Convective Heat and Mass Transfer

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Module No. # 01

Lecture No. # 16

Fully - Developed Laminar Flows – 2

In the previous lecture, we calculated friction factor for a fully developed laminar flow in regular ducts like a circular tube or an annulus or a rectangular duct or annular sector. We used simple algebraic methods as well as the Fourier method.

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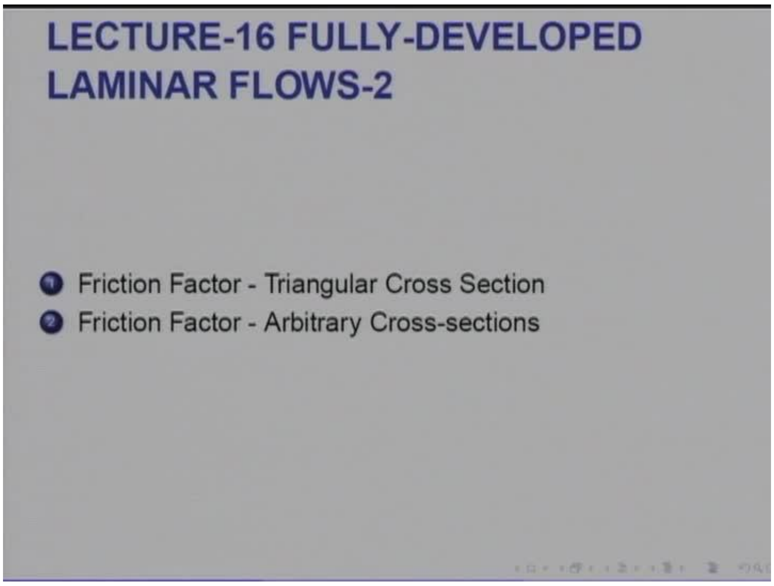

In today's lecture, I am going to consider ducts of even more complex cross-section, such as a triangular duct and a duct of any arbitrary cross-section. So, let us begin with the duct of triangular cross-section and apply one method called the control, which is a variational method.

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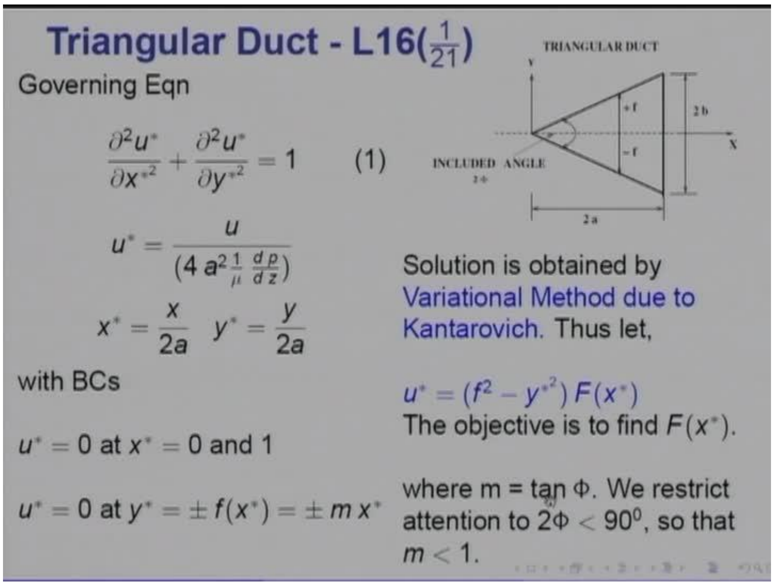

The figure shows a triangular duct of base 2b and height 2a with an included apex angle 2 phi. The governing equation would be d 2 u dx square plus d 2 u dy square equal to 1 over mu d p dz. I can non-dimensional it, in this manner as u star equal to u divided by 4 a square 1 over mu d p d z and x star equal to x divided by 2a and y star equal to y over 2a.

x is measured from the apex and also the y. So, the origin is the apex of the triangle of course. This is a Poisson's equation with right hand side equal to constant, but notice that the boundary conditions given at x star equal to 0. It is the apex itself and x star equal to 1, which is the base. U star is also equal to 0, where y is equal to plus minus f x and that f is a function x itself.

We have a boundary, which does not have a constant y, which is a function of x. In this case, a linear one with m equal to tan phi. This angle is 2 phi and so phi is half the angle. For such an equation, solution can be obtained by variational method due to Kantarovich.

Let u star be a function of x and y. We have given it as f square minus y square multiplied by some function of x star. The objective is to find out what is F x star, by satisfying the boundary conditions and the equation. We will restrict our attention to 2 phi less than 90, so that m is always less than 1 and remember, m is equal to tan phi.

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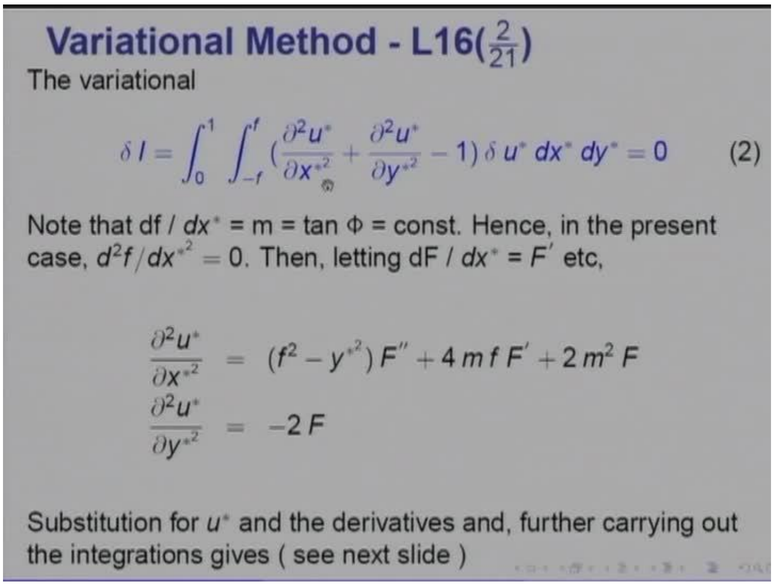

The variational principle is stated like this: the variational I is 0 to 1into integral minus f to plus f the equation itself multiplied by variation and u star into dx star into dy star is equal to 0. Now, df by dx star is equal to m equal to tan phi is equal to constant. Hence, in this case, d 2 f dx star square will be 0.

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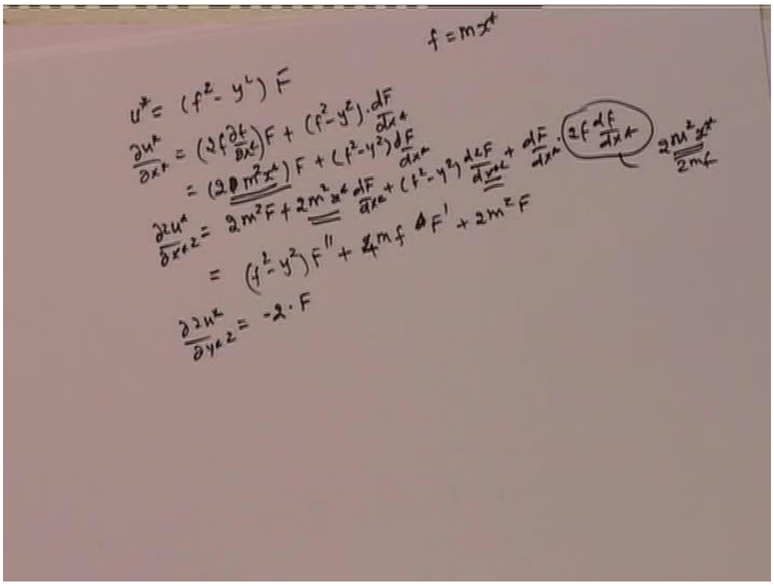

You will see that the equation u star is equal to f square minus y square F. Therefore, du star by dx star will be 2f df by dx into F plus f square minus y square into dF by dx star.

Remember, f is equal to m times x star and therefore, this simply becomes 2 F m into m square x star into F plus f square minus y square dF by dx star. Now, d 2 u dx star square will become 2 m square F plus 2 m square x star dF by dx star plus f square minus y square d 2 f star by dx star square plus dF by dx star into 2 f dF by dx star. This is equal to 2 m square x star. Therefore, you will see that this equation d 2 u dx star square would be written as f star square minus y square F double prime plus this 2 m square x star, which is essentially 2 m f and so this is 2 m f. This becomes 4 m f dF by dx star, which is F dash plus 2 m square F. Likewise, d 2 u star by d y star square will be simply be equal to minus 2 times F. If I substitute this in this equation minus 1 delta u star and I carry out the integration first with respect to y star.

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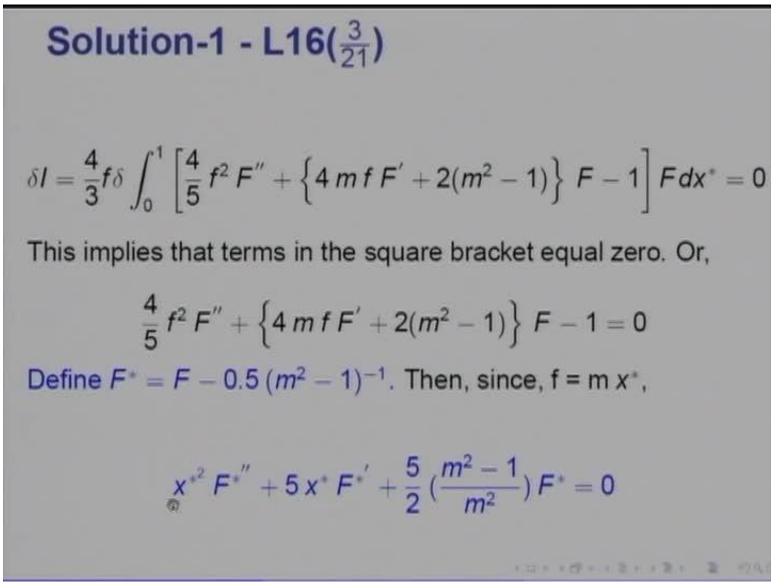

I will get 4 by 3 f delta variation of 0 to 1 into 4 by 5 f square F double prime plus 4 m f F prime plus 2 m square minus 1 whole square into F minus 1 into F dx star is equal to 0. This shows that the term in the square bracket must be 0 or 4 by 5 f square F double prime plus this term minus 1 equal to 0. If I now define F star, it is equal to F minus 0.5 divided by m star minus 1. Since f is equal to m x star, You will see that I get x star square plus F double prime plus 5 x star F star prime into this plus prime.

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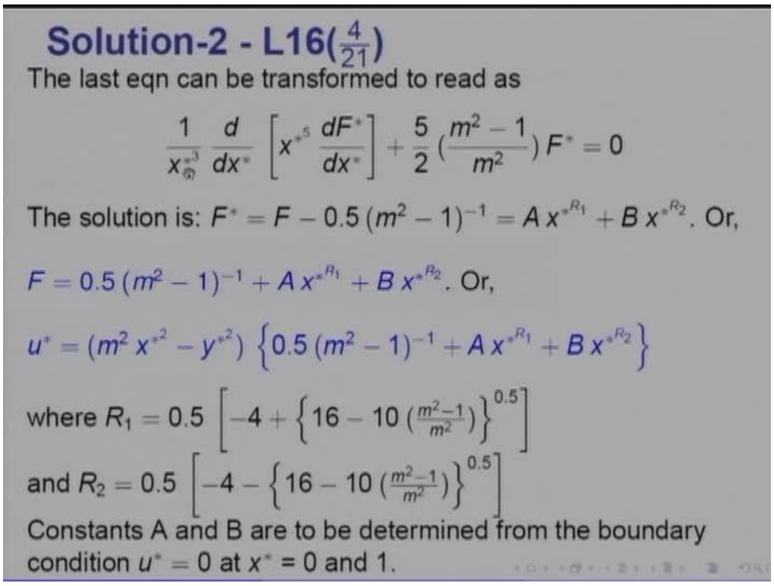

I got the second order equation in F star. This equation can be cast in this form - 1 over x star cube. Remember, this equation can be cast in this form - 1 over x star cube d by dx star x star 5 over d phi by dx star into 5 by 2 function of m and F star equal to 0.

The solution is, F star equal to F minus 0.5 into m square minus 1 whole power minus 1 is equal to A x star raised to R 1 plus B x star square raise to R2 or F itself. It is just 0.5 m square minus 1 raised to minus 1 into A x star raised to R 1 plus A x star raised to R2. U star is given by this expression (Refer Slide Time: 08:38). Remember, u star is related to f in our equation in this manner. Therefore, we have obtained F and y. So, R 1 will turn out be this and R 2 will turn out be this. Constants A and B are to be determined from the boundary condition, u star equal to 0 at x star equal to 0 and 1. It is the apex and the base of the triangle.

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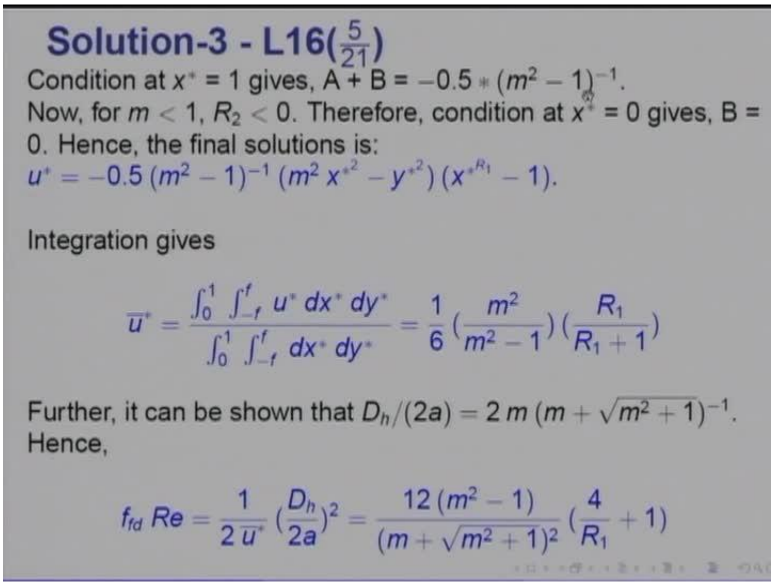

Condition at the base x star equal to 1 gives A plus B equal to that. I have put x star equal to 1 and I get this A plus B equal to this; for m less than 1 because we are restricting to 2 phi less than 90. So, tan phi or m would always be less than 1. R 2 is less than 0 and remember, R 2 will be less than 0. Therefore, at x is equal to 0 would tend to infinity, which is of course unacceptable and therefore, B itself must be 0.

As a result, u star turns out to be a function with x star raised to R 1 minus 1. Integration of this gives us the u bar star. This evaluates to this quantity, R 1divided by plus R 1 plus 1. Further, the hydraulic diameter for this particular triangle can be shown equal to as 2 m into m plus square root of m plus 1 raised to minus 1. Hence, friction factor multiplied by Reynolds number is a function of m. It is the tangent of the half angle and a function of R 1 and R 1. As you remember, it is again a function of m. In this particular problem (Refer Slide Time: 10:47), after considerable mathematical manipulations, we have shown that the friction factor would be function of the included angle.

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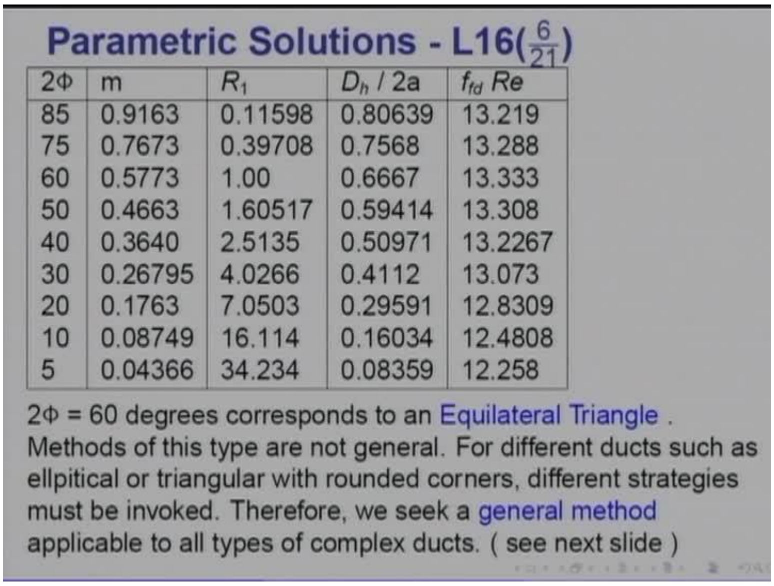

Parametric Solutions - L16($\frac{6}{21}$)

2Φ	m	R_1	$D_h / 2a$	$f_{fd}\,Re$
85	0.9163	0.11598	0.80639	13.219
75	0.7673	0.39708	0.7568	13.288
60	0.5773	1.00	0.6667	13.333
50	0.4663	1.60517	0.59414	13.308
40	0.3640	2.5135	0.50971	13.2267
30	0.26795	4.0266	0.4112	13.073
20	0.1763	7.0503	0.29591	12.8309
10	0.08749	16.114	0.16034	12.4808
5	0.04366	34.234	0.08359	12.258

2Φ = 60 degrees corresponds to an Equilateral Triangle . Methods of this type are not general. For different ducts such as ellpitical or triangular with rounded corners, different strategies must be invoked. Therefore, we seek a general method applicable to all types of complex ducts. (see next slide)

Here are some solutions, I begin with 85, 75, 60, 50, 40, 30, 20, 10 and 5. The value of m is the tan phi. The evaluated value of R 1 is also given. The hydraulic diameter values are also given. Here, the friction factor versus Reynolds number is evaluated for each geometry.

A special case is 2 phi equal to 60 would straightaway give us an equilateral triangle. The friction factor is 13.33 is an often quoted value. What is less often quoted are the values, which I have shown for other angles. Methods of this type are not general for different ducts such as elliptical or triangular with rounded corners. It is very commonly encountered in practical heat exchangers different strategies. It is the trial function, whose variation is to be taken.

Considerable algebra is required, before one gets solution. For each duct, one has to treat it as a special case and go through on developing the solution. Our interest now turns to more complex ducts and such methods do not apply. Therefore, we seek a general method, which can be applied to ducts of arbitrary cross-section of triangle or an ellipse or any other. Even a circular duct would be a special case and we begin with that method.

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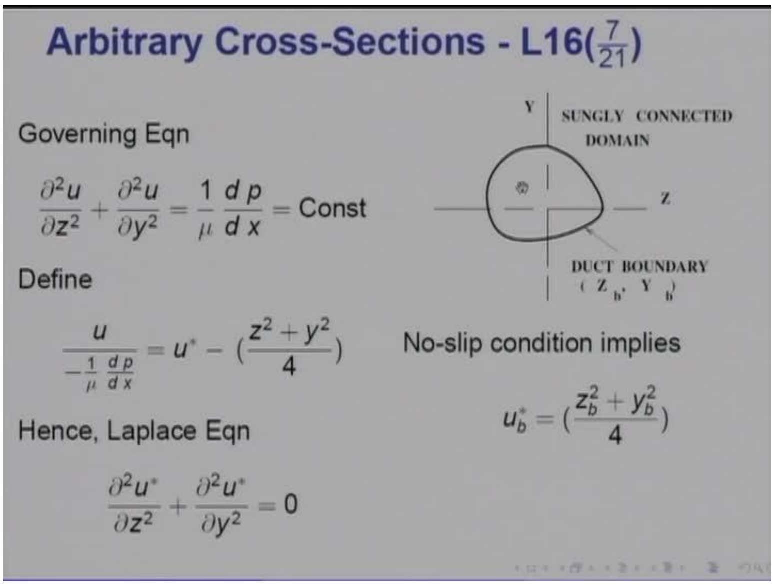

Consider a duct of arbitrary cross-section and the coordinates of the boundary. Z b, Y b are known. The flow is in the x direction. It is into the plane of your screen and the only requirement is that the domain must be singly connected.

Singly connected domain implies that once I put a pencil at any point on the domain. It does not matter, whether I go anticlockwise or clockwise, I must return to the point of origin without lifting the pen. It is the only constrain on this method. In other words, if I had a circular rod inside a solid rod, it would not apply for this method. It has to be a singly connected domain, whose boundary coordinates are known - Z b, Y b. The governing equation would be d 2 u dz square plus d 2 u dy square equal to 1 over mu d p dx equal to a constant.

If I define u divided by minus 1 over mu d p dz equal to u star minus z square plus y square by 4, then substitution of this in this equation would readily show that a Laplace equation results. In other words, whose right hand side is 0. Since u is 0 on the boundaries, u star would be equal to or u star boundary would be equal to z b square plus y b square by 4. U star b is a fictitious function, which takes finite values on the boundaries.

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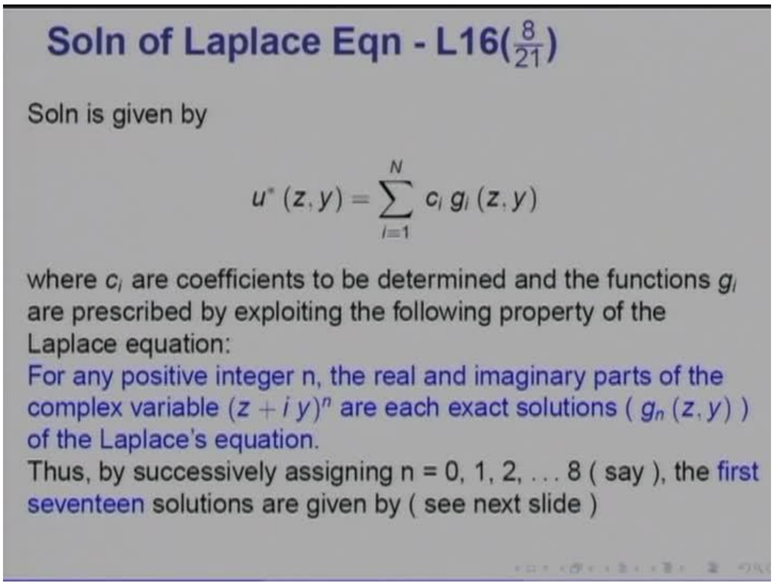

How do we solve this Laplace equation on a domain, which is completely arbitrary? The solution for such a Laplace equation is given as u star z y is equal to sum of coefficient c i multiplied by some functions g i of z, y. So, c i's are the coefficients to be determined and g i's are prescribed by exploiting a very special property of the Laplace equation. The following is the property: for any positive integer n, the real and imaginary parts of a complex variable, z plus i y raised to n are each exact solutions of the Laplace's equations.

If I assign successive values as n equal to 0, 1, 2, 8 etc, I would get the first seventeen solutions, which is usually for most complex geometries. Of course, you are welcomed to take even 10 or 12 and generate 21, 22, 25 solutions. So, the functions g i would read as follows and I will show you how they are evaluated.

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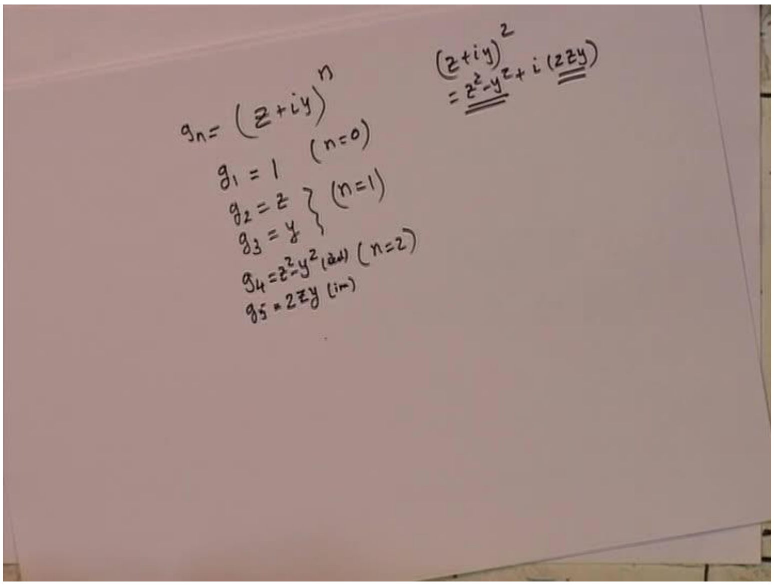

Remember, each z plus i y raised to n is a solution. If I take n equal to 0, I would simply get g 1, which is what I have shown here. Now, g 1 is equal to 1 and that is for n equal to 0. As you can see here, there is no imaginary part or a real part and therefore, g 1 would simply be equal to 1 for any value of n because n is equal to 0. If I take n equal to 1, then the real part will give me z and the imaginary part will give me y.

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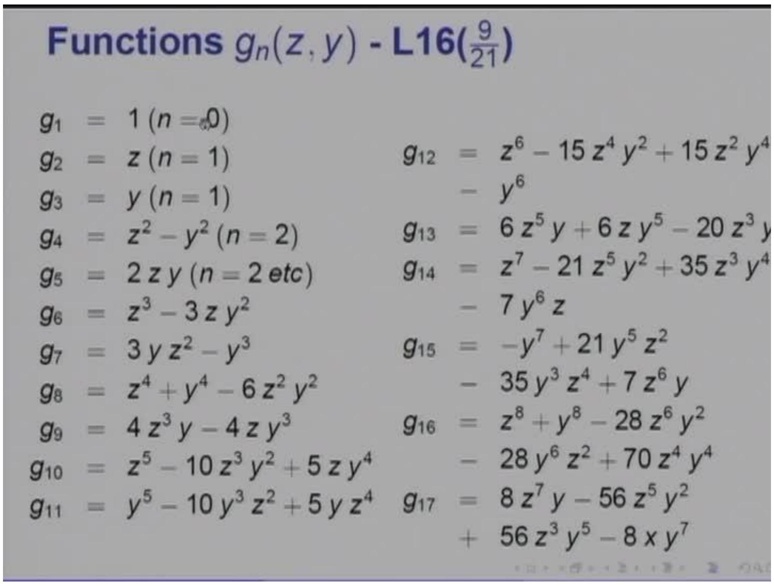

As you can see here on the screen, g 2 is equal to z and g 3 is equal to y and these two solutions for n equal to 1. If I take n equal to 2, then I will get g 4 equal to z square minus y square and g 5 will be equal to simply 2 z y. This is the imaginary part and this is the real part because z plus i y square is simply z square minus y square plus i times 2 z y. So, we take both these as solutions and this is for n equal to 2.

Likewise, you can go on taking. Here, g 6 and g 7 are the solutions for n equal to 3. G 8, g 9 are the solutions for n equal to 4, g10 and g 11 are solutions for n equal to 5, g 12 and g 13 are solutions for n equal to 6, g 14 and g 15 are solutions for n equal to 7, g 16 and g 17 are solutions for n equal to 8. As I said earlier, you could take n equal to 9 and n equal to 10 and in each case, you will get two extra terms in the equation.

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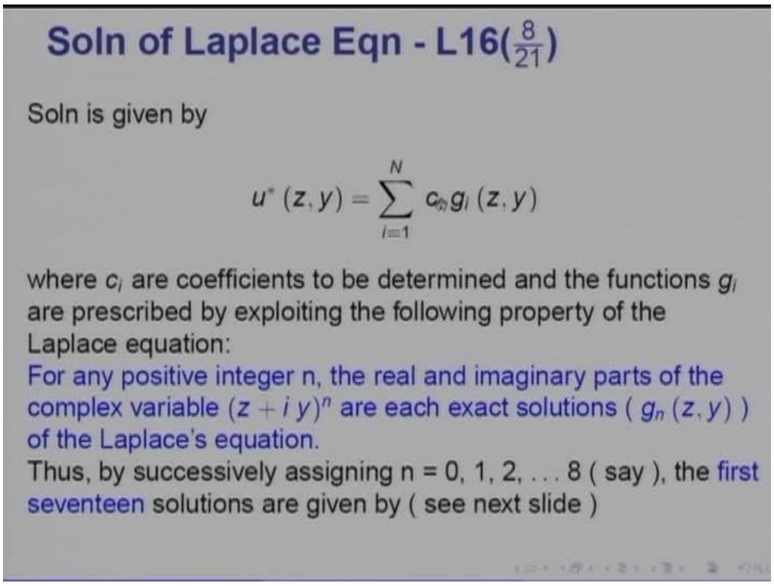

We have g i of z y’s as known and the only thing that is unknown is c i.

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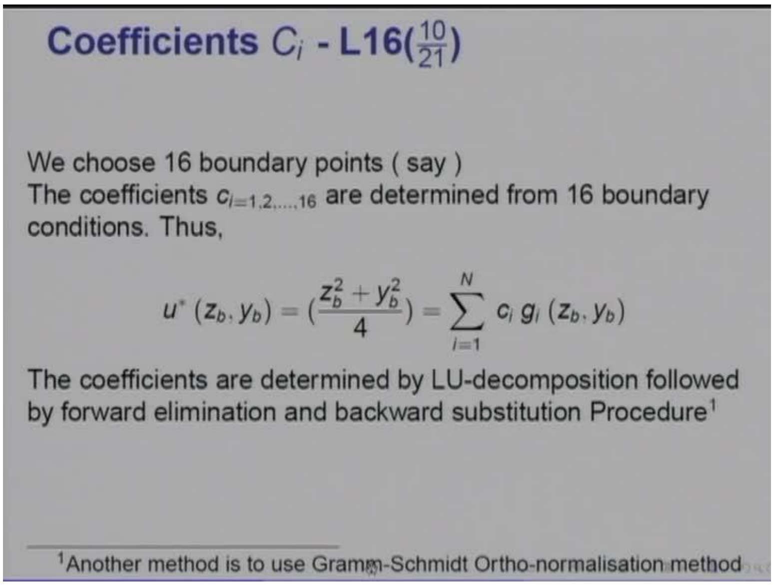

Suppose, I choose a particular duct with 16 boundary points, then I take first 16 terms in that expression. My task is to determine c i equal to 1, 2, 3 up to 16. So, 16 coefficients are to be determined from 16 boundary conditions. Thus, u star z b, y b is nothing but z b square plus y b square by 4. It is known because we know the coordinates of the boundary points equal to c I into g i of z b, y b. These are the functions that are evaluated at the boundaries.

So, essentially you get 16 equations and 16 unknowns and they have to be evaluated. One way in which way this can be done is by the method of LU-decomposition. This is a matrix solution method, followed by forward elimination and backward substitution procedure, which you must have studied in your numerical analysis course. Other method for this type of equation set is what is called as Gramm-Schmidt Ortho-normalization. I will be following LU-decomposition procedure and all the results that I will show are with LU-decomposition method. I will apply this method to a variety of ducts.

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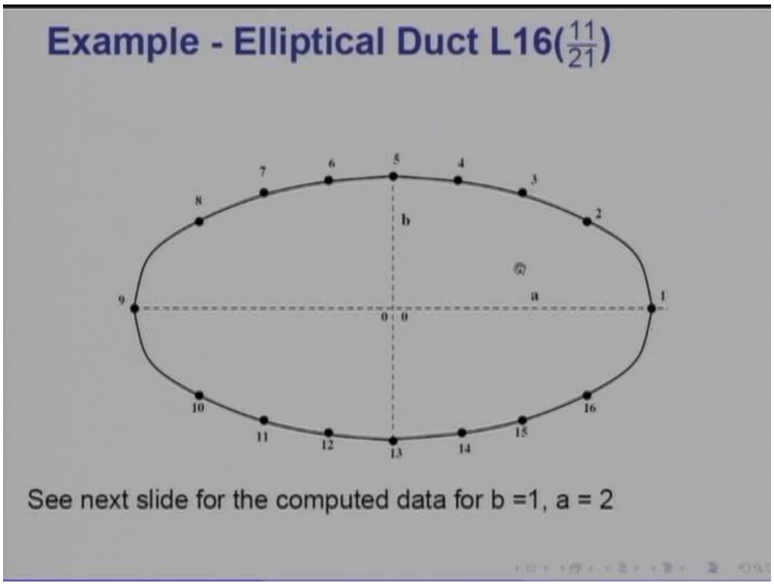

Let us take elliptical duct as shown. The origin 0, 0, major axis is a and its minor axis is b. In this particular case, I have taken a equal to 2 and b equal to 1. As you can see, I have chosen 17 boundary points. They need not be equally spaced, but I have taken them to be equally spaced. They are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16. So, I have essentially taken 16 points and applied.

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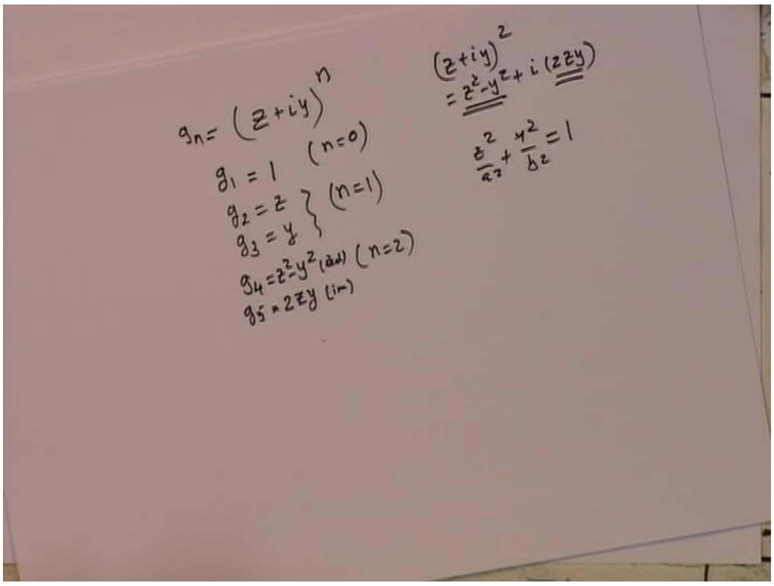

I know the coordinates of 16 points because the equation of an ellipse is simply z square by a square plus y square by b square equal to 1. That is the equation of an ellipse and so

I know exactly, if I choose the values of z, then I will know the values of y b for each of these points.

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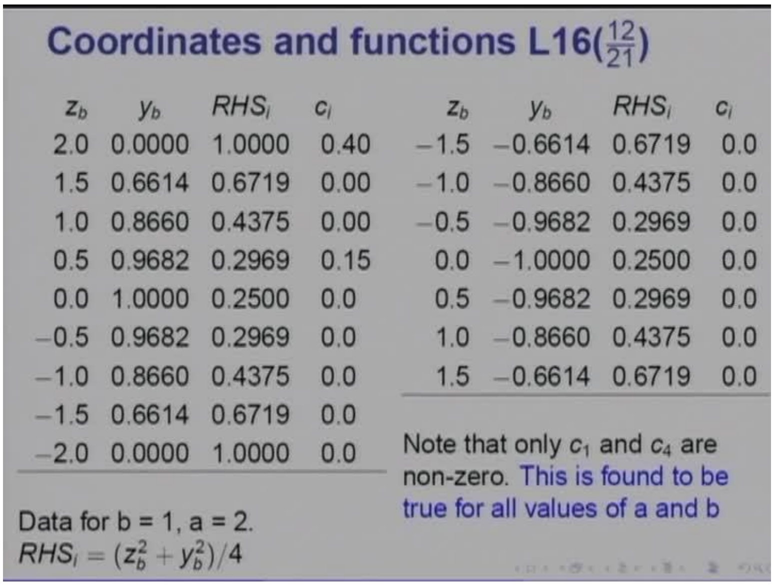

Coordinates and functions L16($\frac{12}{21}$)

z_b	y_b	RHS_i	c_i	z_b	y_b	RHS_i	c_i
2.0	0.0000	1.0000	0.40	−1.5	−0.6614	0.6719	0.0
1.5	0.6614	0.6719	0.00	−1.0	−0.8660	0.4375	0.0
1.0	0.8660	0.4375	0.00	−0.5	−0.9682	0.2969	0.0
0.5	0.9682	0.2969	0.15	0.0	−1.0000	0.2500	0.0
0.0	1.0000	0.2500	0.0	0.5	−0.9682	0.2969	0.0
−0.5	0.9682	0.2969	0.0	1.0	−0.8660	0.4375	0.0
−1.0	0.8660	0.4375	0.0	1.5	−0.6614	0.6719	0.0
−1.5	0.6614	0.6719	0.0				
−2.0	0.0000	1.0000	0.0				

Data for b = 1, a = 2.
$RHS_i = (z_b^2 + y_b^2)/4$

Note that only c_1 and c_4 are non-zero. This is found to be true for all values of a and b

Here are the evaluated constants c i. I have taken 16 boundary points 2.0, z b equal to 1.5, 0, minus 1.5. In each case, y b has been evaluated. The right hand side, which is z b square plus y b square by 4 is also mentioned here. After evaluation by LU-decomposition method, what do I find? I find that only constant c 1 and c 4 are finite. All other constants are identically 0. In fact, no matter of what value you take for a and b, you will always find this for a special case of an ellipse that only c 1 and c 4 are finite. The rest of the coefficients turn out to be 0, which makes for a very elegant solution.

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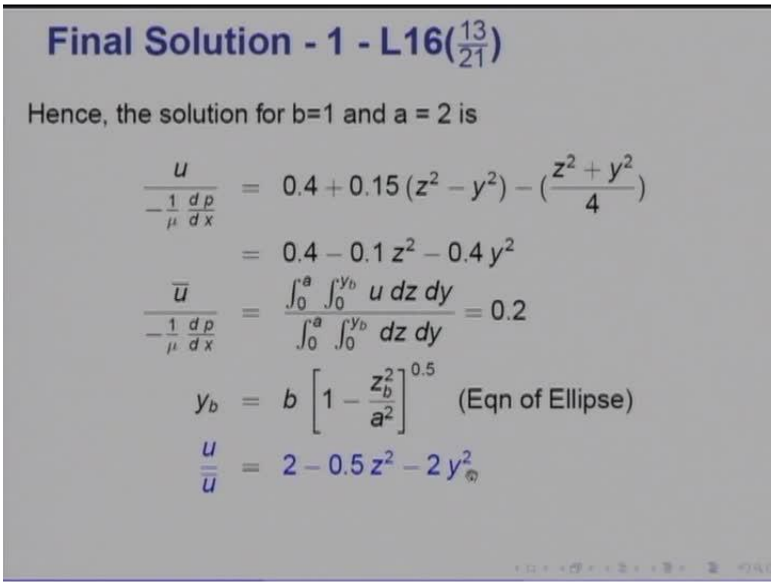

Which makes for a very elegant solution for example the final solution then is simply remember, u is equal to u star which is c 1 plus c 2 c 1 plus c 4 into the function 4 g 4. We see that (Refer Slide Time: 23:00) g 4 was z square minus y square and therefore, the solution is 0.4 plus 0.15 z square minus y square minus z square plus y square by 4, which was the postulated solution. U equal to u star was z square minus plus y square by 4 and we found u star from the analysis. Essentially, u itself becomes 0.4 minus 0.1 into z square plus 0.4 y square in this case integration is done from to 0 to a 0 to y b u dz dy divided by 0 to a 0 to y b dz dy and y b from the equation of an ellipse is simply 1 minus z b square by a square raised to half.

So, if you carry out these two integration, you will see that will turn out to be 0.2. As a result, in this particular case for b equal to 1 and a equal to 2, the solution is u over u bar plus 2 minus 0.5 z square minus 2 y square.

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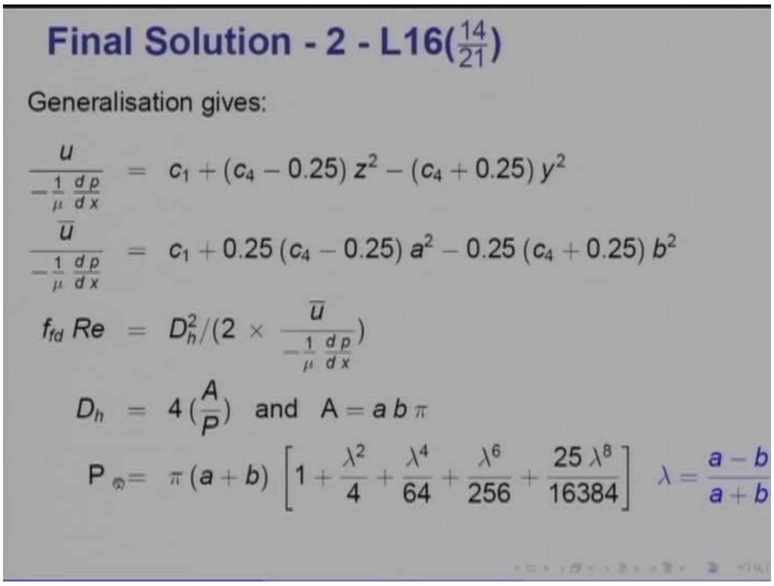

If I were to generalize the solution is like this. As I said, although c 1 and c 4 are functions of a and b, they are the only once that remain finite. All others are 0 and so the general solution is this and u bar would be then given by this. Calculating friction factor into a Reynolds number product is D h square by 2 u bar over 1 over mu d p dx, which we have shown in our previous lecture.

D h square for an elliptical duct is given as 4 times area divided by wetted perimeter. Of course, area is simply a b into pi, but the perimeter is given by this polynomial expansion. Lambda is a minus b divided by a plus b. If you wish to see this, you can look up any mathematical table in which, elliptic integrals evaluate perimeter equal to that.

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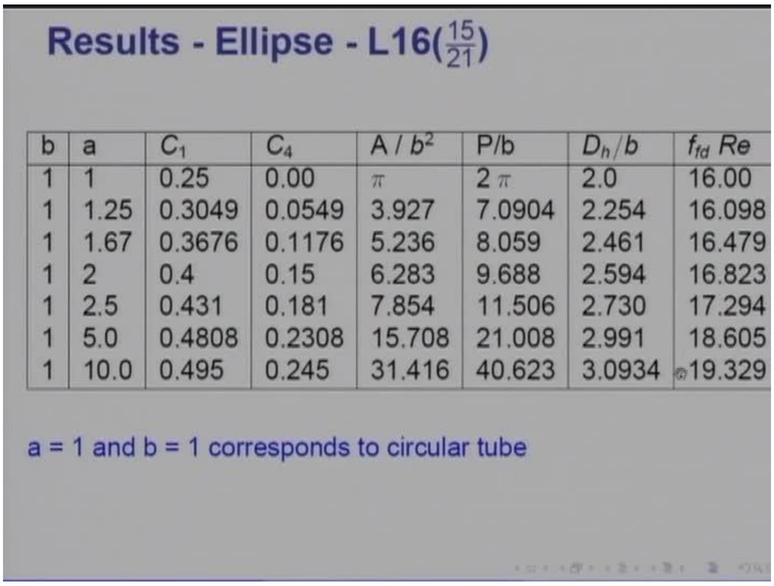

b	a	C_1	C_4	A / b^2	P/b	D_h / b	f_{fd} Re
1	1	0.25	0.00	π	2π	2.0	16.00
1	1.25	0.3049	0.0549	3.927	7.0904	2.254	16.098
1	1.67	0.3676	0.1176	5.236	8.059	2.461	16.479
1	2	0.4	0.15	6.283	9.688	2.594	16.823
1	2.5	0.431	0.181	7.854	11.506	2.730	17.294
1	5.0	0.4808	0.2308	15.708	21.008	2.991	18.605
1	10.0	0.495	0.245	31.416	40.623	3.0934	19.329

Here are the results for variety of values of b and a. Now, b is equal to 1 in all cases, but a is chosen to be 1, 1.25, 1.67, 2, 2.5 and 10, which is a very oblong ellipse. You will see in each case, c 1 and c 4 have been evaluated and also the values of d h by b are given. The special case is when, a is equal to 1 and b is equal to 1 and that is a case of a circular tube. As you would expect, in this case, c 4 is 0, only c 1 is finite and the f Re is equal to 16 as we except. For all other cases, f Re goes on increasing. friction factor increases with the increase in oblong of the elliptical duct. These are very useful solutions because many heat exchangers use elliptical tubes for a heat transfer.

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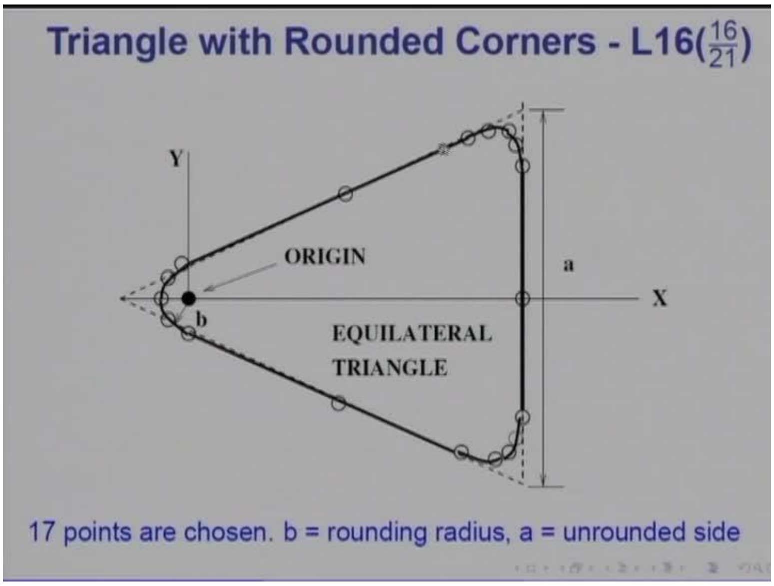

Let us look at another problem that is a triangular duct with rounded corners. Rather than sharp corners, manufacturing wise rounded corners are often more conducive to manufacture. We wish to account for the effect that rounding corners. The rounding radius in this case is b and the unrounded side is a. So, it is really an equilateral unrounded triangle in which, rounding is given at the three corners. So, b is a rounding radius, a is the unrounded side, the origin is here, this is y and this is x. Let us see, how many points I have chosen in this case. These are the points that are chosen. As you can see, I have concentrated on a point, where the curvature is taking place and only one point on the whole entire side.

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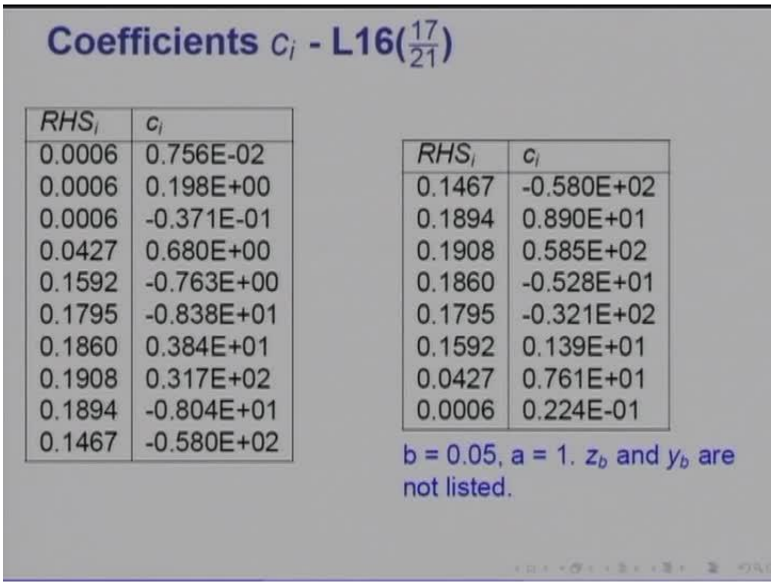

Coefficients c_i - L16($\frac{17}{21}$)

RHS_i	c_i
0.0006	0.756E-02
0.0006	0.198E+00
0.0006	-0.371E-01
0.0427	0.680E+00
0.1592	-0.763E+00
0.1795	-0.838E+01
0.1860	0.384E+01
0.1908	0.317E+02
0.1894	-0.804E+01
0.1467	-0.580E+02

RHS_i	c_i
0.1467	-0.580E+02
0.1894	0.890E+01
0.1908	0.585E+02
0.1860	-0.528E+01
0.1795	-0.321E+02
0.1592	0.139E+01
0.0427	0.761E+01
0.0006	0.224E-01

b = 0.05, a = 1. z_b and y_b are not listed.

Now, I am not giving the values of z b, y b and there are very obvious. Notice that in this case, all coefficients like 1, 2, 3, 4, 5 for 17 are all finite coefficients. None of them is 0 and some are small like this. For example, it is minus 0 371, whereas this is minus 58 which is very large. So, they are uneven magnitudes, but some of the coefficients are very small and some coefficients are very large. So, all the 17 terms in our expansion would have to be retained to express the solution.

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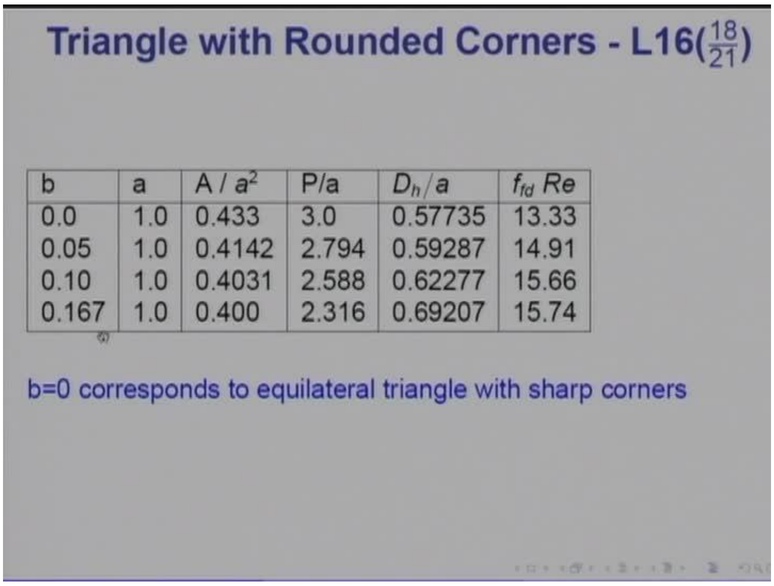

Triangle with Rounded Corners - L16($\frac{18}{21}$)

b	a	A/a^2	P/a	D_h/a	$f_{fd}\,Re$
0.0	1.0	0.433	3.0	0.57735	13.33
0.05	1.0	0.4142	2.794	0.59287	14.91
0.10	1.0	0.4031	2.588	0.62277	15.66
0.167	1.0	0.400	2.316	0.69207	15.74

b=0 corresponds to equilateral triangle with sharp corners

To evaluate friction factor and Reynolds number, you must evaluate the u bar from this solution. It has to be done on a computer because hand integration becomes very difficult. So, you have to do that on a computer and here are the results. In each case, I have taken a equal to 1. The rounding radius is 0, which of course corresponds to an equilateral triangle with sharp corners 0.05, 0.1 and 0.167. You can see, I have evaluated cross-sectional area divided by a square perimeter divided by a and hydraulic diameter is divided by a.

Now, you can see that for a sharp corner it is 13.33, which we had also seen in the Kantorovich method. As the rounding radius increases, you see that the friction factor versus Reynolds number seems to increase. Now, based on hydraulic diameter, the hydraulic diameter goes on increasing with the rounding radius.

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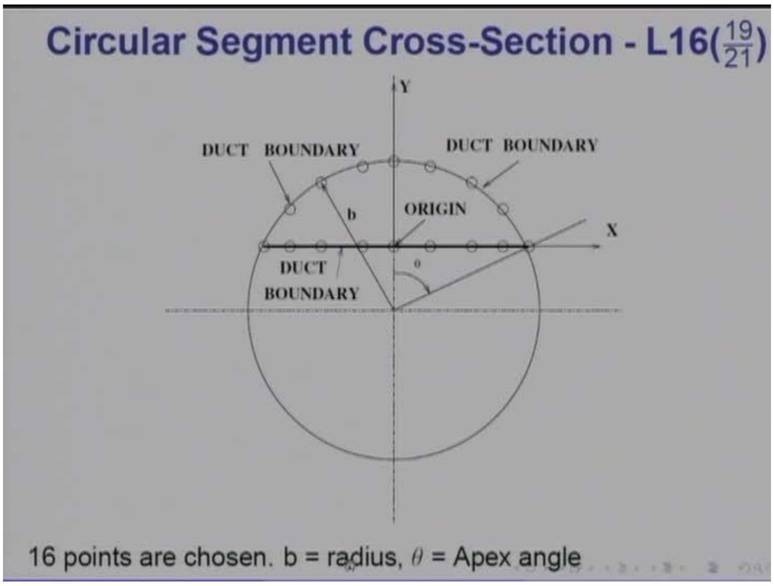

Let us take another problem. This is a circular segment at cross-section. The duct itself has a flat side and a part of a circle, the apex angle is theta. So, the duct boundary is the flat side and a curved side. Now, you can imagine such geometry would be extremely difficult to handle by any other method because it is so irregular. Again, I have chosen 16 points: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 and 16. The solutions are obtained for a prescribed value of the apex angle - theta and radius b.

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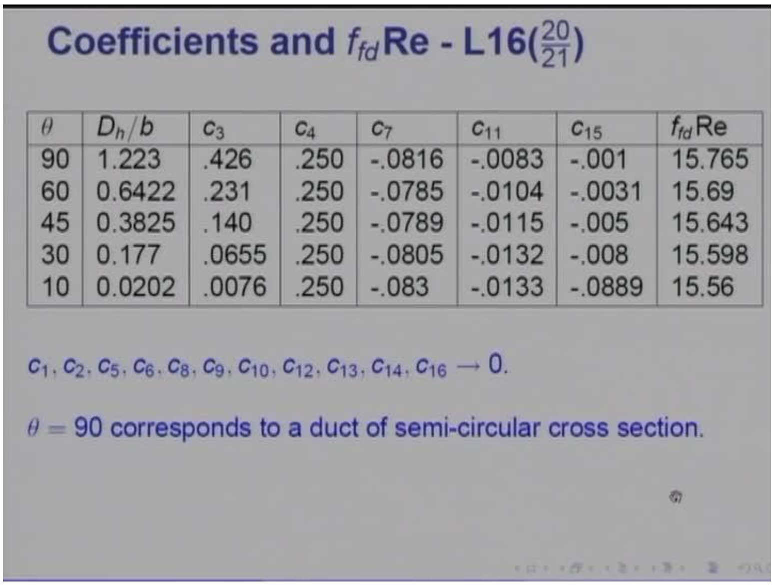

θ	D_h/b	c_3	c_4	c_7	c_{11}	c_{15}	f_{fd}Re
90	1.223	.426	.250	-.0816	-.0083	-.001	15.765
60	0.6422	.231	.250	-.0785	-.0104	-.0031	15.69
45	0.3825	.140	.250	-.0789	-.0115	-.005	15.643
30	0.177	.0655	.250	-.0805	-.0132	-.008	15.598
10	0.0202	.0076	.250	-.083	-.0133	-.0889	15.56

Here are the results: theta equal to 90, 60, 45, 30 and 10. In each case that hydraulic diameter and 2 b, the radius is given. Now, in this case, c 3, c 4, c 7, c 11 and c 15 are the only finite values, all others are almost 0. When you actually take printout from your computer after LU-decomposition, you may find these values to be of the order of 10: minus 8 or 10: minus 9, which are not worth considering at all because they are very small. So, only 1, 2, 3, 4 and 5 coefficients are finite in this particular case.

Again, using these five coefficients, you evaluate the average velocity u bar, which enables you to calculate friction factor multiplied by Reynolds number product. Theta equal to 90 is a very special case because it represents its semicircular duct of a semi-circular cross-section. As you can see from this figure (Refer Slide Time: 31:45), if theta is 90, then I would have a perfect semicircular duct. The value is 15.76, but the value goes on decreasing, but not so much. These accurate values are very useful, when we compare with experimental data. We want to be sure that our calculations and predictions compare very well.

What we have now is a very general method. You can in fact program the entire method for a variety of ducts. All you have to do is to give boundary point coordinates for different types of ducts and that is what I have done. I have a single computer program, which has all the 17 functions stored in it and it has a LU-decomposition subroutine as well as it has the subroutine to calculate u bar, which requires numerical integration. So that is all you require for each different type of duct. All you need to do is to give coordinates of the duct boundaries.

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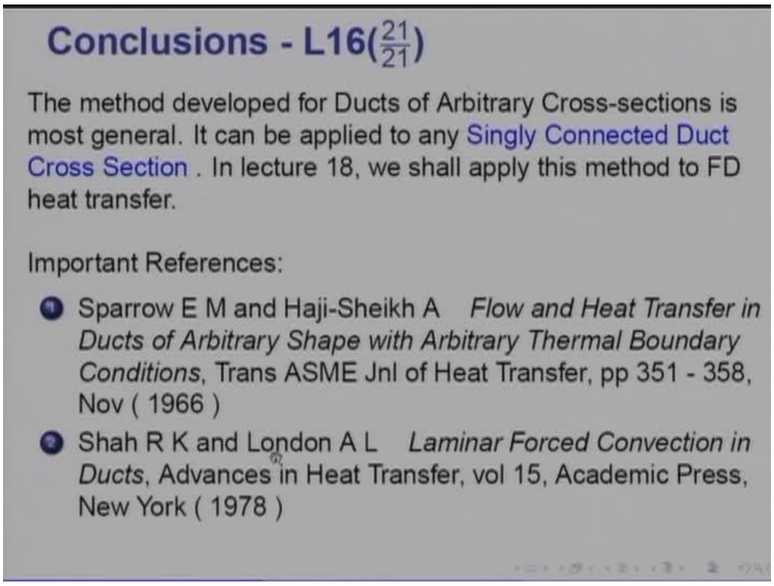

The only constrained is the duct that you choose must be singly connected. It must have singly connected cross-section. As you can imagine, the method does not change with the duct shape. Unlike Kantorovich method or the Fourier series methods and so on, where in each case, you have to develop different functions to satisfy the boundary conditions. Here, you do not have to worry about that. You simply give the boundary point coordinates, it is the general method and it can be applied. Moreover, the method can also be extended to heat transfer. We shall see that in couple of lectures from now. In lecture number 18, you will be able to see that.

This method has been developed in the paper by Sparrow and Haji-Sheikh. It is Trans ASME Jnl of heat transfer, 1966. Several non-circular ducts of variety of cross-sections have been mentioned in earlier lecture and it is moon shaped sine duct and so on. Results for them are given in this companium by R K Shah and London Laminar Forced Convection in Ducts - Advances in Heat Transfer of volume 15 in 1978. I might say that the results, which are presented here, have been collaborated in this companium. Thank you.