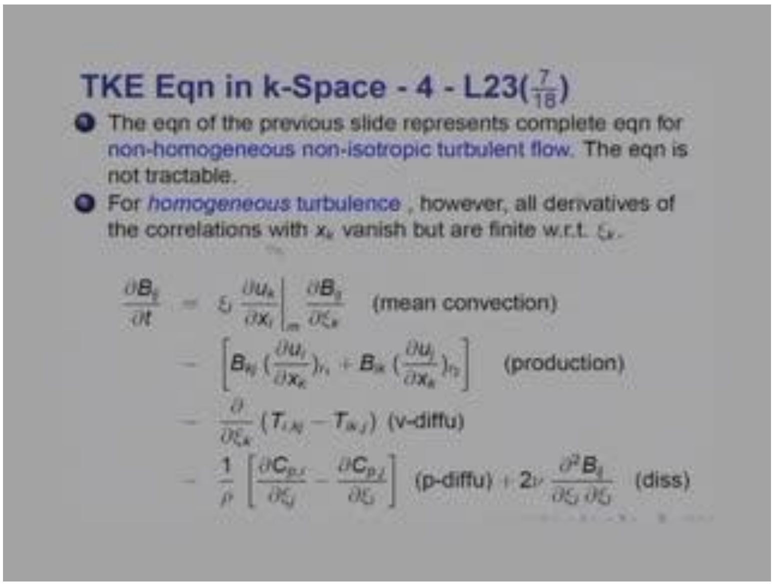

d by d xi k into tau i k j minus tau i k j is this term (Refer Slide Time: 18:20). That is the diffusion of B i j due to velocity fluctuation. So, we say that is v diffusion. Then, the pressure diffusion terms, which are given by these (Refer Slide Time: 18:37) are the pressure diffusion terms. These vanish, but these survive because these are the derivatives with respect to xi.

Then finally, this term (Refer Slide Time: 18:48) survives because it is the derivative with respect to xi l xi l. That is, 2 nu times d 2 B i j by d xi l d xi l, which is really the dissipation due to viscosity. So, you have fairly complex equation.

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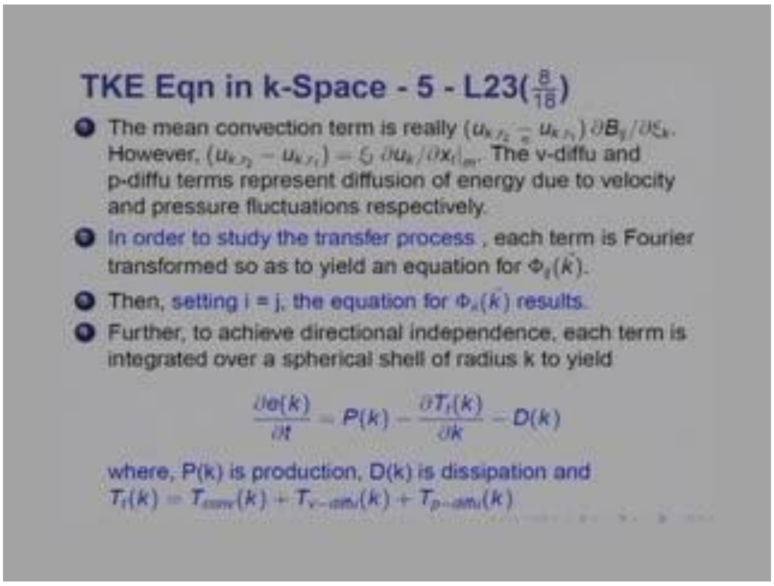

Now, the mean convection terms xi l times du k by dx l into dB i j by d xi k is really the u k r 2 minus u k r 1 dB i j by d xi k as I explained. However, u k r 2 minus u k r 1 will be xi l du k by dx l at m. Therefore, we have written that as that. This is (Refer Slide Time: 19:30) u k r 2 minus u k r 1.

The v diffusion and p diffusion terms represent diffusion of energy due to velocity and pressure fluctuations respectively. In order to study the transfer process, each term of this equation is Fourier transformed so as to yield an equation for phi i j k bar. So, we will have a differential equation in Fourier space of phi i j k bar. If you then set i equal to j, we would get an equation phi i i k bar.

Further, to I achieve directional independence, each term is integrated over a spherical shell. I left out all the algebra here to yield de k by dt equal to P k minus dT t k by dk minus D k, where P k is production; D k is dissipation and T t k is the transfer term due to convection due to v diffusion and due to p diffusion. That is, the diffusion due to pressure fluctuation; diffusion due to velocity fluctuation and this is diffusion due to mean convection.

Essentially, then we get an equation in scalar k space. It is a one-dimensional equation because there is only one independent variable in wave number space k; dk. It is a transient equation de k by dt equal to production minus d T t k by dk minus D k, which is the dissipation.

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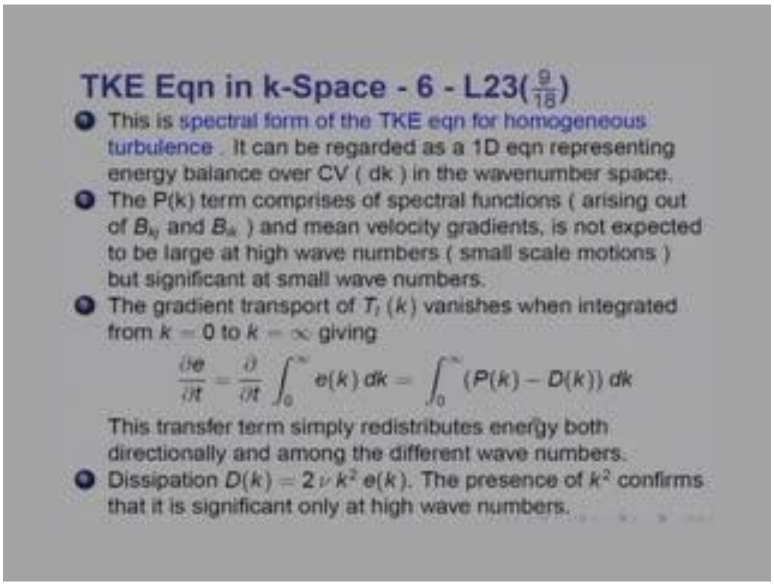

We can look up on this equation, which is the spectral form of the turbulent kinetic energy equation for homogeneous turbulence. It is nothing but a one-dimensional equation representing the energy balance over a control volume dk in the wave number space.

The P k term comprises spectral functions arising out of B k j and B i k and multiplied by the mean velocity gradients, is not expected to be large at high wave numbers; where the velocity fluctuations have almost died down and viscosity has taken over. So, the P k term would largely dominate at small wave number.

The gradient T t k term would vanish when integrated from k equal to 0 to k equal to infinity giving de by dt equal to d by dt integral 0 to infinity e k dk equal to integral 0 to infinity P k minus D k d k. This transfer term simply redistributes energy both directionally in the components as well as along the wave numbers. That is, along the sizes. Therefore, T t means total transfer term.

The dissipation term after Fourier transforming of this term (Refer Slide Time: 22:53) would be written as 2 nu k square e k. The presence of k square confirm that it would be significant only at very high wave numbers; k square suggests that it would be significant at very high wave numbers.

As I said, I have left out the algebra because it is considerably long. However, the essential ideas that we had seen in RANS equations from which we derived the turbulent kinetic energy in physical space and the equation we have now derived in the wave number space have very similar characteristics is that a rate of change of energy is equal to the rate of its production minus net rate of its transfer to smaller scales and minus dissipation.

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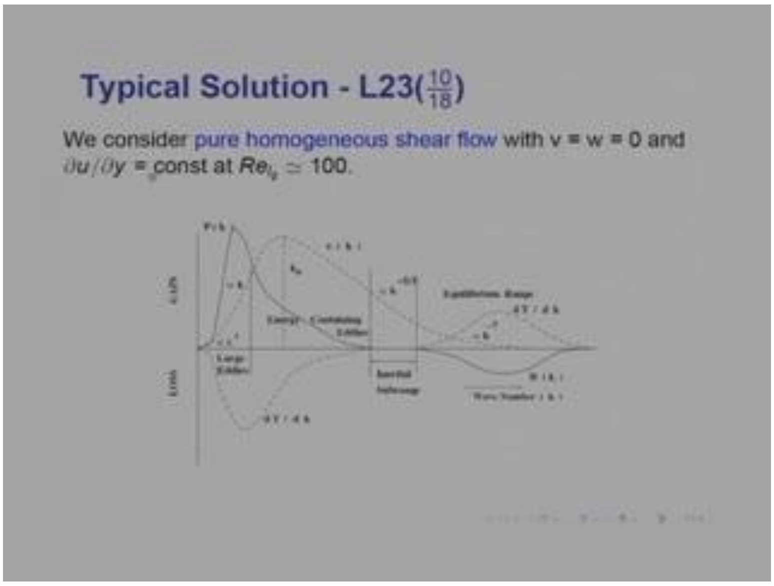

If you consider a flow in which homogeneous shear flow in which v and w are 0, the mean velocities are 0 and du by dy is a constant, then here are the results computed at the turbulence Reynolds number based on transfers Taylor micro-scale of about 100. Looks something like this. You see that loss means negative contribution to energy balance; gain means positive contribution to energy balance. You see that the production term is indeed very high at small wave numbers. This is the wave number access at small wave numbers. In fact, it peaks at very small wave numbers. Therefore, it is called large eddies; as I said, very large eddies at which the production dominates. However, it declines as the wave number increases. In this range, the transfer term is negative, which means energy is being sucked out from large eddies and being push towards smaller eddies; energy is continuously being fed to smaller eddies.

A point is reached where the production almost dies down and so does the transfer almost died down. There can be a region called the inertial sub range about which I will talk in a minute. However, dissipation takes over. As you can see, it is shown as negative dissipation, which occurs at very large wave numbers. The transfer term becomes positive at large wave numbers and again dies down. As we would expect the area under the negative sign here (Refer Slide Time: 25:49) of the transfer term must equal the area under the positive part of dt by dx k so that the net area is 0 in this.

The energy itself raises to very large value e k and peaks at some wave number, which is designated by k sub e and then begins to decline;. energy itself is goes on declining. Then, it falls here at the rate proportional to k to the power of minus phi by 3 and finally, keep very rapidly to k to the power of minus 7.

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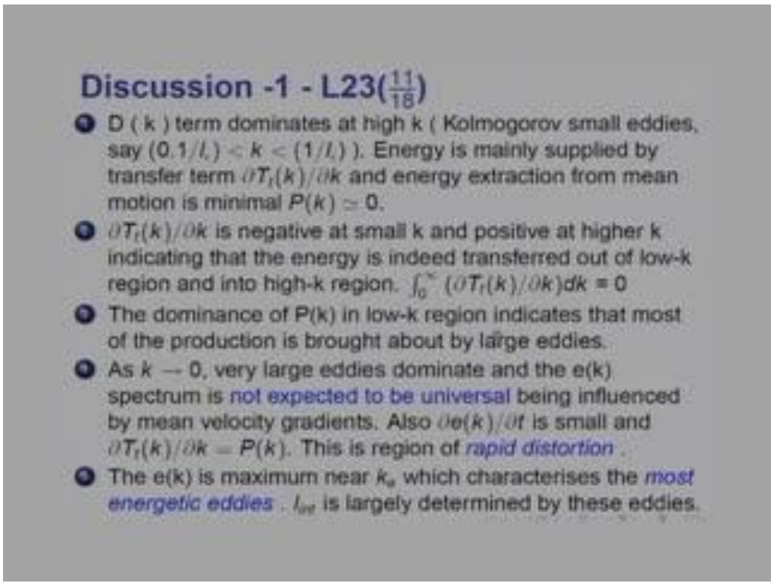

What does all these indicate? The D k term dominates at very high k, which means large wavelengths or Kolmogorov small eddies; say of the order of 0.1 divided l sub epsilon k to 1 divided by l sub epsilon. That is the range of wave numbers in this part here (Refer Slide Time: 26:52) 0.1 divided by 1 over l epsilon to 1 over l epsilon.

Energy is mainly supplied by the transfer term dT t k by dk and the energy extraction from the mean motion is minimal as we saw here (Refer Slide Time: 27:07). The energy for dissipation is essentially supplied by the transfer term and the production term itself contributes nothing to energy dissipation.

The transfer term is negative at small k and positive at high k indicating that energy is indeed transferred out of low-k region into high-k region. Therefore, the integral of dT t k by dk into dk is 0.

The dominance of P k in the low k region indicates that most of the energy production is brought about by large eddies. As you saw here (Refer Slide Time: 27:44), most of the production is brought about by large eddies. That is why I have said large eddies here. As k tends to 0, very large eddies dominate and the e k spectrum is not expected to be universal at all, being influenced by the mean velocity gradients. Also de k by dt is small and dT t k by dk equals production. This means all the energy produced simply goes into its transfer to smaller and smaller eddies or bigger and bigger wavelengths. Therefore, this region (Refer Slide Time: 28:29) is called the region of rapid distortion.

The e k is maximum near k sub e which characterizes the most energy-containing eddies. The l integral where we saw in the previous lecture is largely determined by these eddies. The characteristic eddies of this region (Refer Slide Time: 28:52) recall the energy containing eddies this is where the l integral corresponds somewhere to k sub e.

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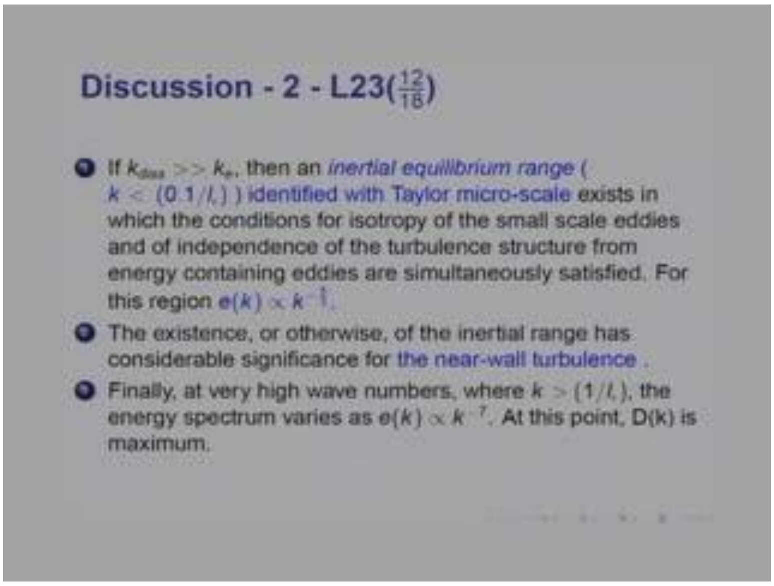

If k dissipation is much greater than the energy containing wave number k sub e, then an inertial sub range exists at k less than 0.1 divided by l sub epsilon. As you can see (Refer Slide Time: 29:20), if k dissipation is much bigger than K sub e, then a range exists. As you can see here, (Refer Slide Time: 29:31) k less than 0.1 divided by l epsilon identified with Taylor micro-scale exists in which the conditions of the isotropy of the small scale eddies and of independence of turbulence structure from energy containing eddies are simultaneously satisfied.

In other words, this range of wave number (Refer Slide Time: 29:53) is small enough for isotropy to prevail; is a range in which the isotropy of the large scale and the independence of the turbulence structure from energy containing eddies are simultaneously satisfied. In this range, e k is proportional to k to the power of minus 5 by 3. This is very important. Many experiments always want to ensure that they are turbulent and sufficiently vigorous so that they can take it as fully turbulent or not. This means they are the large wave numbers, where dissipation takes place and the small wave numbers; the energy containing wave numbers, where the production takes place. Are they sufficiently separated or not? Because if they are separated, then an inertial sub range must exists with k raised to minus 5 by 3 as the energy spectrum. If it exists, they would declare such turbulence as being truly representative of vigorous turbulence.

The existence, or otherwise, of the inertial sub range has considerable significance for the near-wall turbulence. This is the matter, which is employed in turbulence modeling, which may be used a little later. Finally, at very high wave numbers, where k is greater than 1 over l epsilon, the energy spectrum varies as e k is proportional to k raised to minus 1. At this point, D k is maximum as we saw earlier.

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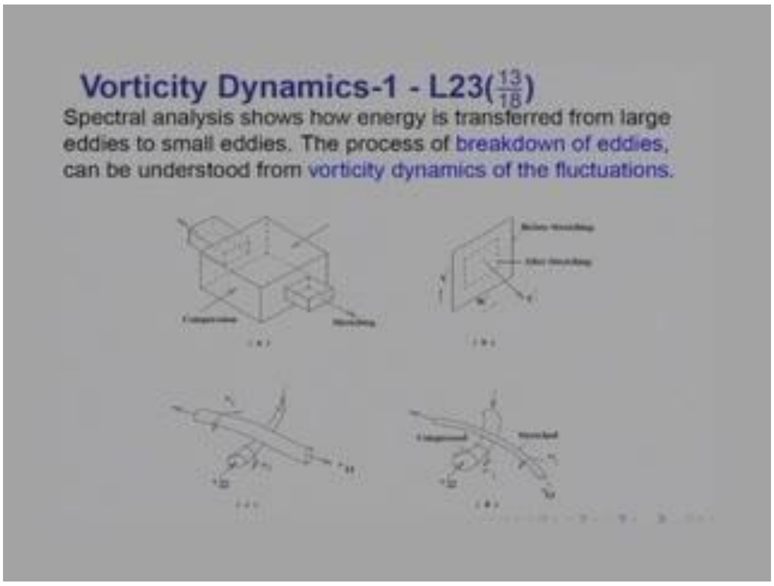

Spectral analysis gives you how continuously the energy is transferred from large eddies to small eddies or from small wave numbers to large wave numbers. Vorticity dynamics is another way of explaining the same phenomenon. That is, by the process of breakdown of eddies. For example, consider a three-dimensional cubic element here. This is direction 1, direction 2 and direction 3. Imagine that this element is being stretched because of the strain rate in direction 1. If the turbulence was very high where viscosity would have very little influence, then you will see that the plane of cross-section must shrink as I have shown here. Because of this stretching in x direction, the cross-section in y and z direction or the length scales associated with y and z direction must decrease.

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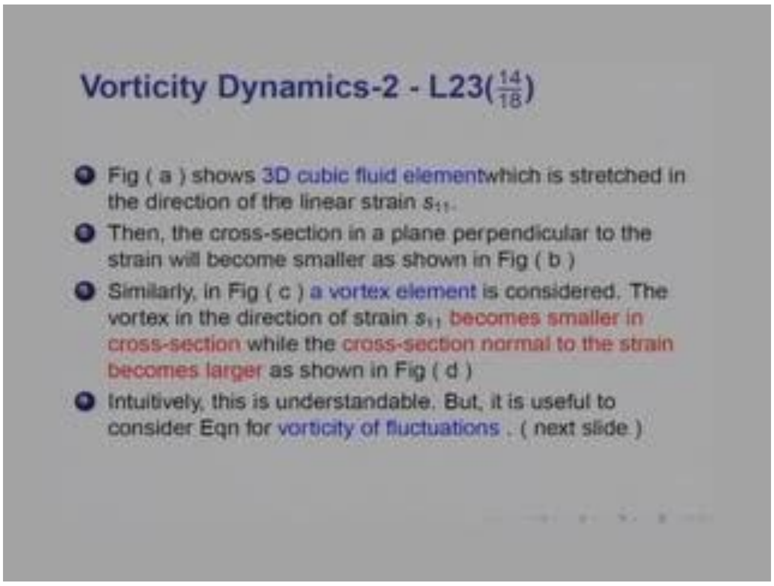

Figure a shows 3D cubic fluid element, which is stretched in the direction of the linear strain s 1 1. Then, the cross-section in a plane perpendicular to the strain will become smaller as shown in figure b.

Similarly in figure c, I now consider a vortex element rather than a cubic element. The vortex in the direction of strain s 1 1 becomes smaller in cross-section as you can see here (Refer Slide Time: 33:19). This is S 1 1 and the vortex is being stretched. So, it becomes smaller in cross-section while the cross-section normal to the strain becomes larger as shown in figure d. So, the cross-section in this direction (Refer Slide Time: 33:39) would become larger while in the stretch direction, it would become smaller from this to there.

Now, intuitively, this is understandable, but it is useful to consider equation for vorticity of fluctuations on the next slide. I will return to 3D cubic element after this explanation.

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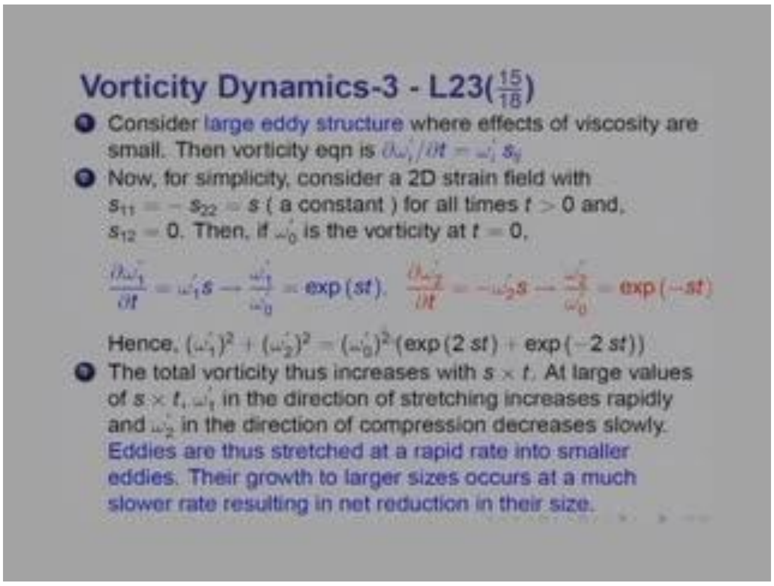

Consider large eddy structure, where effects of viscosity are small. Then vorticity equation will read as d omega dash i by dt equal to omega dash i into s i j. Again, vorticity equation is simple; I have not derived here the full form of it. However, when viscosity goes to 0, you could very well take it as that at the moment for the purpose of discussion.

Now, for simplicity, consider a 2D strain field in which only s 1 1 is equal to minus s 2 2; s 1 1 being stretching, tensile; s 2 2 being compressive; is equal to s and that is equal to constant. These are strain fields for all times t greater than 0 and s 1 2 is equal to 0. That is, the strain rate. If omega dash naught is the vorticity at time t equal to 0, then from this equation (Refer Slide Time: 35:07), we say that d omega dash 1 by dt would be equal to omega dash 1 s. This is because s 1 1 is positive, which would give the solution omega dash 1 divided by omega dash naught equal to exponential of plus st. Likewise, d omega dash 2 divided by dt will be equal to minus omega dash 2 s equal to omega dash 2 divided by omega dash naught equal to exponential of minus st. Therefore, if I were to consider omega dash 1 square plus omega dash 2 square as the total vorticity; that would be equal to omega dash naught square exponential of 2 st plus exponential of minus 2 st.

Therefore, the total vorticity increases with s multiplied by t at large values of s multiplied by t omega dash 1 in the direction of stretching increases very rapidly because of the plus. Omega dash 2 in the direction of compression decreases slowly because of the minus. What does is tell us? This tells us that the eddies are thus stretched at a rapid rate into smaller eddies. However, their growth to larger eddies occurs at a much smaller rate resulting in the net reduction in their size. This is a very important observation to make from vorticity dynamics.

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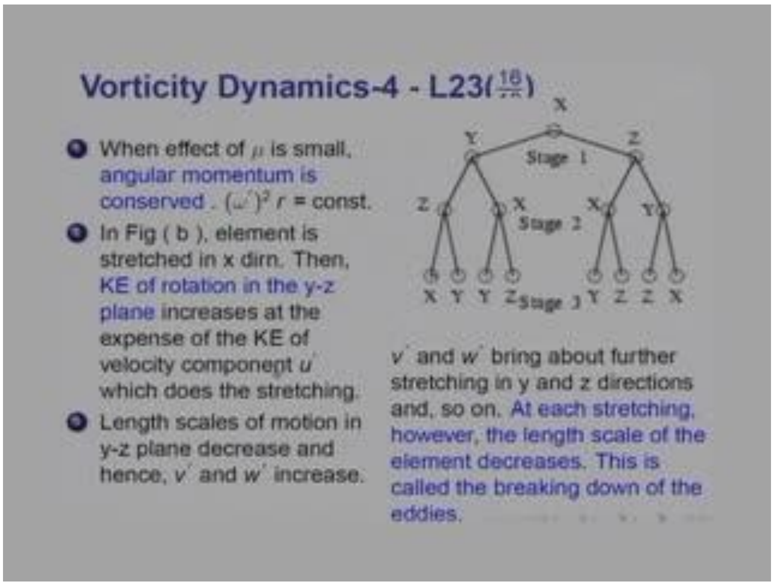

Now, for the same case when effect of mu is small, then the angular momentum must be conserved or omega dash square r equal to constant.

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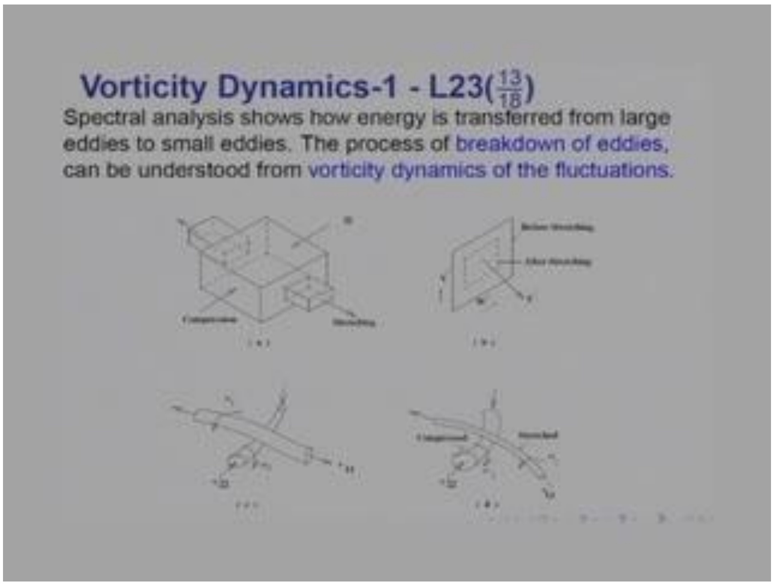

If we now go back to cubic element, I am stretching it in x 1 direction. Therefore, it shrinks in the y-z plane. However, since angular momentum must reduce length scale, which means increase intensity of vorticity. This would mean v dash and w dash must now increase compared to the state before stretching, whereas u dash is doing the stretching.

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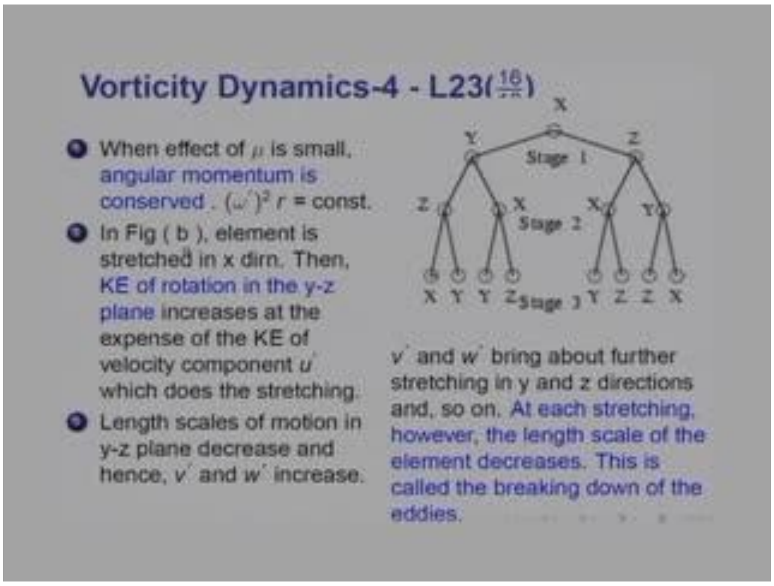

Say in figure b, element is stretched in x direction. Then, the kinetic energy of rotation in the y-z plane increases at the expense of the kinetic energy of the velocity component u dash, which does the stretching. Therefore, the length scales of motion in y-z plane decrease and hence, v dash and w dash increase.

Now, this increased v dash and w dash will bring about further stretching in y and z directions and so on. So, you can see that stretching in x direction brings about intensification of velocities in y and z directions, which do further stretching. So, intensification in y will do stretching, which will increase the intensity in z and x direction while reducing the scale. Similarly, intensified w dash would bring about reduction in size in x and y directions and so on and so forth. However, at each stretching, the length scale of the element will go on decreasing. This is called the breakdown of the eddies.

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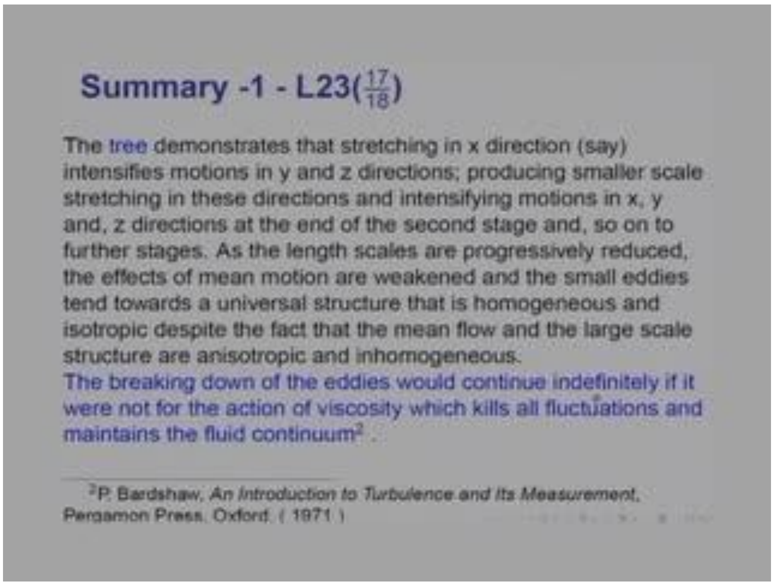

In summary, I can say the tree demonstrates that the stretching in x direction intensifies motions in y and z directions; producing smaller scale stretching in these directions and intensifying motions in x y and z directions at the end of the second stage and so on to the further stages. As the length scales are progressively reduced, the effects of mean motion are weakened and small eddies tend towards a universal structure that is homogeneous and isotropic despite the fact that the mean flow and large scale structure are anisotropic and inhomogeneous.

The breakdown of eddies would continue indefinitely if it were not for the action of viscosity, which finally, kills the fluctuations and maintains the fluid continuum. This is the notion, which we started with; we questioned - whether the fluid continuum survives?

In our previous lecture, the first slide - whether the fluid actually spiked up? By observing that the scales at the molecular levels and the scales at the turbulence level being so far separate, we concluded there that the turbulence in no way influences the molecular motions and vice versa. Therefore, there is no question of continuum being brought under question.

As the analysis shows, continuum indeed prevails at small in turbulence. Therefore, as we showed here, (Refer Slide Time: 40:14) the breaking down of the eddies would continue indefinitely if it were not for the action of viscosity, which kills all fluctuations and maintains the fluid continuum.

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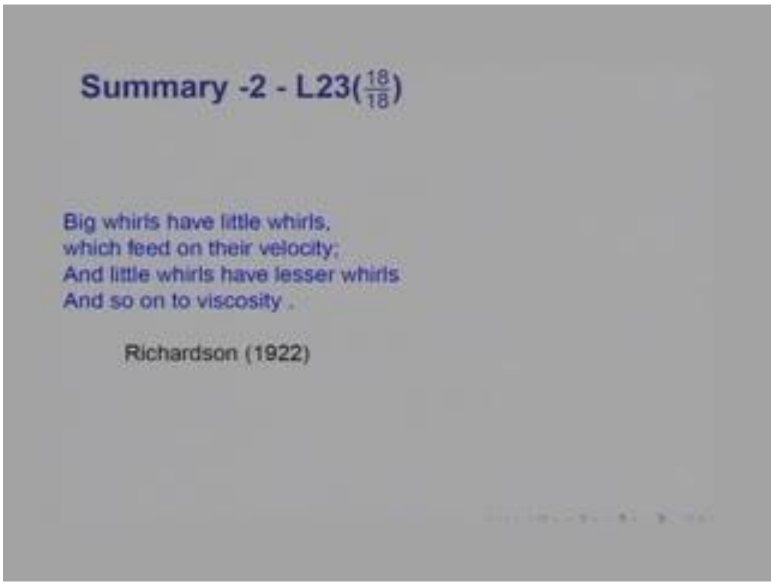

Now, this idea is very well captured in a beautiful poem by Richardson. Big whirls have little whirls, which feed on their velocity; and little whirls have lesser whirls and so on to viscosity. This poem was written in 1922. The turbulence modules while appreciating the poem very much because it explains to the layman how turbulence sustains itself, they often point out that the existence of the inertial sub range is not revealed in this poem.

However, this was the poem and not science. The existence of inertial sub range in which you have independence from energy containing eddies and the isotropy of the dissipating eddies is not made explicit in this poem. That is well taken, but nonetheless to a layman at least. It is a good take-home message to remember that big whirls have little whirls, which feed on their velocity; and the little whirls have lesser whirls and so on to viscosity. So, big whirls indicating large-scale eddies, which creates small-scale eddies and small-scale eddies creates smaller-scale eddies and so on to viscosity.