Convective Heat and Mass Transfer

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Module No. # 01

Lecture No. # 13

Superposition Theory and Application

In the previous lecture, we saw how to account for the pressure gradient variation. In principle, that method can also be extended to other types, where vw is present or where viscous dissipation is present, but the algebra simply becomes extremely complex. Therefore, I will not deal with it, but there is a simple method also to account for wall temperature variation, which we have not considered in the previous lectures.

My purpose today, is to consider how to account for wall temperature variation in integral analysis and one makes use of what is called a superposition theory. What is superposition theory? You will know very shortly.

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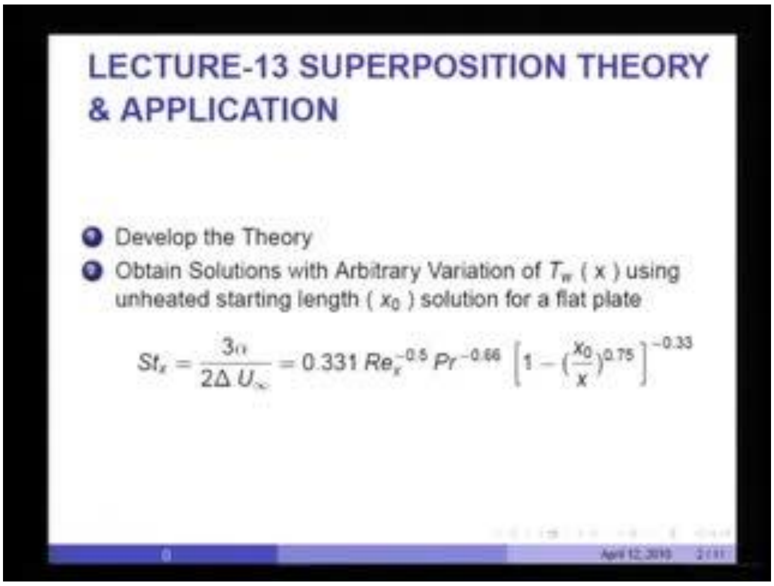

So, I will develop the theory very briefly and then obtain solutions with arbitrary variation of wall temperature using unheated starting length x naught solution for a flat plate. Just to recall your memory in the solution, for the flat plate with unheated starting length x naught, read as Stanton x equal to 0.331 Re x to the minus 0.5 Prandtl raise to minus 0.66 1minus x naught by x 0.75 raise to minus 0.33.

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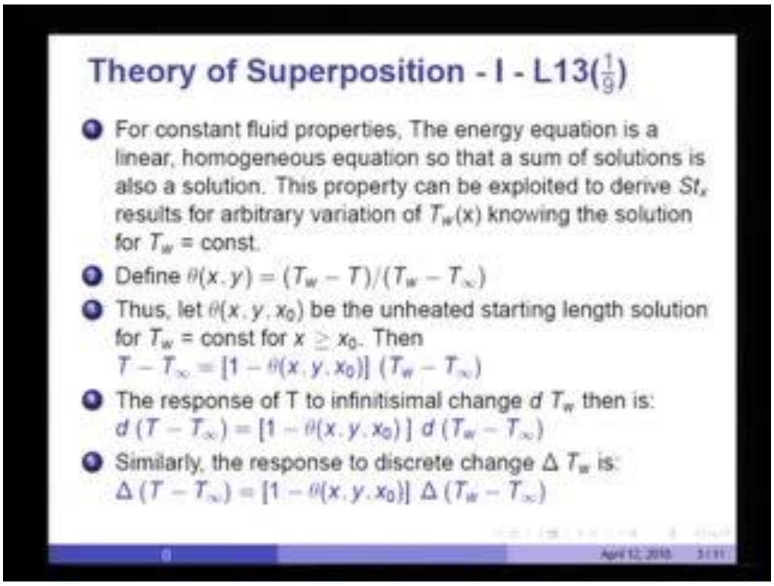

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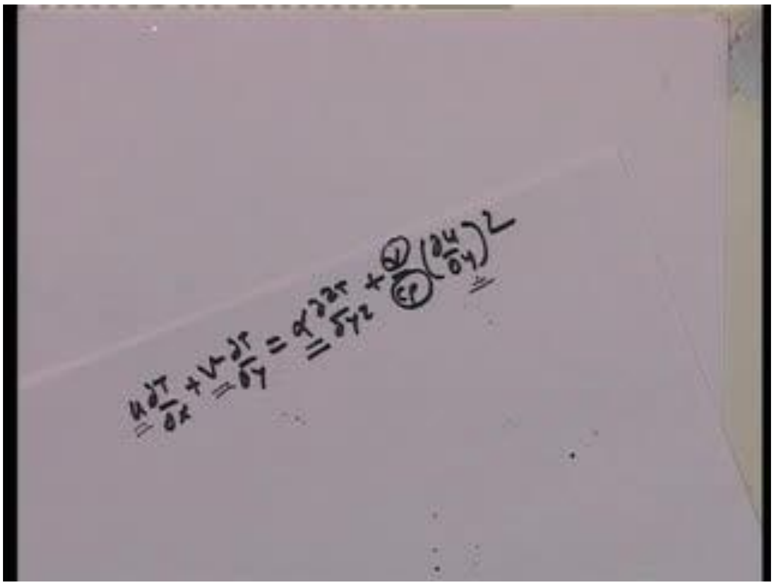

For constant fluid properties, you will recall our energy equation is actually u dT by dx plus v dT by dy equal to alpha d2T dy square plus nu by Cp du by dy whole square. If the fluid properties are constants - alpha, nu and Cp, if these are constants, then the temperature solution is independent of the velocity solution.

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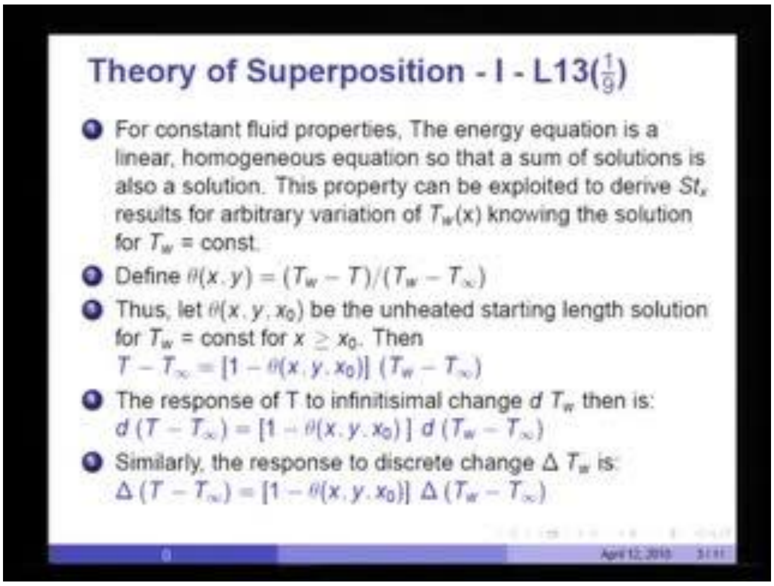

So, even u, v and du by dy whole square, all these are given in the energy equation. Therefore, what remains is a simple homogeneous equation in temperature T; such an equation is called linear homogeneous equation.

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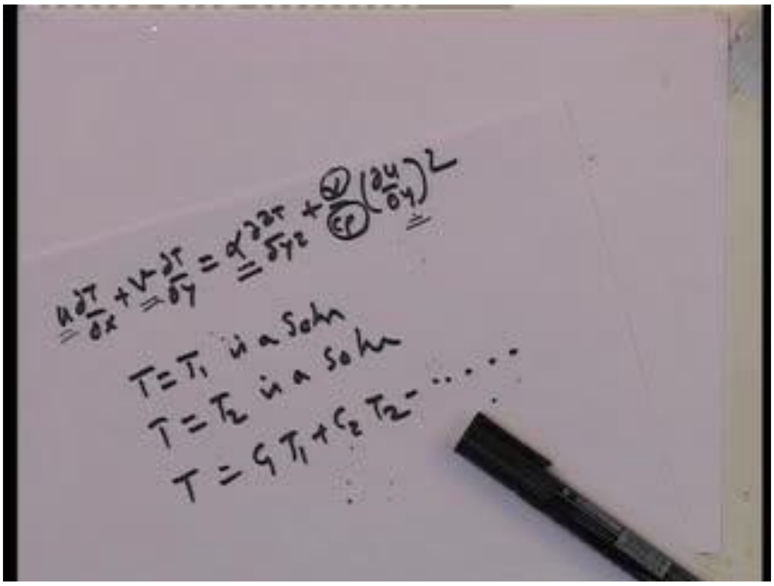

In that if T equal to T 1 is a solution, T equal to T 2 is a solution, then T equal to C 1 T 1 plus C 2 T 2 and so on so forth is also a solution. So, the sum of the solutions is also a solution. This property of a linear equation is exploited to derive Stanton x results for arbitrary variation of T w x knowing the solution for T w equal to constant.

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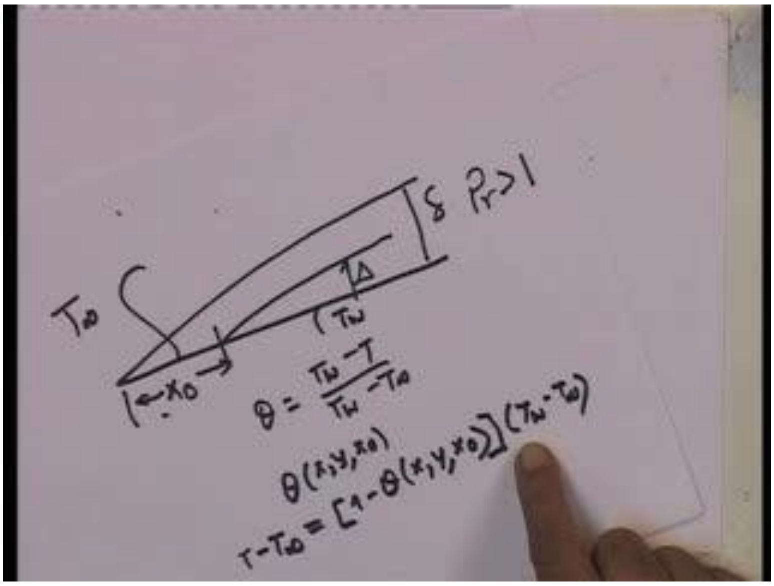

To do that we shall define theta (x, y) equal to T w minus T over T w minus T infinity as we have done before. Thus, theta (x, y, x naught) will be the unheated starting length solution for T w equal to constant, for x greater than x naught. The situation is being what we had described earlier, so the situation is very clear. This is the flat plate, the velocity boundary layers grow well, but the temperature boundary layer starts here. So, this is delta, Prandtl number is greater than 1 and this is the unheated starting length x naught.

We have defined theta is equal to T w minus T, as I said, this is T w, but here it is T infinity over T w minus T infinity. Therefore, the solution for this particular case characterized by x naught choice of x naught will be theta (x, y, x naught). Then, that solution will read as T minus T infinity equal to 1 minus theta (x, y, x naught) into T w minus T infinity.

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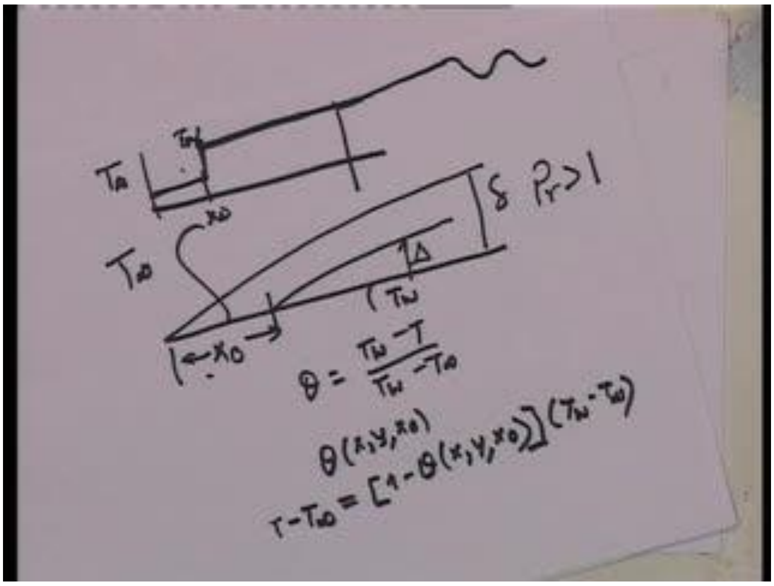

Now, you want to capture the effect of variation of T w minus T infinity. Now, in reality, T w can change abruptly like a step change or it can change in a continuous manner; both are possible. But, the case we have here is precisely the case of a step change, because if I plot T w, well it was equal to T infinity up to x naught and has been suddenly raised to T w from x equal to x naught.

In other words, we have a solution which responds to a step change in wall temperature, but I may also have, after some length x, it may vary linearly or it may go like that; any variation is possible. These sorts of variations occur over gas turbine blade surfaces.

When you are cooling electronic boards like printed circuit boards, there are wires carrying currents, there are condensers there are capacitors, there we generate heat at different temperatures. You have a cooling air flowing over it, so the wall temperature can either vary abruptly or can vary continuously. We want to be able to predict how would the heat flux change or how would heat transfer coefficient change along this length, when the wall temperature varies in an arbitrary fashion.

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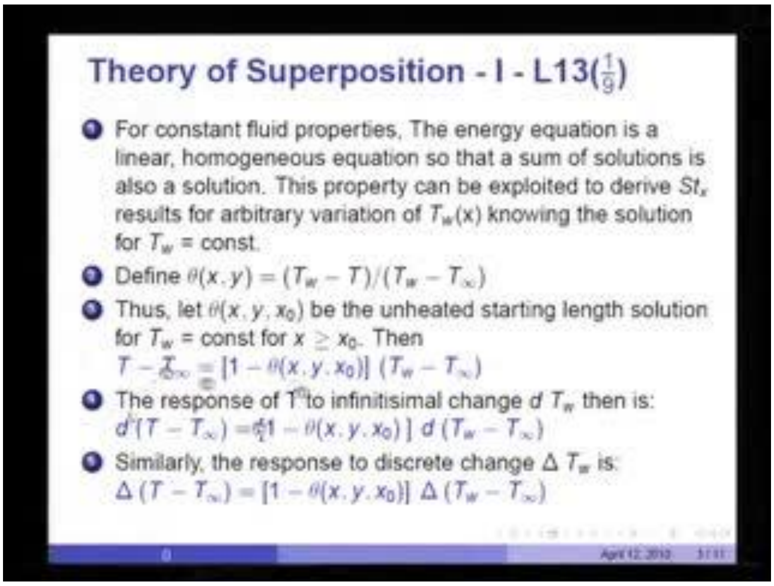

For continuous variation, the response of this function to an infinitesimal change dT w will be simply d T minus T infinity equal to 1 minus theta (x, y, x naught) dT w minus T infinity. This is straight forward; I have just differentiated this and this part keeping this constant.

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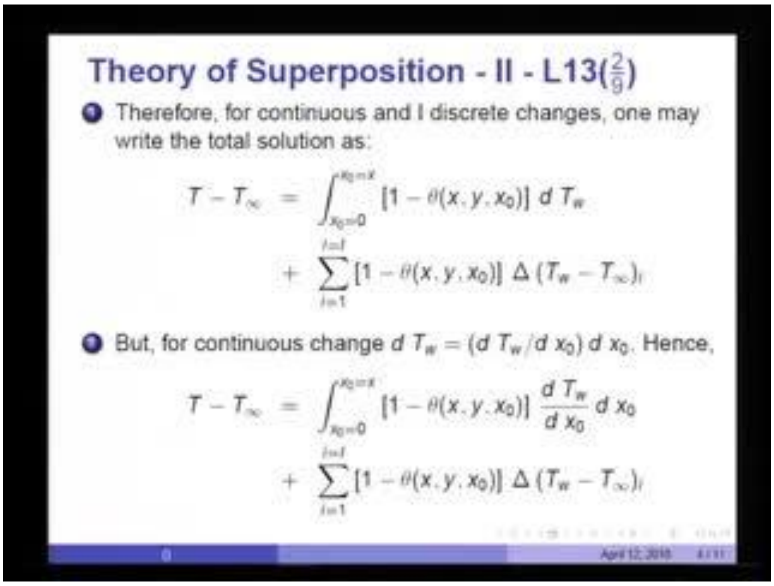

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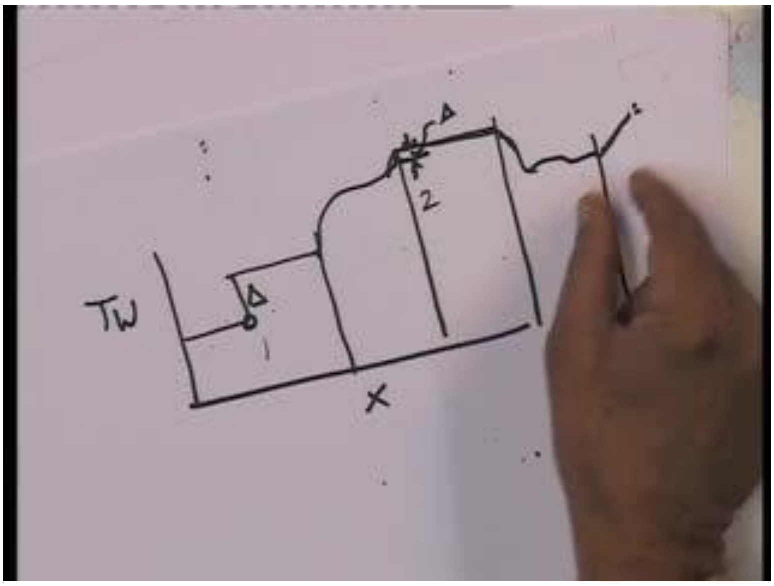

However, for a finite change or a discrete change, the response will be delta times T minus T infinity 1 minus theta (x, y, x naught) delta times T w minus T infinity. These are response functions to a small change, either discrete or continuous. Therefore, suppose, I have a temperature variation of T w with x, let us say it is first a step change, then another step change, then some continuous variation, then constant, then another continuous variation and so on so forth.

So, then I have several situations of - there is a discrete change here, there is a change delta here, but there is a continuous change here, then there is a constant value, then there is a continuous change like that and then there is a continuous change again here. But, from here, I have a little continuous discrete change here, let us say another delta somewhere (Refer Slide Time: 08:08).

So, I may have variation in which there are 1, 2, 3, 4, 5 discrete changes and there are several portions over which the change is continuous.

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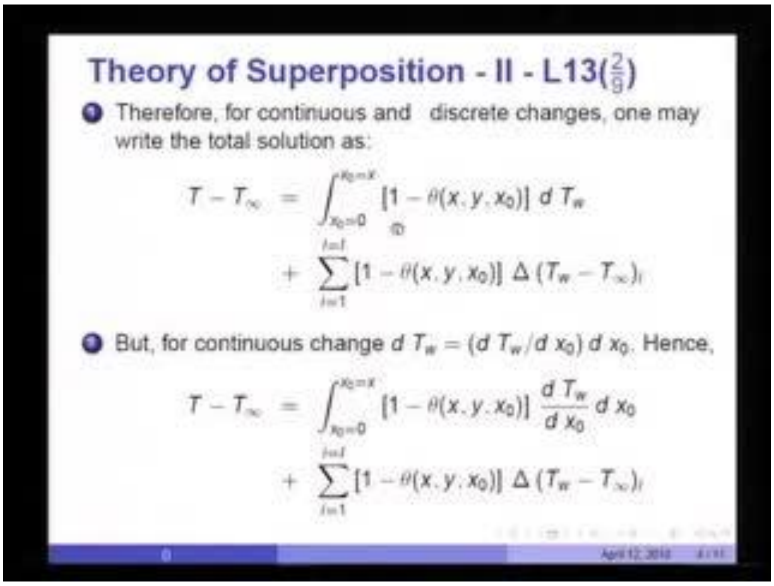

So, T minus T infinity x naught equal to 0 to x equal 1 minus theta x naught plus dT w plus the total response for such a change would be little bit from the continuous change and a little bit from discrete changes i. So, sum of i equal to 1 to i equal to i and to T w minus T infinity, but notice that is in the continuous part dT w, is nothing but dT w dx naught into dx naught. Therefore, I can also write the total response temperature function as x naught equal to 0 to x naught equal to x into 1 minus theta (x, y, x naught) dT w by dx naught dx naught into 1 minus theta (x, y, x naught) into delta T w minus T infinity i.

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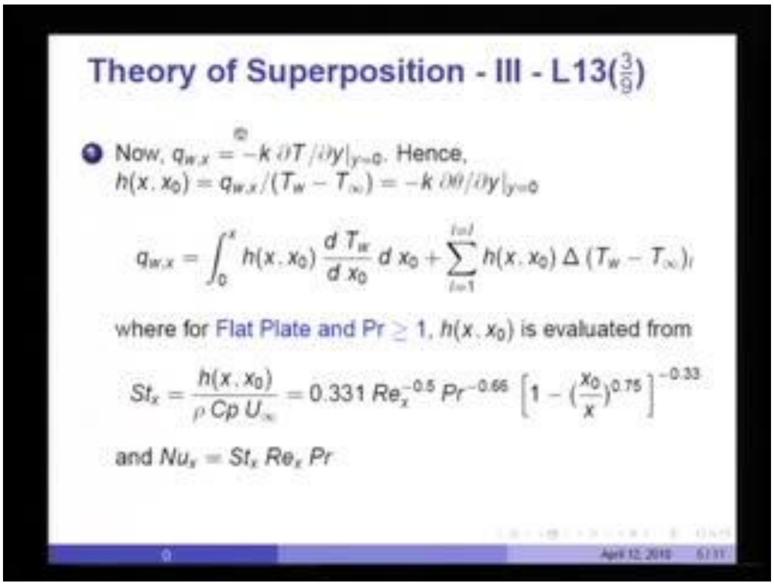

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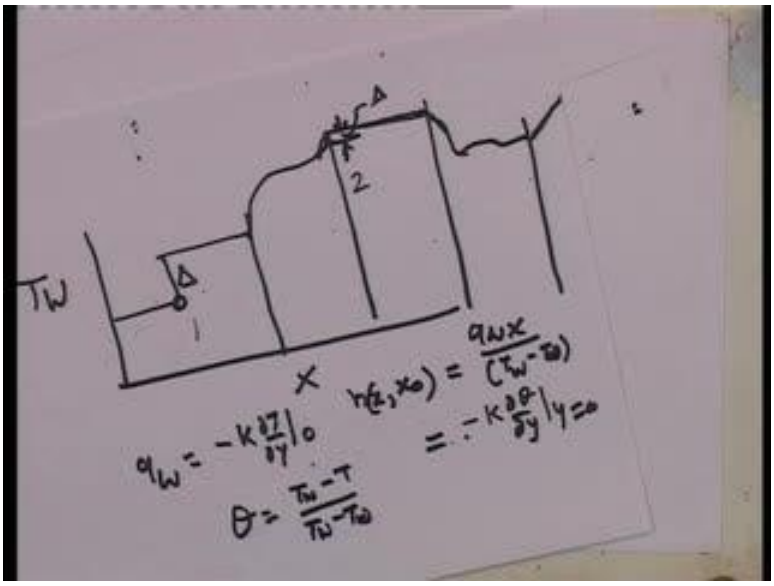

If I differentiate this equation with respect to y, dT by dy at y equal to 0 with a negative sign. Then q w will be simply minus k dT by dy equal to 0 h (x, x naught) will be simply q wall x over T w minus T infinity and that will become equal to minus K d theta by dy y equal to 0, because the way we have defined theta. Theta was defined as T w minus T over T w minus T infinity. Therefore, you will get this as d theta by dy equal to 0.

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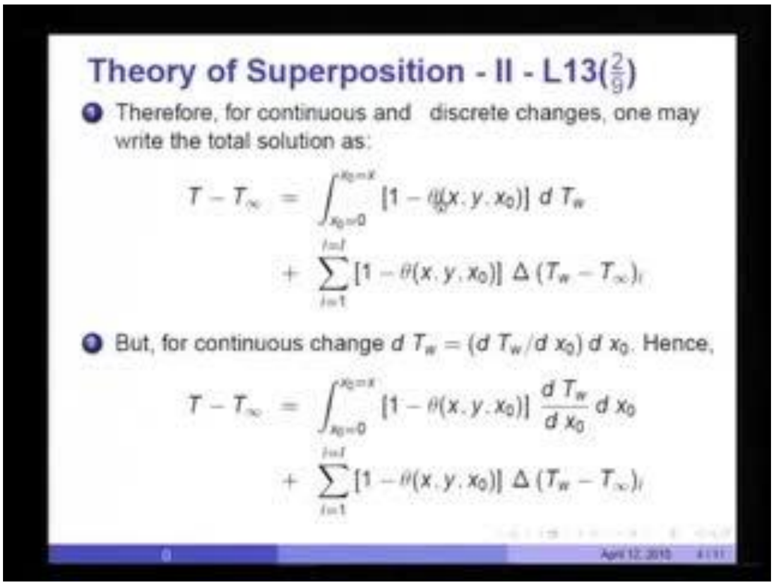

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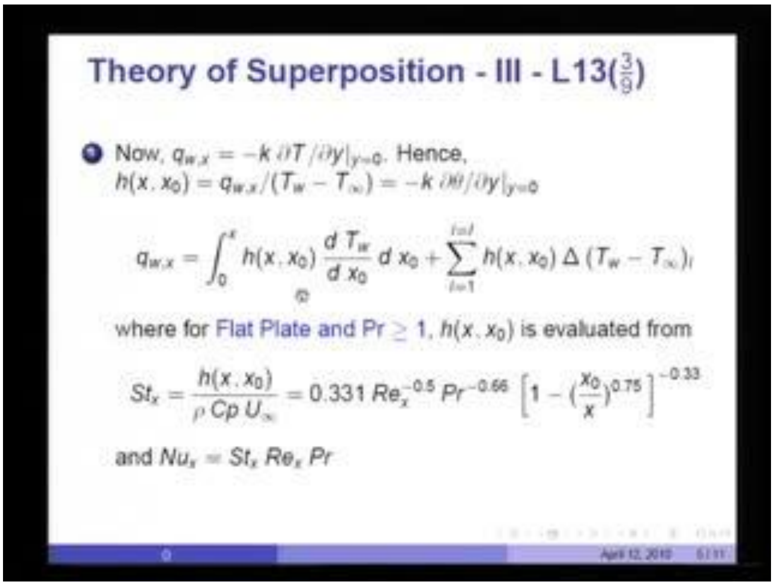

Here, this will turn to - when I differentiate this temperature with respect to y, I will get minus d theta by dy (x, y, x naught) and here also minus d theta by dy (x, y, x naught), which will be nothing but the heat transfer coefficient (x, x naught) as a continuous variation dT w by dx naught h (x, x naught) into delta T w minus T infinity i. Where for flat plate and Prandtl number h (x, x naught) will be evaluated from our relationship here, so it will be simply h (x, x naught) rho Cp U infinity into 0.331 Reynolds x to the power minus half Prandtl to the power minus 0.6 and so on so forth.

So, I can get now for a given wall temperature variation, the response of the heat flux as a function of x. Once, I know the response of the heat flux, I simply divide by the local value of T w minus T infinity to get the heat transfer coefficient variation over the entire surface.

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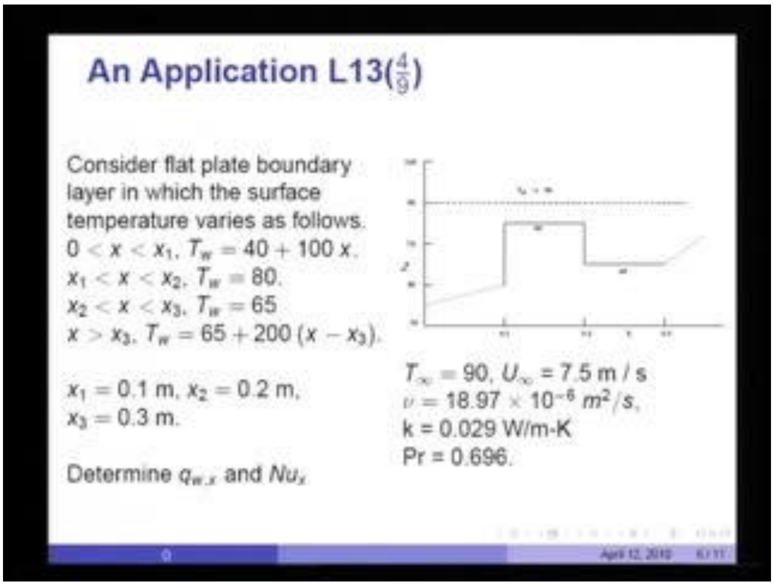

How do we do this now? I am going to take a very simple problem here. It shows a flat plate boundary layer in which the surface temperature varies as, for 0 to x 1 where x 1 is 0.1 meters. The temperature is varying linearly as 40 plus 100 x. T infinity is 90, so there is a hot gas flowing over a colder surface. So, remember the temperature here is 40; it will change to 50 at x equal to 0.1. Now, at x 1, the temperature suddenly rises to 80 and remains so up to 0.2.

So, there is a constant temperature, but still less than the free stream temperature. For x 2 to x 3, the temperature is lowered now to 0.265 degree centigrade. For x greater than x 3, the temperature again raises linearly, let us say as 200 x minus x 3 plus 65.

Remember, the wall is throughout at a lower temperature and then at the free stream temperature. Therefore, one would expect the heat flux to be continuously pouring into the surface. But, you will see, because of the wall temperature variation in x direction, this will not be so. There will be situations where, although T infinity is greater than the T wall, the heat flux will actually flow out of the surface.

These kind of situations, we considered even in similarity solution, where T w minus T infinity was allowed to vary as c x raise to gamma. U infinity is given as 7.5 meters per second. The kinematic viscosity is this, thermal conductivity is this, and Prandtl number is 0.696 (Refer Slide Time: 14:03).

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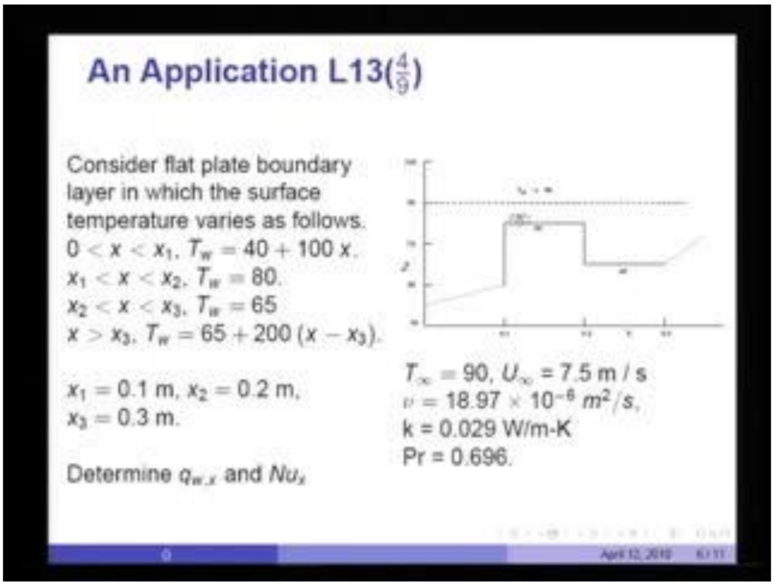

Remember, although I said our analysis is valued for Prandtl greater than 1, essentially where delta by delta is less than or equal to 1. So, 0.696 is not too bad one; can use the same results even for air, then no harm done. We would still capture the essential features of the solution.

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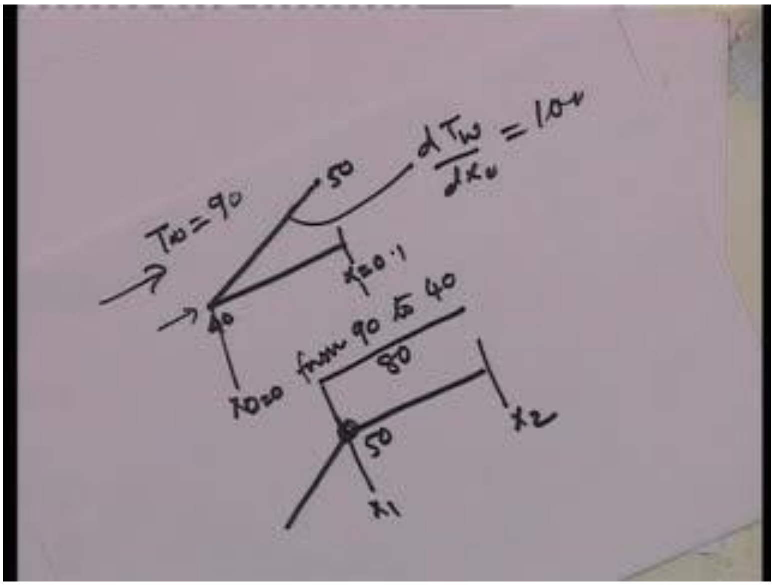

So, our task is to determine how the wall heat flux and the Nusselt number would vary with x in response to such a variation of wall temperature. All we know is, for a single step change, how the heat transfer coefficient should vary.

For 0 x x 1 the temperature varies like this. This is x 1 equal to 0.1 meters, so the temperature is 40 and it goes up to 50 (Refer Slide Time: 14:58). The free stream temperature T infinity is 90, so I can say that if x naught equal to 0 itself, there is a step change from 90 to 40 and then there is a continuous change.

The continuous change part would be written as 1 minus x naught by x 0.75 into minus 0.33 dT wall by dx naught in this region is 100. I will show you the previous slide, see 100 dT w by dx is 100, but there is also a step change 40 minus 90 equal to minus 50, so delta T w0 will be minus 50 and this is down. A would be 0.331 R e x to the half Prandtl to the minus 0.6 would remain as a multiplier and I would integrate this from 0 to x where x naught varies into dx naught delta T w.

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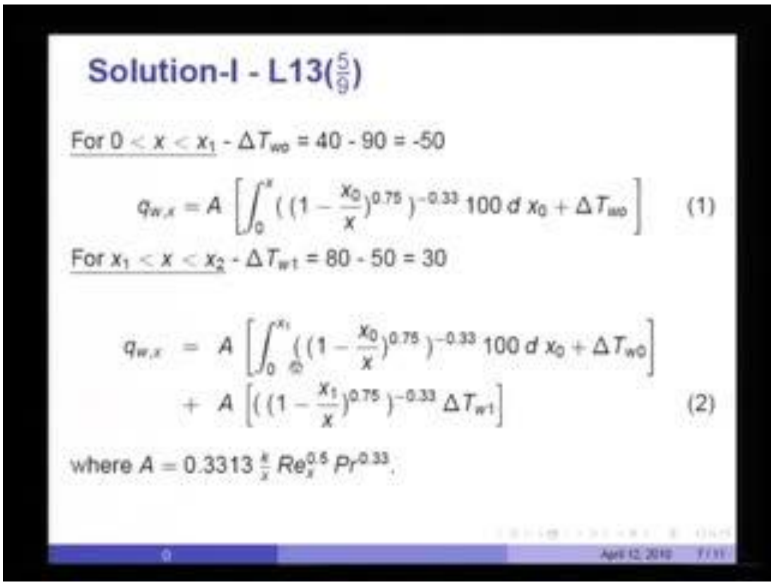

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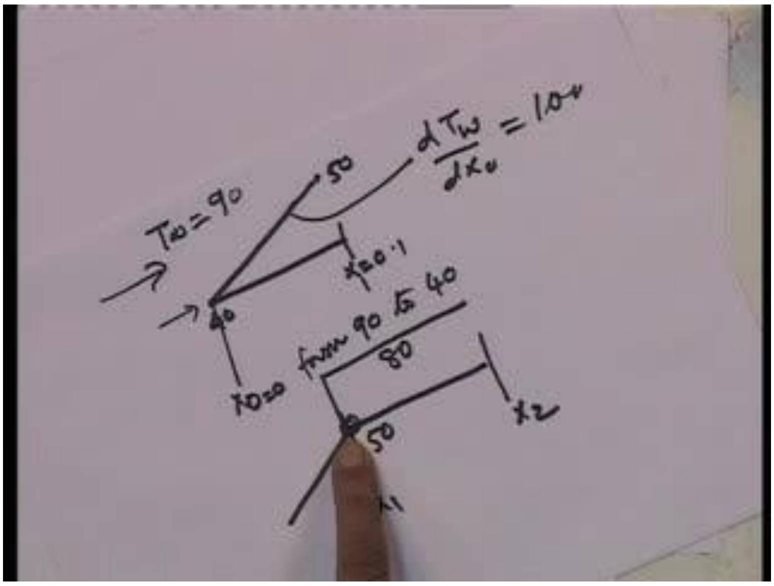

The temperature has increased from 50 to 80. In this region, the effect of the previous solution would survive, A into 0 to x 1 part; that is the first part. That would still be the solution at x 1; this is the solution at x 1 plus, now the solution, sorry there should be a plus sign here, plus delta T w1 and plus A 2; this is the solution x 1 to x integral, then there is an error here, which we shall correct plus delta T w1. Delta T w1 is 80 minus 50, in fact 80 minus 50 should be read correctly as 80 minus 90 minus 50 minus 90 equal to 80 minus 50, which is plus 30.

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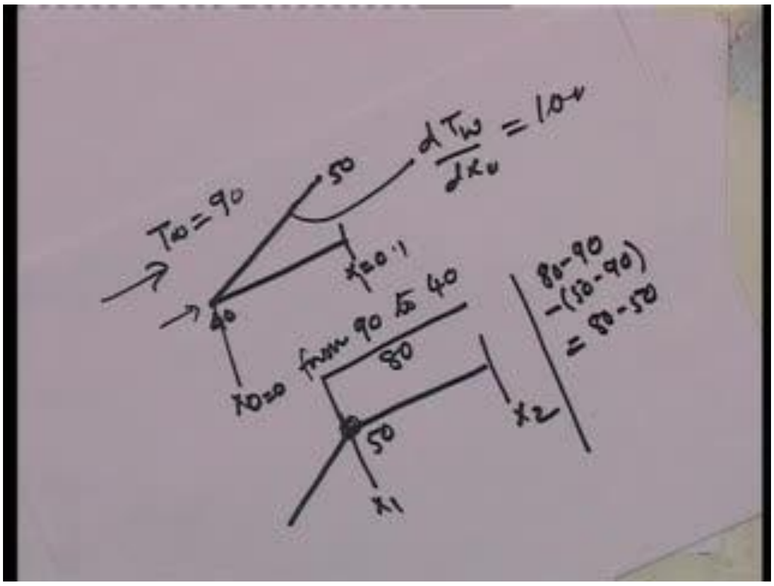

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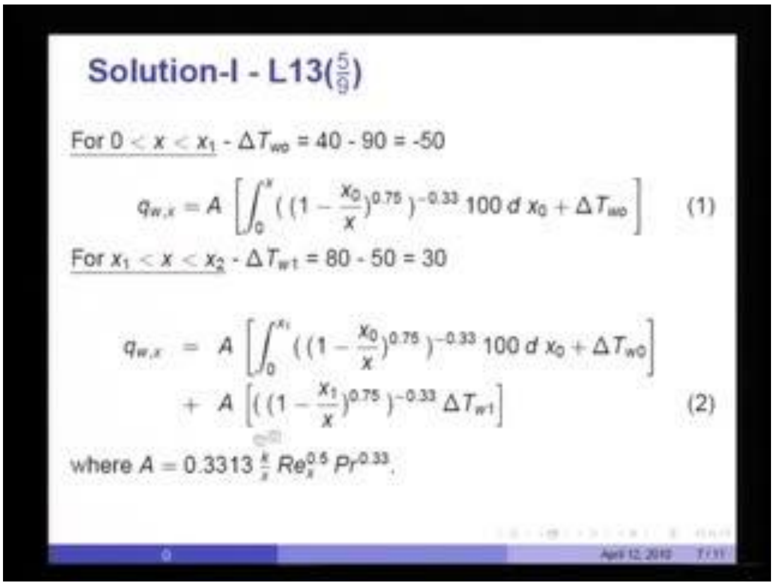

So, remember, there is an error here; it should be 0 to x 1, 1 minus x 1 by x 0.75 dx 1 plus delta T w1. There is no error, because between x 1 and x 2 there is no continuous variation, only a step change and therefore this result is correct. Where it is 1 minus x 1 by x 0.75 0.33 into delta T w1 (Refer Slide Time: 17:48). So that is absolutely correct and a will be as before.

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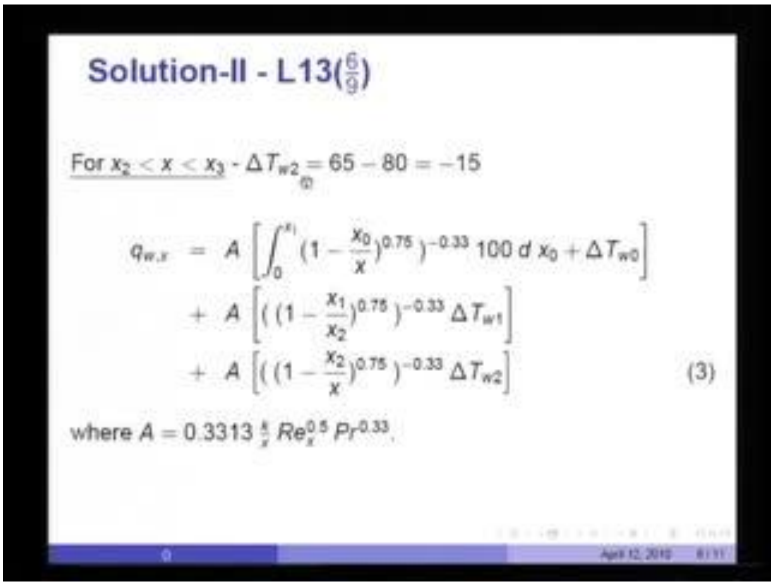

From x 2 to x 3, now there is a delta T w2, which is 65 minus 80 is minus 15.

So, remember here, the solution up to x 1 is correct. This is the solution up to x 2 now, this is x 1 divided by x 2 0.75 0.33 delta T w1 (Refer Slide Time: 18:25). Now, you have another step change, which is 1 minus x 2 by x 0.75, it will go up to x 3. x will go from x 2 to x 3 with a step change delta T w equal to minus 15, because 65 minus 80 is minus 15, A is again that.

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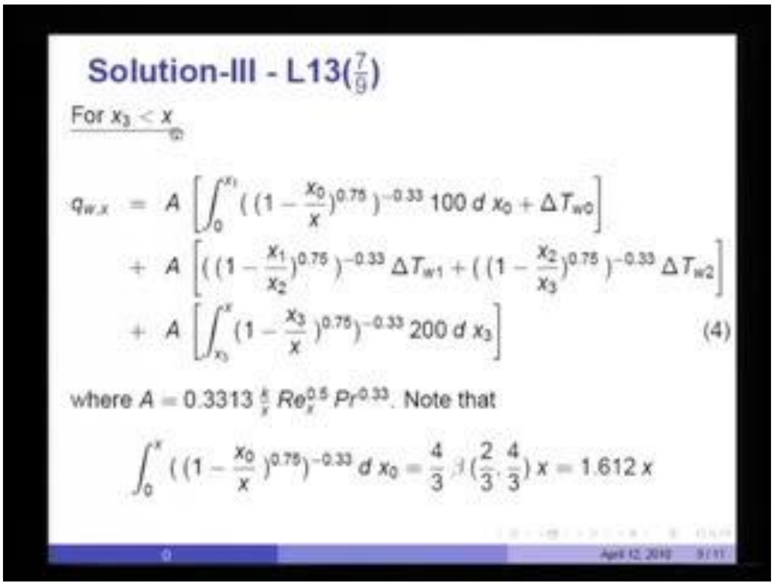

This is the solution up to x 1, this is the solution up to x 2 and from x 2 onwards, but less than x 3, this is the solution (Refer Slide Time: 19:05). For x greater than x 3, this is the solution up to x 1, this is the solution up to x 2, this is the solution up to x 3, because x 2 as gone up to x 3. You will have simply integral x 3 to x 0.75. There is no discrete change here; simply from 65 there is no discrete change, is only a continuous change.

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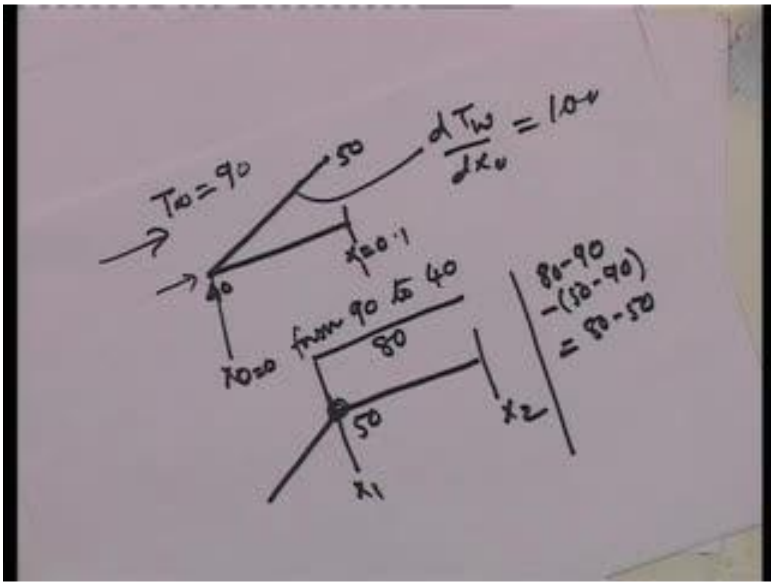

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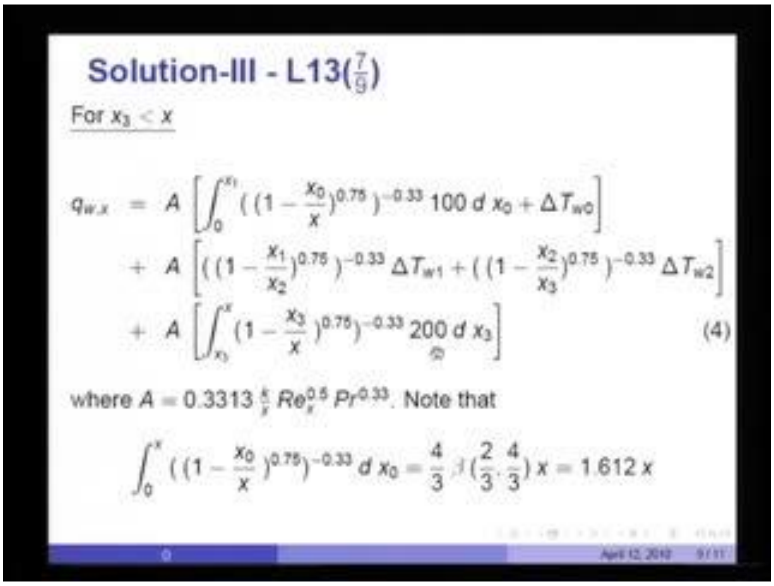

Therefore, in the last part, you will have 20 times dx 3, this is nothing but dT w by dx naught and 1 minus x 3 by x 0.75, where A is equal to all that.

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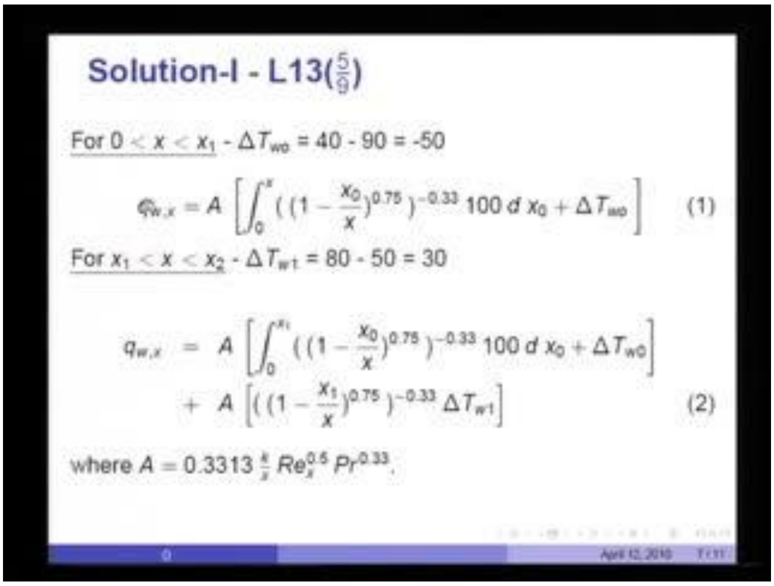

Notice that we have slowly built up the solution by superposition. This is what is meant by superposition, you started with the solution for the first step change, which was valid for x greater between 0 and x 1; so you could integrate this relation.

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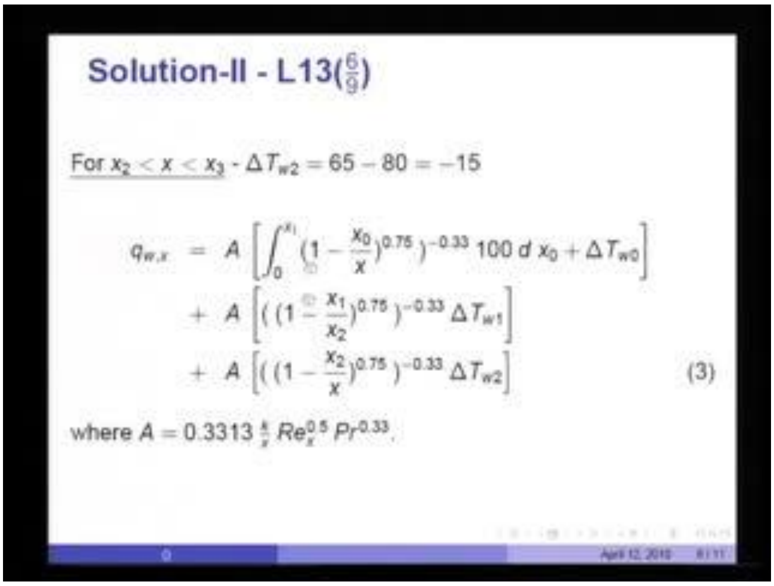

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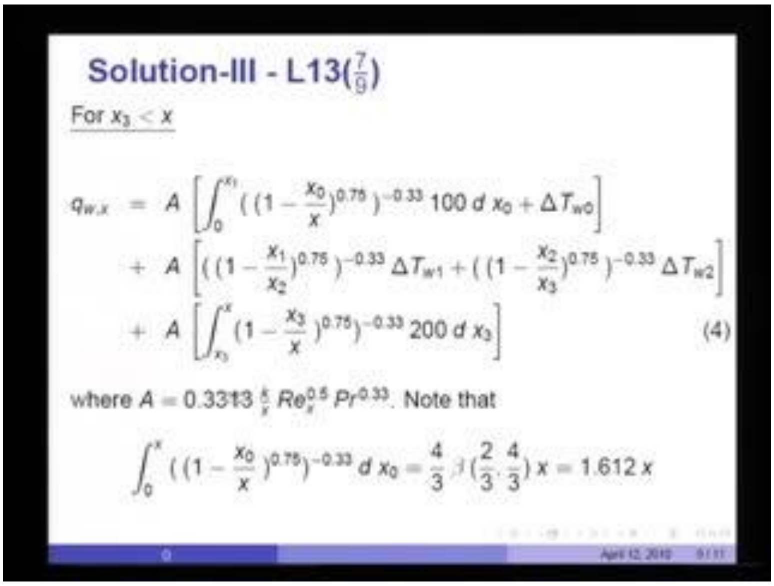

Then, from x 1 to x 2 you had solution at x 1 plus this discrete change solution, where x varies from x 1 to x 2, so the solutions are valid from there, so x is simply taken from x 1 to x 2. Then, from x 2 to x 3 you had discrete solution up to x 1, discrete solution up to x 2 and then a third discrete solution from x 2 to x 3. Then, you had the continuous variation, where this is discrete solution up to x 1, this is up to x 3 and then the continuous solution.

Now, wherever the integral sign involve like in this case, here, the integration 0 to x 1 minus x naught by x raise to 0.75 raise to minus 0.33 dx naught, is actually a beta function. It can be shown that is equal to 4 by 3 beta (2 by 3, 4 by 3) x and that is equal to 1.612 x; the very useful result.

So, one can make use of that here to simply replace these quantities by 1.612 x that is all there is to it. Now, which will be 1.612 x1 to 0 - x 1 minus 0 multiplied by 100 and this would be 200 into 1.612 x minus x3 into 200.

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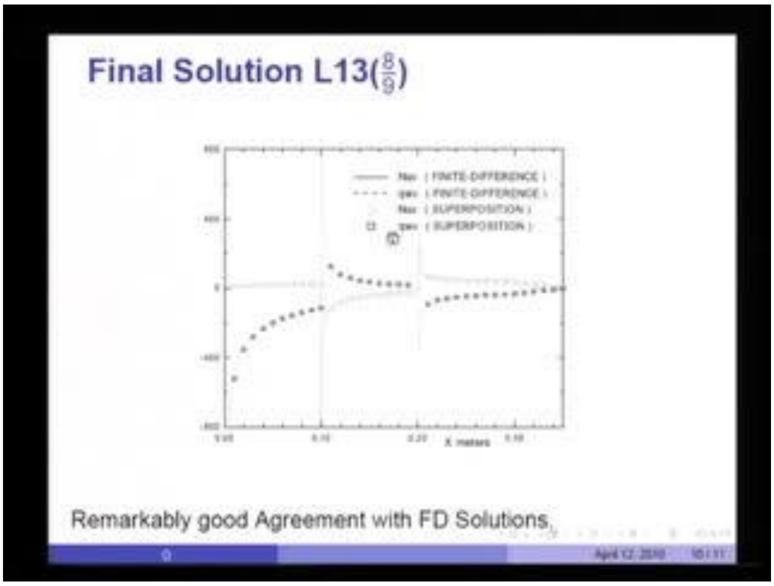

So, these are the solution we would proceed with, I will get q w x variation for different segments of the solution. What would they look like? The squares show the variation of q w x. As you expect, in the first part, the wall temperature varied from 40 to 50, the free stream was at 90 and therefore heat was simply flowing into the wall. So, q w x is negative, but rising, because as the temperature difference goes on reducing, the amount of heat flow is also going on reducing. The moment however the temperature moves - of the wall moves from 50 to 80, remember 80 is less than 90. Then, what happens is that the layers of fluid close to the wall are still at a temperature close to 50.

Although, the wall temperature is 80 now, so the heat actually flows from the wall to the fluid, although the free steam temperature is higher than the wall temperature. So, one would expect heat transfer to be positive from the wall to the fluid, because the fluid layer is close to the wall, are close to 50 degree centigrade.

Therefore, you see suddenly the heat transfer becomes positive and starts to fall in this manner. Then, when you come to from 85 to 65, there is now the wall is at 65, but the fluid close to the wall is close to 80 degree centigrade and it is likely to be at a higher temperature than 65 degree centigrade. Therefore, again the heat transfer goes into the fluid. Ultimately, up to 3 it is negative and then it continues when the wall temperature begins to rise, you get positive heat transfer.

Now, this is the case corresponding to that would be Nusselt numbers; you will see Nusselt number is more or less very slow decline, then that Nusselt number has turned negative and then again positive.

Now, this is the case really that can be handled by numerical methods or the finite difference method. I have done calculations of this type for this case using a computer program for a finite difference method. What I have done is, plotted the finite difference solutions by a dotted line for q wall x and Nusselt x agreement is x.

This shows you the power of the integral method and the superposition theory to capture effects of arbitrary variations of wall temperature. But, the case we have consider is that of a flat plate boundary layer, is very simple case, free stream remains constant.

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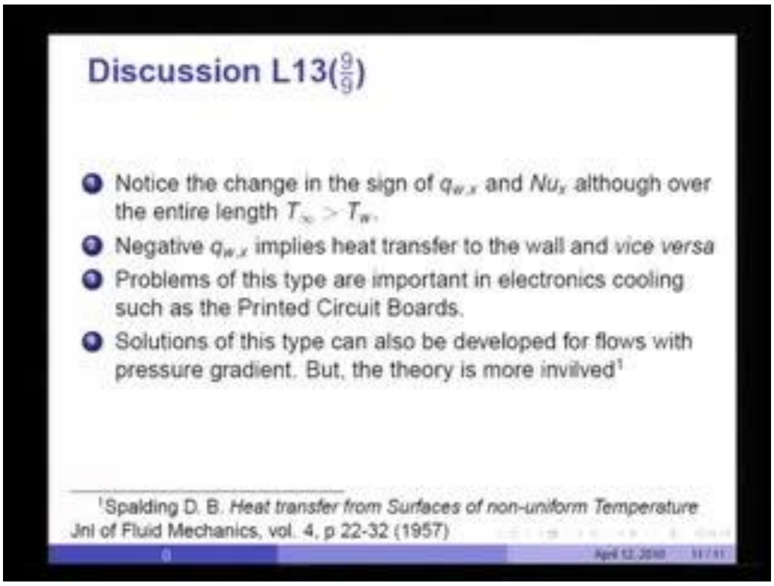

Notice the changes in q wall x Nu x, although over the entire length T infinity is greater than T w. So, you could get situations where actually the heat transfer takes place from the solid surface to the fluid, although the fluid bulk free stream temperature is higher than the fluid.

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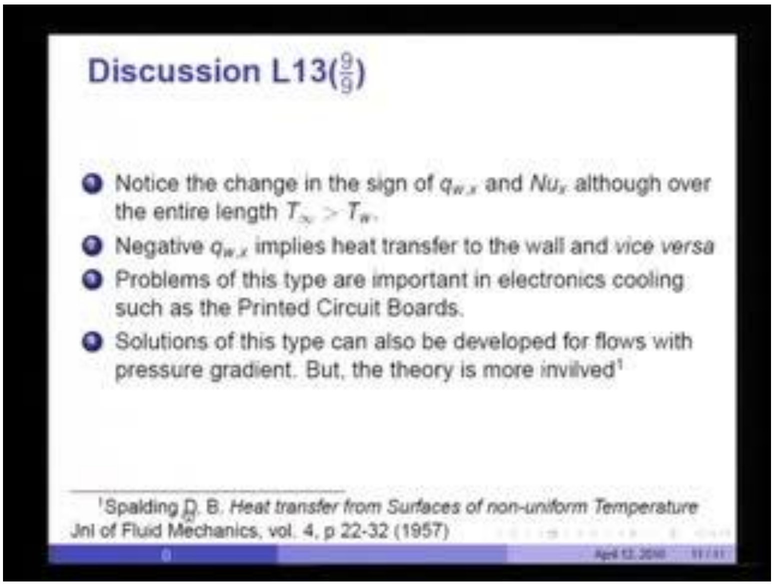

Negative q w x implies heat transfer to the wall and vice versa. So, problems of this type are important in electronics cooling such as printed circuit boards. So that many a time the heat generating surface itself would begin to get if it was in the wake of a certain type of temperature variation. Then, a heat generating condenser would actually instead of giving out heat, would actually receive heat from the gas.

Either, it can go into any adiabatic case or actually positive heat flux from the gas - air to the component and the temperature of that component would suddenly shoot up, a drift would occur. As you know, in electronic equipment, the components should not vary too much in the temperature drifts in order to deliver a certain performance.

Very important consideration in printed circuit board design is the thermal management of how to put the current carrying conductors as well as, the condensers and other pieces of electronic equipment.

Now, as I said, we have developed this solution for a flow over flat plate with arbitrary variation. What about flow over a gas turbine blade, where the free stream will vary or flow over a cylinder, where free stream vary in an arbitrary fashion and so the wall temperature vary in arbitrary fashion.

You have to simply go back to the velocity and temperature equations that we have derived, evaluate the delta 2 in the most laborious way and then allow for dT w by dx term in the integral equation. From there on, calculate the variation of delta 2 with x and from which, you would extract the heat transfer coefficient variation.

The procedure is quite laborious, there are some simplifications possible. One of them is published by a Spalding D B, heat transfer from surfaces of non uniform temperature in journal fluid mechanics in 1957.

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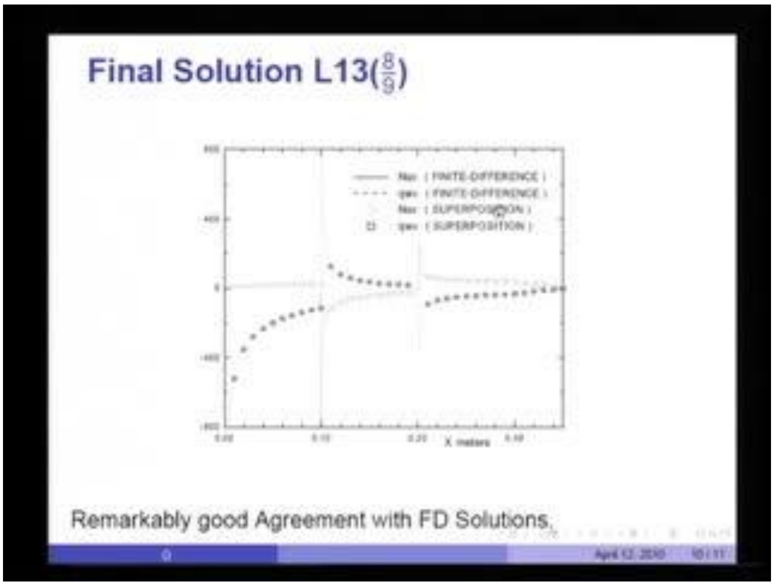

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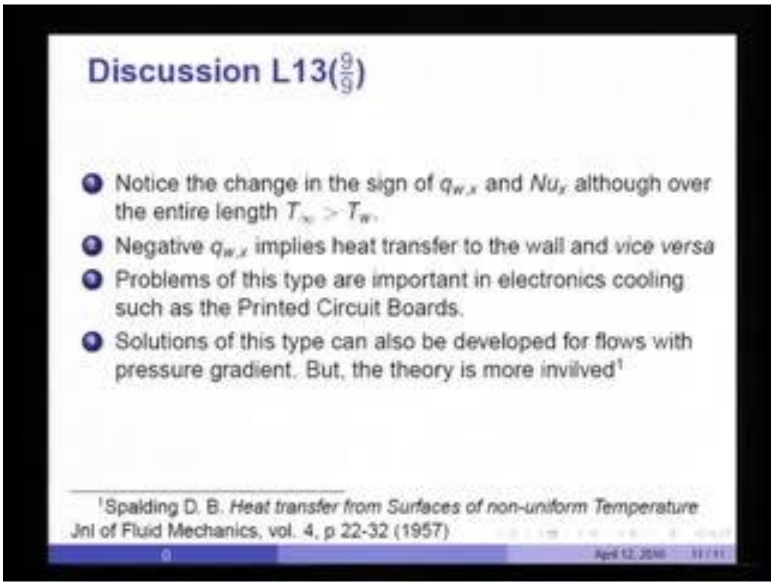

One could look at those equations, but as I must say that today, no one does integral solutions of this complexity any more. People have turned to numerical solutions and that was my purpose in showing you how could the numerical solutions impact on the finite difference solution. For the simple cases, at least the agreement is simply excellent. With this, I complete our discussion on integral solutions to laminar temperature boundary layer equations.