Convective Heat and Mass Transfer

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Module No. # 01

Lecture No. # 15

Fully-Developed Laminar Flows -1

Today, we shall look at Fully-Developed Laminar Flow in a duct. We will begin with definition, calculate friction factor for a simple circular cross-section duct as well as annular cross-section duct and then somewhat more complex ducts like rectangular duct or an annular sector duct.

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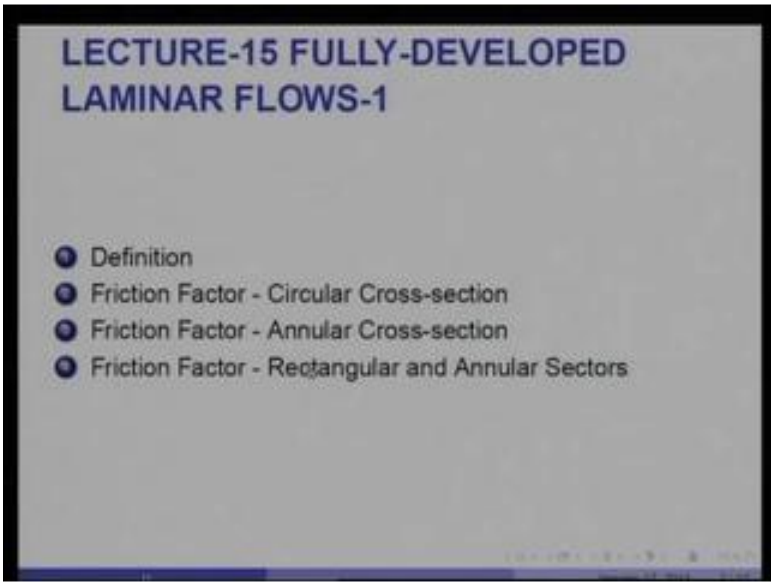

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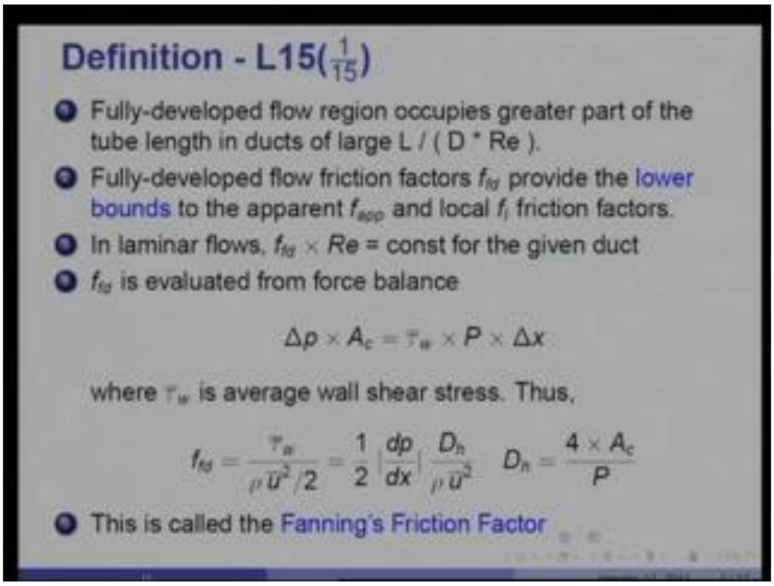

You will recall from lecture 14 where the fully-developed flow region occupies greater part of the tube length in ducts in which length to diameter ratio is very large. This is particularly obtained in micro tubes where diameter is so small that any practical length gives you very high length to diameter ratio. Fully-developed flow friction factor f sub fd provides the lower bound to the apparent friction factor as well as the local friction factor. These two were evaluated for flow between parallel plates during our last lecture.

In laminar flows, as a rule, fully-developed flow friction factor multiplied by Reynolds number is an absolute constant and that constant depends on the duct geometry for the given duct. How is fully-developed flow evaluated? It is simply from force balance over an actual distance delta x of the tube.

So, the pressure drop over length delta x multiplied by the area of cross section of the duct equals the shear stress at the periphery of a cross section multiplied by the perimeter multiplied by delta x. Tau all bar is the averaged shear stress. Now, you know that the friction factor is the ratio of shear stress divided by kinetic energy which means, this case defined as rho u bar square by 2 where, u bar is the mean actual velocity. From this relationship, it will be obvious that would become 1 by 2 modulus of dp dx D h by rho u bar square. I have put a modulus to remains that dp dx is negative in any ducted flow. The hydraulic diameter D h is 4 times area of cross section divided by perimeter of the duct.

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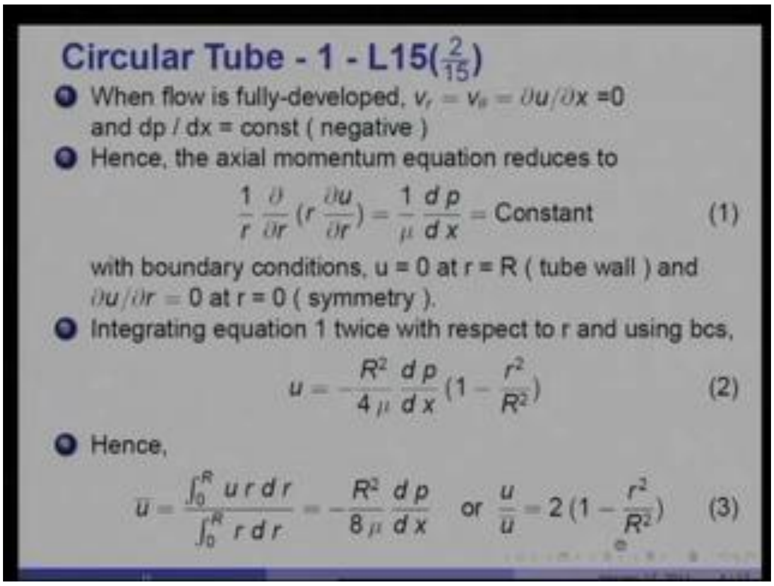

In this definition, friction factor is called the Fanning's friction factor named after Fanning. Let us take the case of a circular tube which you all very well know; this is simply to refresh your memory. When the flow is fully-developed, the radial velocity as well as the circumferential velocity components are 0. The actual velocity gradient is also 0 because the boundary layers have now met which results in the actual pressure gradient to become constant.

Hence, the actual momentum equation reduces to 1 over r d by dr r du by dr equal to 1 by mu d p by d x equal to a constant with boundary conditions u equal to 0 at the tube wall, r equal to R and the gradient du by dr equal to 0 at the access of symmetry. Integrating this equation would gives you once or twice with respect to r and using these two boundary conditions to evaluate two constants of integration will give you u equal to minus R squared by 4 mu d p by d x 1 minus r square by R square.

Hence, evaluating the mean velocity as 0 to R u r d r divided by 0 to R r d r gives you minus R square by 8 mu d p by d x. Notice the negative sign here (Refer Slide Time: 04:40) as well as here which simply allows counts for the fact that d p d x now is of course negative; which gives us u over u bar equal 2 into 1 minus r square by R square; an very well-known result that you will recall from your undergraduate days.

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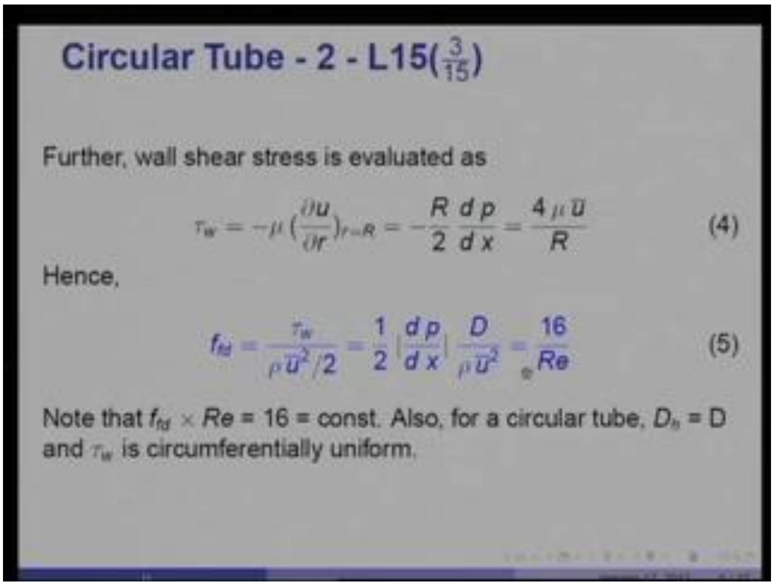

What about the wall shear stress? Well, wall shear stress is tau equal to minus mu du by dr at the wall r equal to R and that becomes equal to R by 2 d p by d x with a negative sign giving you 4 mu u bar by R, that would be circumferentially constant.

Hence, the friction factor which is tau wall over rho u bar square by 2 as we saw on the previous slide, as well as this definition gives you 16 by Re. Note that f fd into a Reynolds number therefore is 16 and constant as we said before. Also, for a circular tube, the hydraulic diameter itself becomes equal to the diameter. Often you have encountered the fact that f fd into Re is taken a 64 rather than 16; that simply follows from the definition that f fd is 2 times d p by d x D by rho u bar square.

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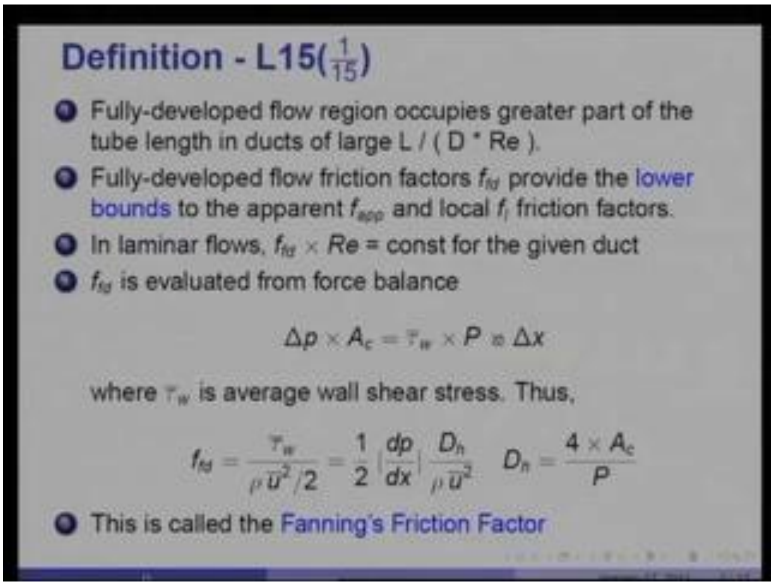

However, as we showed on this slide that the correct definition of f fd is according to the Fanning’s friction factor and which results from the force balance. Sometimes, although the friction factor is taken as 4 times the Fanning’s friction factor, we would continue in our lectures with the Fanning friction factor.

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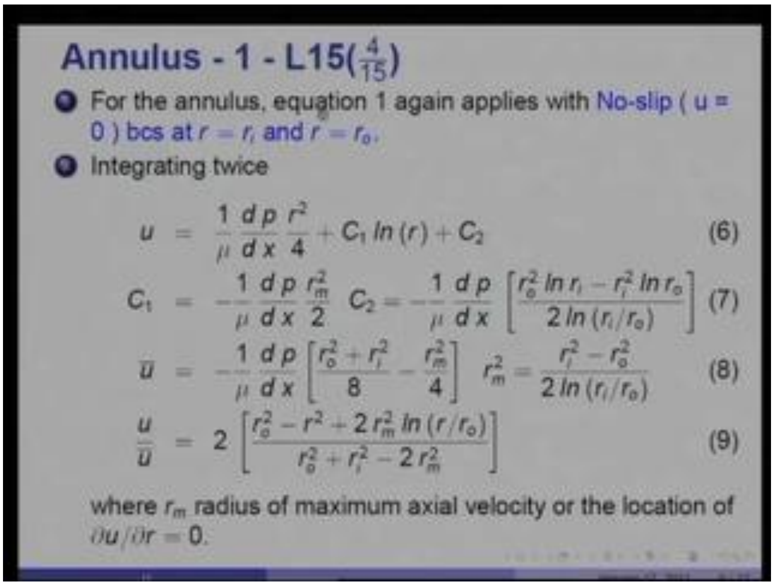

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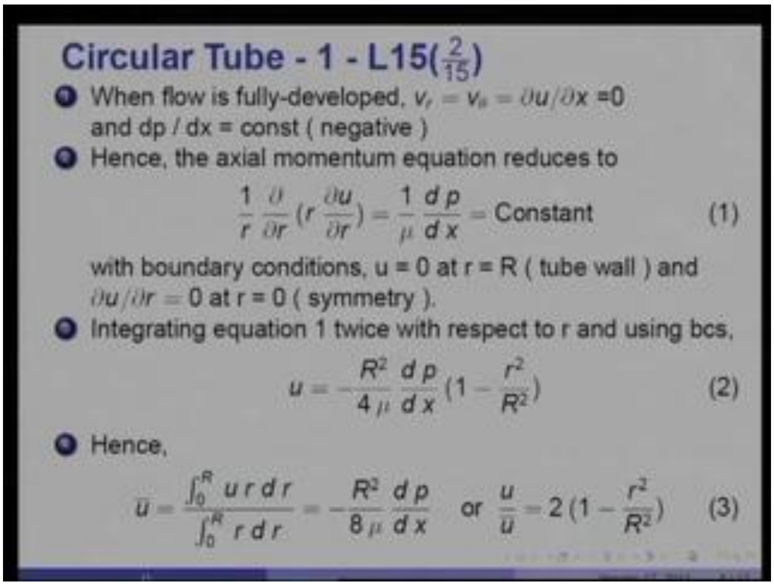

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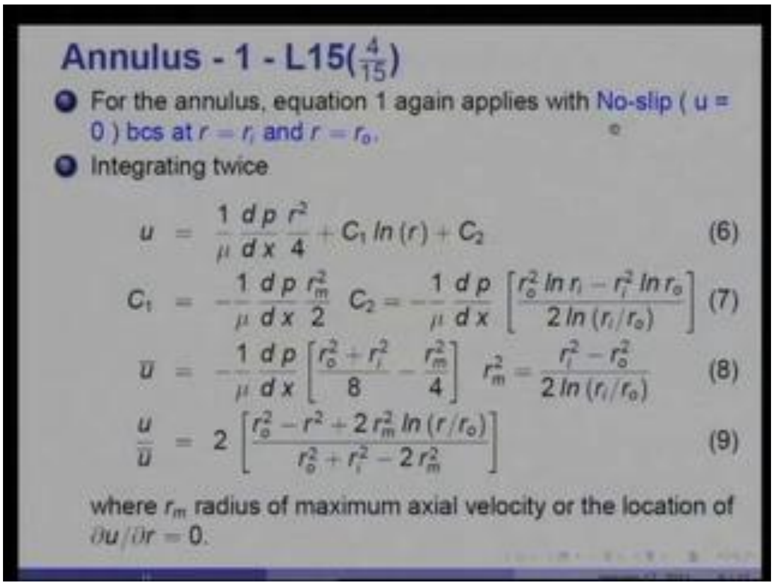

Let us now turn to the Annulus, for the fully-developed flow in Annulus equation 1 will again remain as it is that is no difference there because again v r v theta are 0 and actual derivative of velocity is 0. But, the boundary conditions now will be u equal to 0 at r equal to the inner radius and u equal to 0 at r equal to outer radius. That is what is mentioned here, the no-slip boundary conditions at r equal to r i and r equal to r o.

If I integrate the equation 1 twice, I would get two constants of integration C 1 and C 2. Applying these boundary conditions would give me C 1 equal that C 2 equal to that integrating this equation to evaluate u bar the average velocity gives me that expression. In all these cases, r m is really the velocity is the radius where du by dr is 0 (Refer Slide Time: 07:14).

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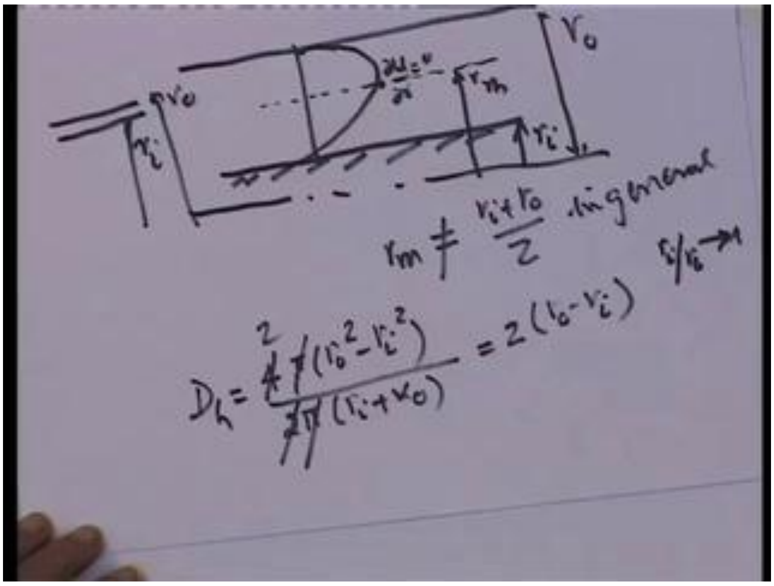

Remember, if I have axis symmetry like this (Refer Slide Time: 07:40), this is the inner radius r i, this is the outer radius r o, then the plane of the velocity profile will look like that and with this 0 velocity both then the point at which du by dr is 0 is the radius r m.

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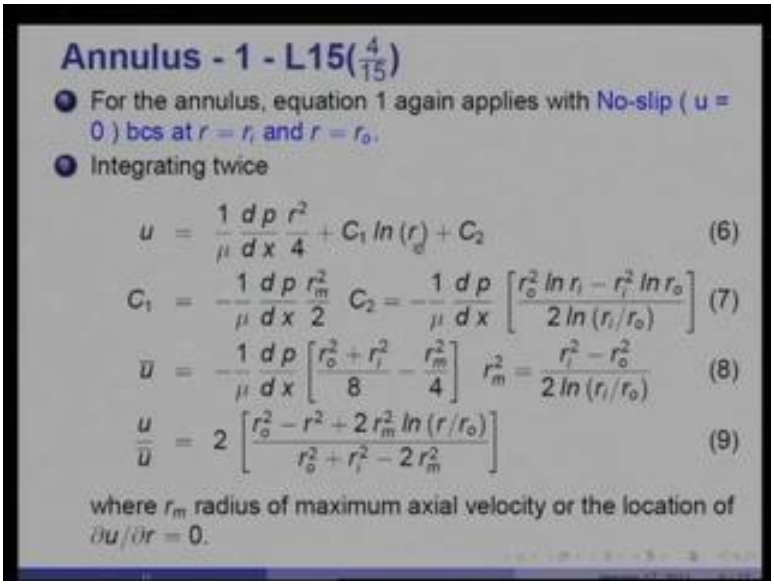

Remember, r m will not equal r i plus r o by 2 in general. In fact, as this expression shows r m is very much a function of the radius ratio r i by r o. So we use locator r m and therefore, u over u bar becomes 2 times r o squared minus r i r square plus 2 r m square ln r by r o divided by r o square plus r i squared minus 2 r m square.

So I as said where r m is the radius of maximum axial velocity or the location of du by dr equal to 0. Mind you, this integration to evaluate u bar is not very straight forward because ln r is involved and one needs to take special care to evaluate ln r r d r properly.

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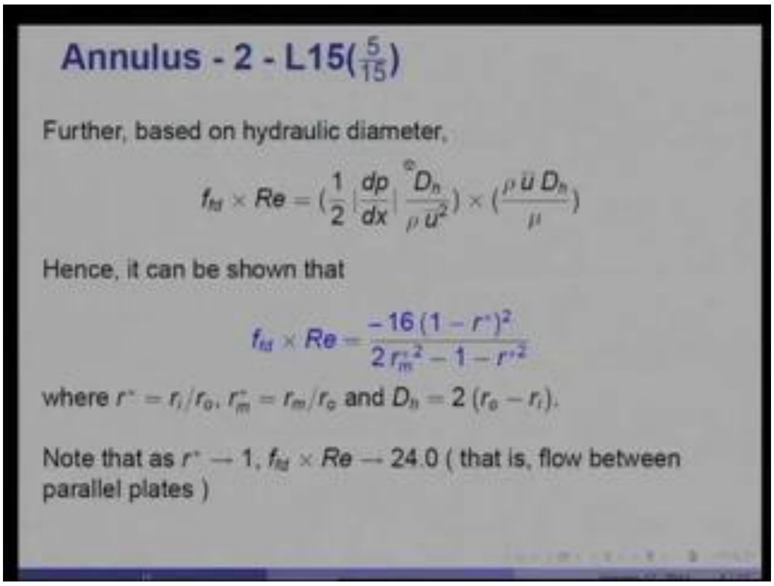

Further, based on hydraulic diameter, now the hydraulic diameter for a duct annuls section will be 4 times a cross sectional area, that will be pi into r o square minus r i square and the perimeter will be 2 pi r i plus r o which gives us - pi and pi gets cancel 2 gives me 2 - 2 times r o minus r i.

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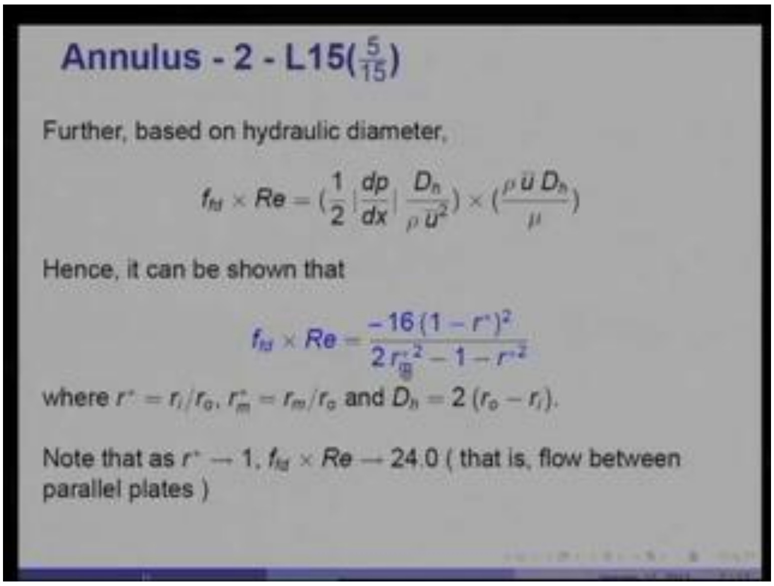

So, twice the flow width r o minus r i is the hydraulic diameter. So if I use that definition of hydraulic diameter it is very easy to show that f into Reynolds number would be 16 times 1 minus r star square, r star is the radius ratio of the duct, r m star is r m by r o and D h is 2 into r o minus r i is the hydraulic diameter.

Now of course, as you can imagine when r i by r o tends to 1 that means the 2 radii are very close to each other. Then we practically have the case of flow between two parallel plates and for which f into Reynolds number are 24 as you derived in your undergraduate class.

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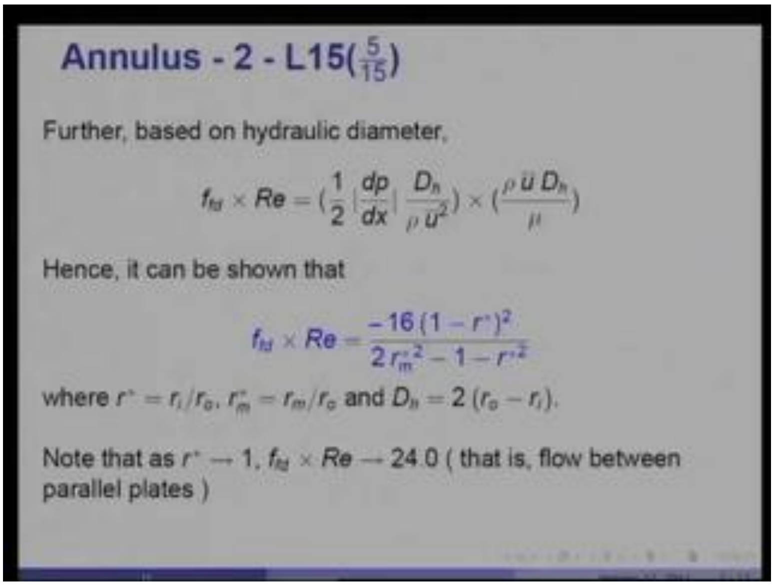

Let us look at the numbers; so for different values of r i by r o you can evaluate the friction factor. In fact as r star tends to 0 you will see f into Reynolds number tends to 16 which is the lower bound indicated by the circular tube.

The r i by r o standing to 1 is the upper bound and that is the flow between parallel plates. For all other radii, product of friction factors Reynolds number would lay between 16 and 24.

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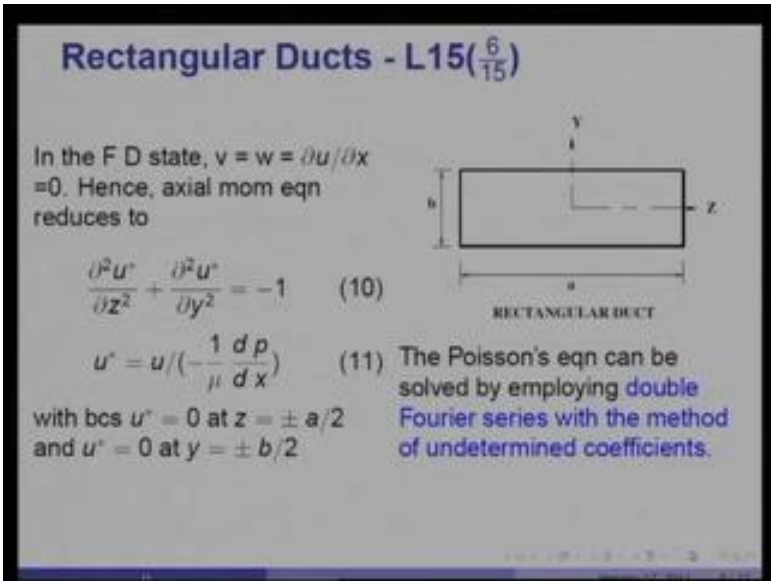

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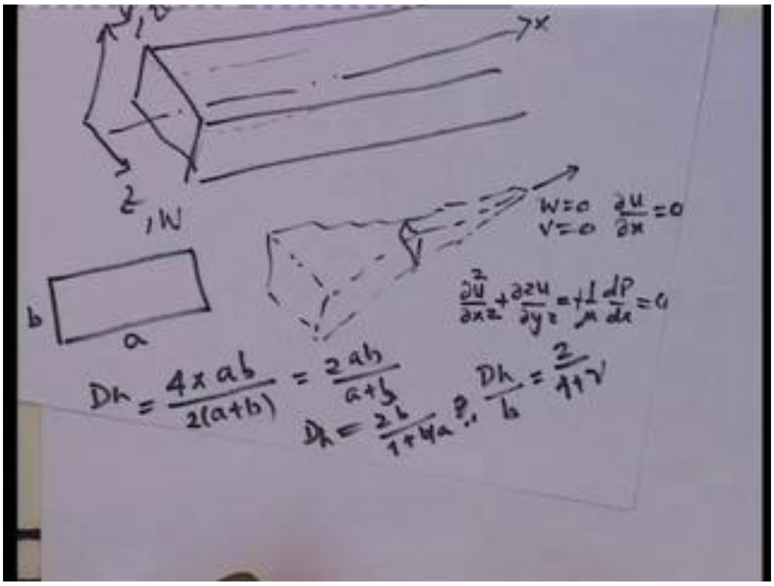

Let us now consider more complicated case and that is flow in a rectangular duct. Now in this case, you will appreciate if I have a rectangular duct like so, with this as the symmetry line in x direction and let us say this is - this direction is z, this direction is y (Refer Slide Time: 11:50). Then the boundary layers will develop on the y max and y 0 wall as well as on z equal to 0 and z equal to z max walls.

Therefore, I would have a structure of boundary layers growing like so and ultimately the boundary layers will meet at some point. After that, components w and v, so w will be 0 v will be 0 and du by dx will be 0; therefore, you are left with an equation of the fully-developed flow in a rectangular duct is simply d 2 u by square plus d 2 u by dy square equal to minus 1 over mu dp by dx equal to constant.

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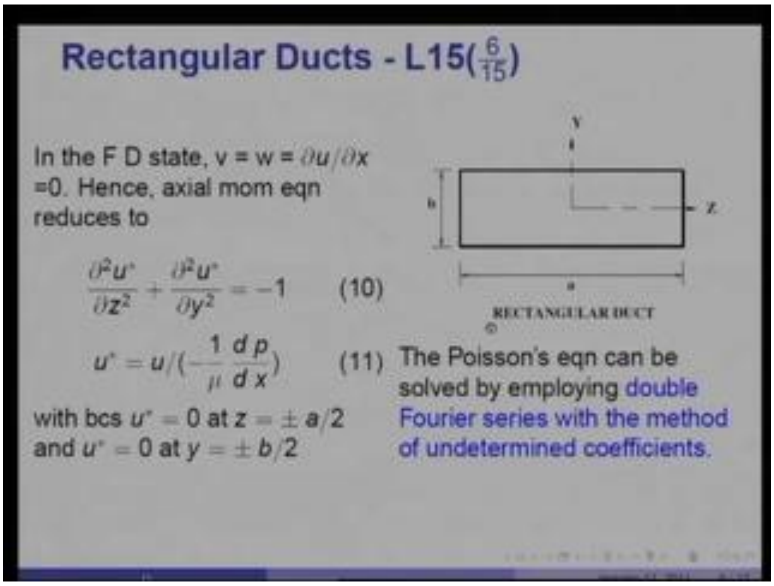

That is what I have shown here, sorry this should be plus dp and that is equal to constant so if you divide u divided by minus 1 over mu d p dx then you get d 2 u star by dz square plus d 2 u star by dy square equal to minus 1. This is a Poisson equation with a constant on right hand side. What are the boundary conditions? If we if you say that the duct dimension is b on the narrower side and a on the longer side then, u star is equal to z at plus and minus a by 2 and it is also equal to 0 at plus and minus b by 2.

Now a Poisson equation of this type can be solved by employing double Fourier series with the method of undetermined coefficient and I will briefly explain what that method is.

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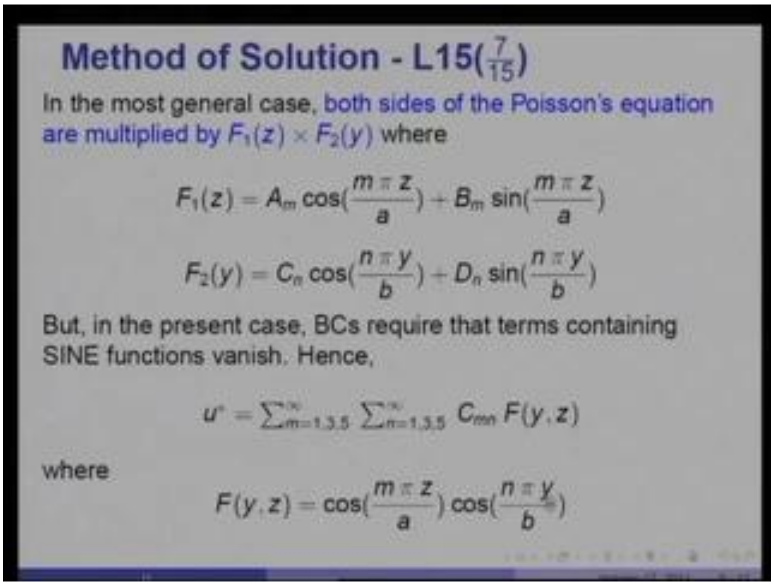

In the most general case both sides of the Poisson equations are multiplied by F 1 into F 2 where, F 1 is a function of z only and F 2 is a function of y; they are taken as Fourier series A m cos m by z plus a by a plus B m sin m pi z by a. So this is a function of z and likewise, this is a function of y again a Fourier series.

In the present case, the boundary conditions required that the term considering sin functions must vanish on the wall the velocity being 0, you will see that sin m pi by 2 would be finite and that would make F 1 finite at the wall which we do not want.

Therefore, the sin must be sin terms must be 0 only the cos terms survive and therefore, our equation would be u star equal to C mn F y z; m equal to 135 to infinity, n equal to 135 to infinity where, F y z is equal to cos m pi z a which is the product of that and that and C mn is really A m into C m as you can see. So we need to determine the constant C mn, so that we can get value of u star. This is the double Fourier series, you must remember what the function is F y z is cos m pi z by a multiplied by cos n pi y by b.

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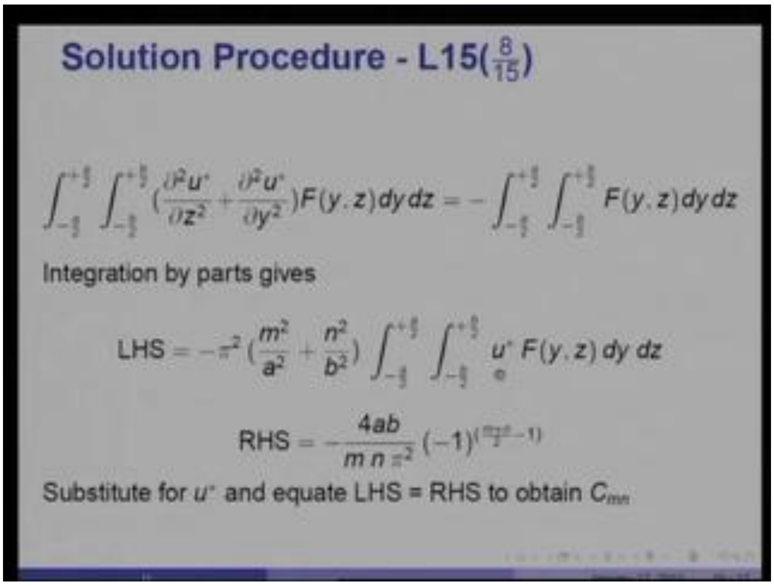

So, the solution procedure is like that; the given Poisson equation is multiplied by F y z on both sides and integrated from limits on z n y. So we have integrated it from minus a by 2 to plus a by 2 minus b by 2 to plus b by 2 d 2 u dz square d 2 u dy square F y z dy dz is equal to the right hand side which is minus 1 into F y z dy dz. Integration by parts would give the left hand side equal to minus pi squared m square by a square plus n square by b square minus a by 2 plus a by 2 minus b by 2 to 2 plus u star F y z dy dz; simply these terms are reduce to then, multiplied by u star into F y z.

The right hand side on the other hand, would give you minus 4ab by m n pi squared minus 1 raise to m plus n by 2 minus 1. So, if you substitute for u star and equate the left and right hand side, what is u star? u star is simply this; so if we substitute for u star in this expression I equate left hand side with right hand side we get, C mn equal to the expression given here and again here substituting for F star square by z would give me that expression for C mn.

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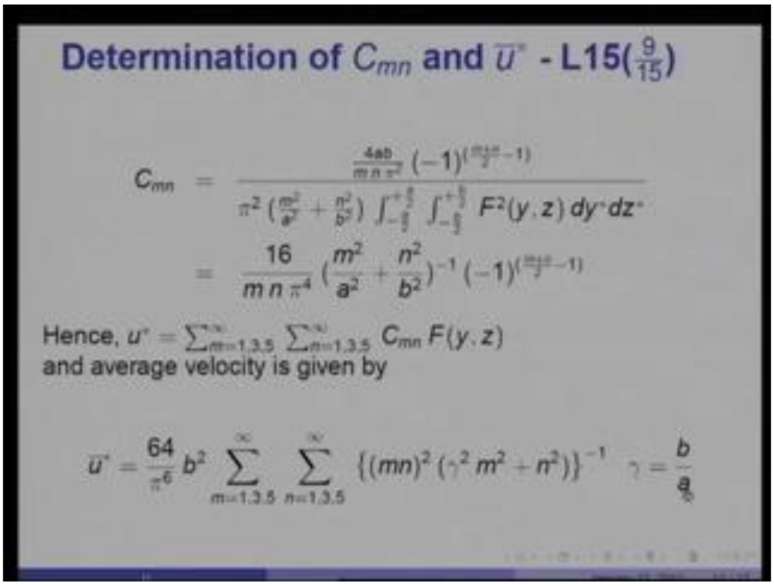

Therefore, the velocity is now given by C mn that expression multiplied by cos m pi z by a plus sin n pi z y by b is the F y z function. Of course, you must now integrate this over the cross section of the duct to obtain u bar star which comes out to be 64 by pi raise to 6 into b square into all this and now I use gamma as the aspect ratio. The ratio of the shorter site to the longer site - gamma is the aspect ratio of the duct.

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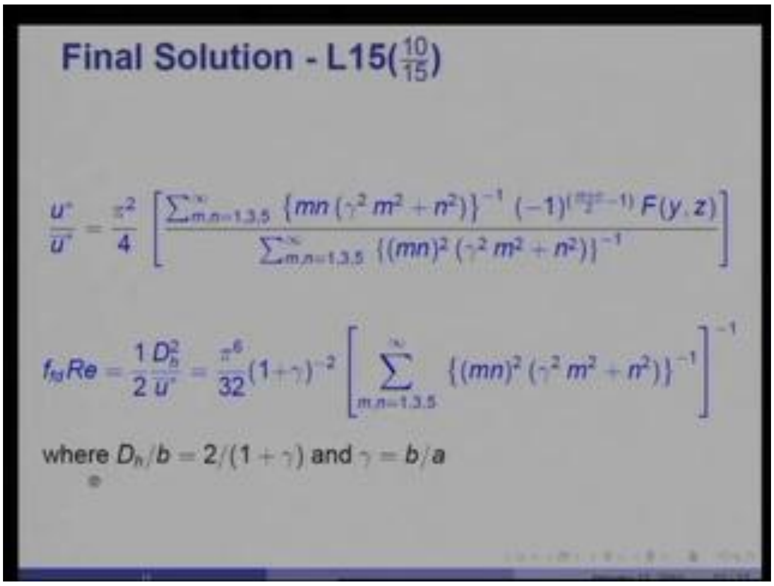

So, the final solution then is this where F y z again as I said is product of cos m pi z by a and cos n pi y by b. Therefore, the friction factor into Reynolds number would be given by 1 by 2 D h square by u bar equal to that (Refer Slide Time: 18:30).

D h of course, for a hydraulic diameter in this particular case - in the rectangular case would be b and a so 4 times area of cross section ab divided by 2 times a plus b which gives me 2 ab divided by a plus b or if I have to divide this by a, then I get this is equal to hydraulic diameter. So I get this as 2 b divided by 1 plus b by a, which is D h by b will be equal to 2 divided by 1 plus gamma. That is what I have shown here D h by b would be 1 plus gamma and gamma is b by a - the aspect ratio.

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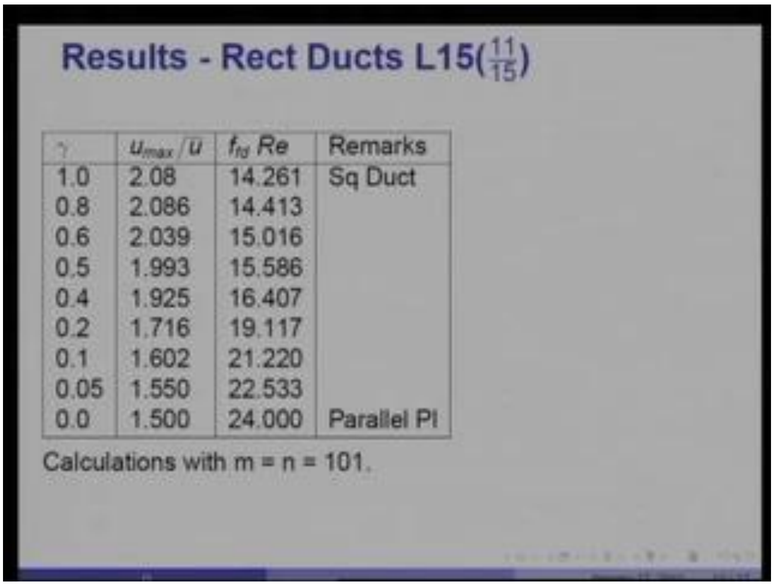

Results - Rect Ducts L15($\frac{11}{15}$)

γ	$u_{max}/\bar{u}$	f_{fd} Re	Remarks
1.0	2.08	14.261	Sq Duct
0.8	2.086	14.413	
0.6	2.039	15.016	
0.5	1.993	15.586	
0.4	1.925	16.407	
0.2	1.716	19.117	
0.1	1.602	21.220	
0.05	1.550	22.533	
0.0	1.500	24.000	Parallel Pl

Calculations with m = n = 101.

If we evaluate friction factor Reynolds number by substituting different values of gamma we obtain this results that are shown here. So, when gamma is equal to 1 that is the aspect ratio is exactly equal to 1 then, we get a square duct. This both shorter and longer side are equal length and we get a square duct in which case the maximum velocity would be 2.08 u max divided by u bar would be 2.08.

The fully develop friction factor into Reynolds number will be 14.26 and that is a case of a square duct. As you go on reducing aspect ratio, you see that the maximum to u bar ratio in general goes on falling. The friction factor in Reynolds number goes on increasing when we come to 0 that is when the shorter side is much smaller than a then of course, you approach small as the flow between parallel plates and you can see you max over u bar as you recall would be 1.5 and F Re is 24 which is very well known results.

In all these cases you have to choose the values of m and n in this series and I have chosen m and n equal to 101. Of course, it varies with duct shape but, once you evaluate these things on a computer you can give any value and once the series has converge it is good enough to take any value of m and n beyond 101 the results will not change.

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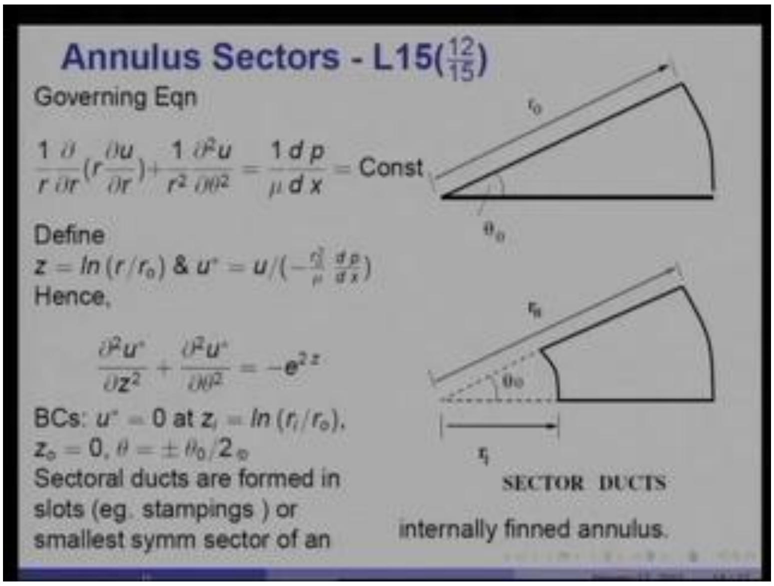

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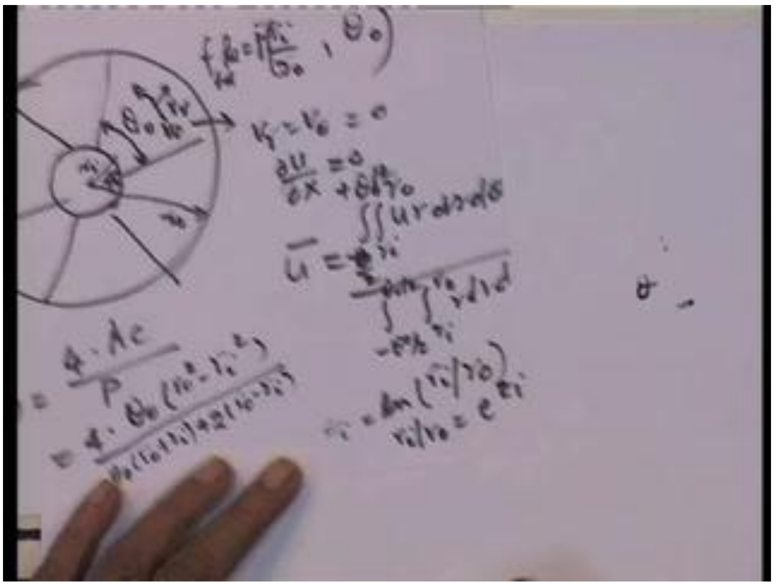

Now, let us look at even more complicated duct and that is a sector of an Annulus or a circle whichever way. Typically you get ducts which are of this type (Refer Slide Time: 21:41), this is the inside radius r i and this is the outside radius r o and this is the included angle which I have called here as theta naught.

So we have two parameters here, r i by r o and theta naught. So, we expect from friction factor fd into Reynolds number to be a function of both these parameters. In the fully-developed state, the radial velocity v r as well as the tangential velocity v theta will be 0. This is v r equal to v theta will be equal to 0 and the du by dx in the actual direction will now be 0 which renders the momentum equation in the cylindrical polar coordinates to the form shown here.

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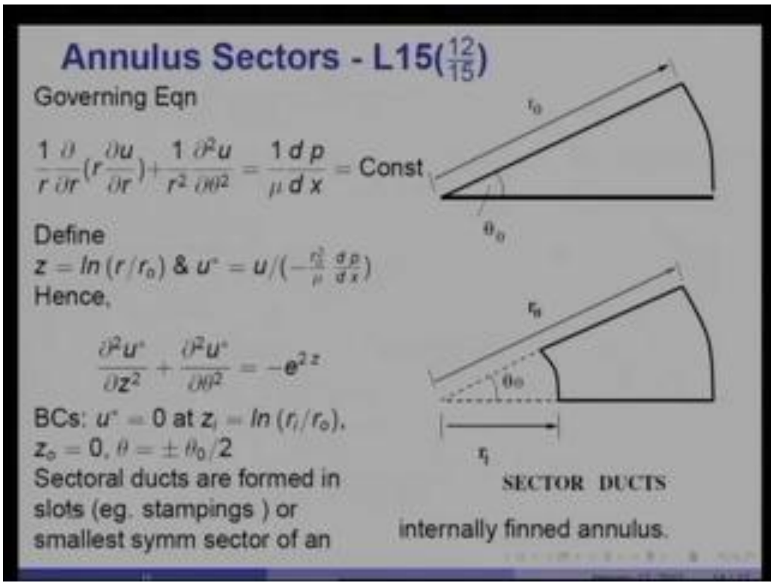

1 over r d by dr r du by dr plus 1 over r square d 2 u by d theta square equal to 1 over mu d p d x equal to a constant and here is a little trick which you must bear in mind Z equal to ln r by r o and u star equal to u divided by minus r o square by mu d p d x; then you will notice that d 2 u star by dz square plus d 2 u star by d theta square equal to minus e raise to 2 z.

Again we get a Possion equation but, unlike the case of a rectangular duct the right hand side is now a function of z or the function of radius r, z is ln r by r o. Therefore, the right hand side is function only of the radius, because u star is equal to 0 at z i ln r i by r o and z o is equal to 0 at theta plus minus theta naught by 2 as at the 2 walls.

This particular case is the one in which the inner radius is 0; here the inner radius is finite. Such sectoral ducts as I said are formed in slots of electrical stampings or the smallest symm sector of internally finned annular duct. Now, the method of solution for this case is exactly same as in the case of the rectangular duct that I showed you earlier.

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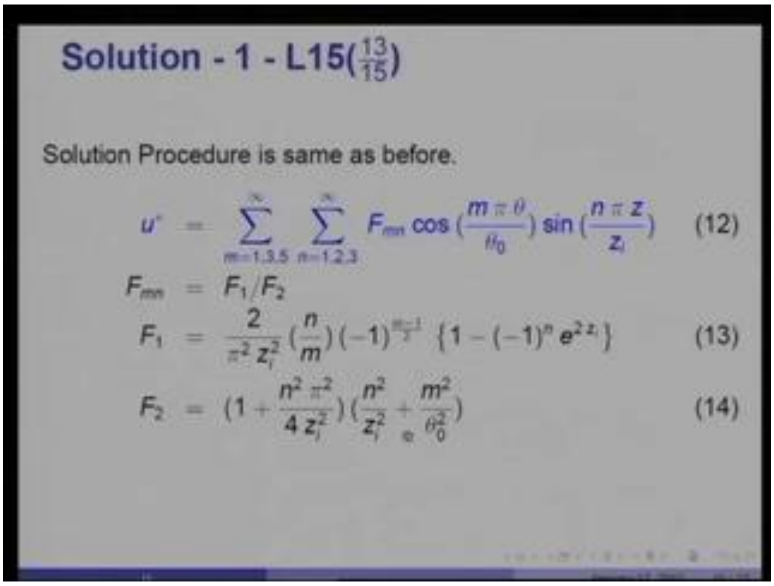

So, we multiplied both sides by an appropriate Fourier series function cos m pi theta by theta naught sin n pi z by z z i where these functions are obtained by noting the boundary conditions and F mn is the Fourier coefficient. We can evaluate that in the manner I explained earlier to F mn would be equal to F 1 divided by F 2 F 1 of course, is given by that z i as you know is ln r i by r o and F 2 would be given by this which also includes the included angle theta naught.

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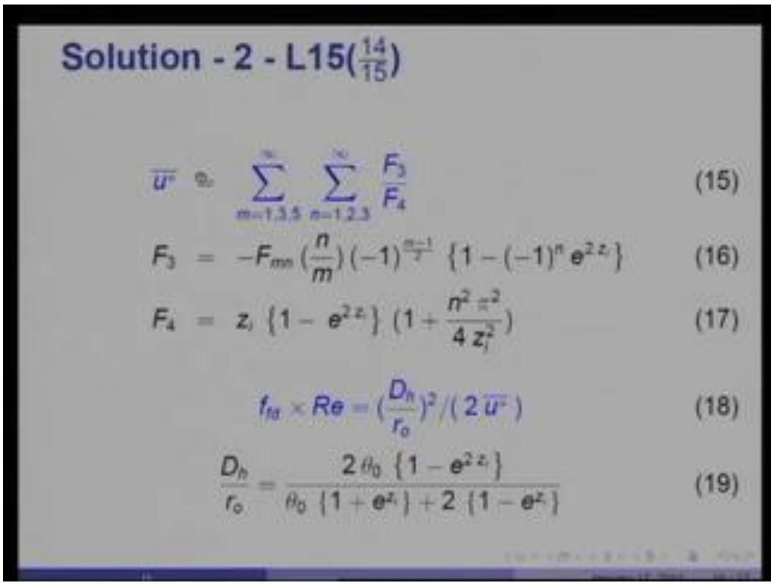

Now, we can evaluate u bar, u bar star would be simply ur dr d theta r i to r o 0 to theta naught or if you like minus theta naught by 2 to plus theta naught by 2 divided by minus theta naught by 2 to plus theta naught by 2 r i by r o, r dr d theta.

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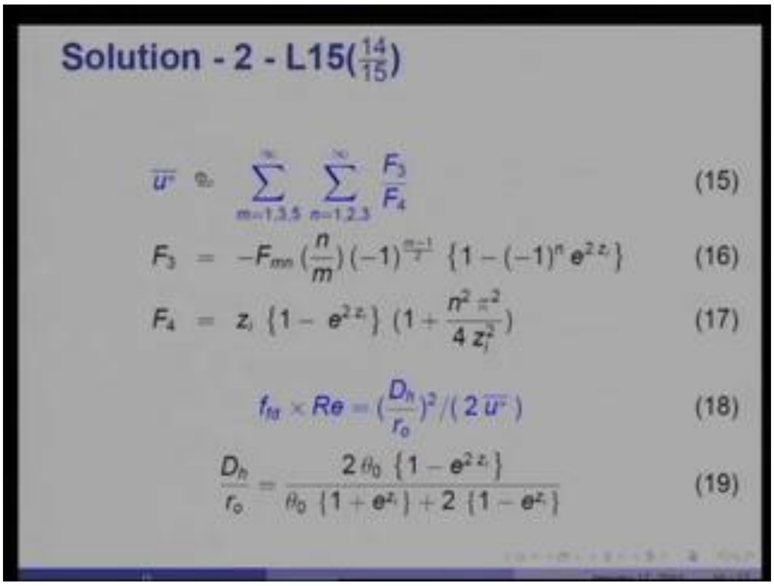

If you carry out that evaluation you get this equal to F 3 by F 4 at the sum given by that. Remember m varies 1 3 5 to infinity, n on the other hand varies where is as n equal to 1 2 3 to infinity and F 3 is given by that and F 4 is given by that. Therefore, the friction factor multiplied by Reynolds number is simply D h by r o square divided by 2 u bar star square by u bar is taken from there (Refer Slide Time: 26:00).

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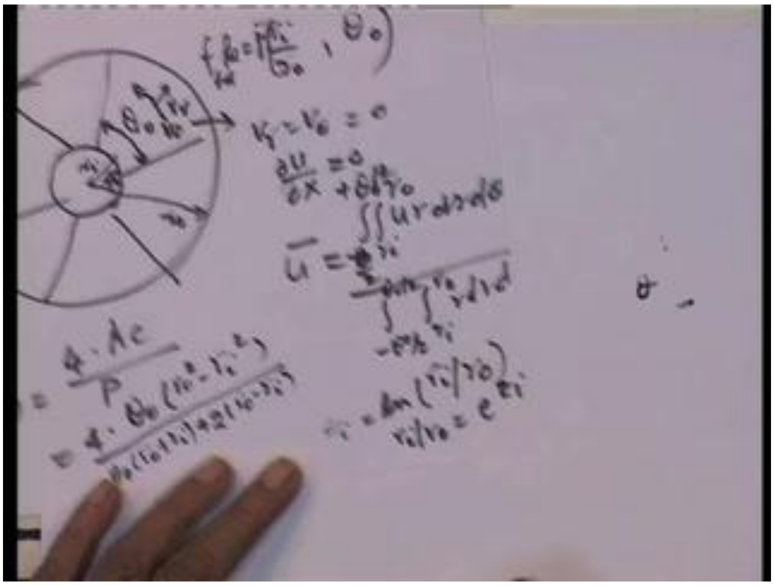

The hydraulic diameter in this case would be, as you can see it, looks a very complicated evaluation but, very simple idea is D h is equal to 4 times the cross sectional area divided by perimeter. So, 4 times theta naught into r o square minus r i square divided by theta naught into r o plus r i plus 2 times r o minus r i; that is what it is all about.

If I have to divide this through by r o then note that z i is simply ln of r i by r o and therefore, r i by r o will be equal to exponential of z i and that is what I have shown here.

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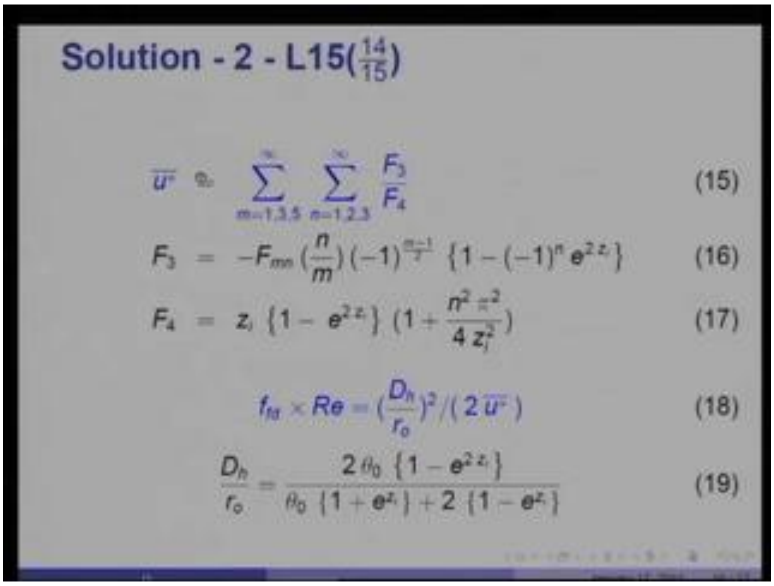

So, it is possible to evaluate friction factor Reynolds number as a function of these two parameters r i by r o and theta naught.

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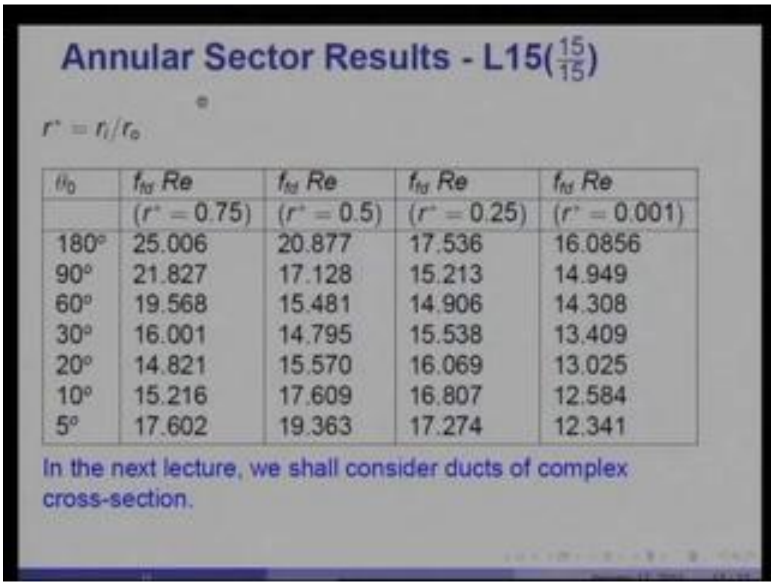

θ_0	$f_{fd}\,Re$ ($r^* = 0.75$)	$f_{fd}\,Re$ ($r^* = 0.5$)	$f_{fd}\,Re$ ($r^* = 0.25$)	$f_{fd}\,Re$ ($r^* = 0.001$)
180°	25.006	20.877	17.536	16.0856
90°	21.827	17.128	15.213	14.949
60°	19.568	15.481	14.906	14.308
30°	16.001	14.795	15.538	13.409
20°	14.821	15.570	16.069	13.025
10°	15.216	17.609	16.807	12.584
5°	17.602	19.363	17.274	12.341

Here are the evaluations for a few values of radius r star equal to 0.75, r i by r o - 0.5, 0.25 and 0.001 which is a very small value. Of course, theta naught equal to 180 degrees we will give you semicircular duct. As you can see here, if theta naught was 180 degrees that would give you a semicircular duct and then 90 degrees would give you that kind of a duct and so on and so forth, coming down to very narrow angle of 5 degrees (Refer Slide Time: 28:08).

You can see the friction factor at 0.75 for the ample is 25 for a semicircular duct going down to 17.6; here at r star equal to 0.5 it goes down to 9, goes down and then increases to 19, this is 0.25 (Refer Slide Time: 28:44). This of course, would correspond to r star equal to 0 again for a semicircular duct 16 down to 12 this is very monotonic decline in it. So, these ducts the annular sector family as well as rectangular duct family, is amenable to relatively easy solutions using Fourier series.

However, now a days, we extremely compact heat exchangers; extremely in electronics for example, where the flow passages are so miniaturized - that the compactness requires that the ducts are often highly quash or curved as I said in lecture 14, they can be moon shape they can be sinusoidal shaped and so on and so forth.

These methods that I described is ok as long as, the boundary shape is nice and regular, which is describable by a nice function but, in order to deal with ducts, which have very complex shapes. We need to consider certain special methods and that is what I will turn to in my next lecture on complex duct cross section, with fully developed laminar flow in ducts of complex cross section.