Convective Heat and Mass Transfer

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Module No. # 01
Lecture No. # 10
Integral Equations of BL

How to derive and solve similarity equations to obtain friction factor and Nusselt number for two-dimensional laminar boundary layers? Now, we will move to the next method; that is, the integral method for solving boundary layer equations. Therefore, the task for this lecture is to derive the integral equations of a boundary layer.

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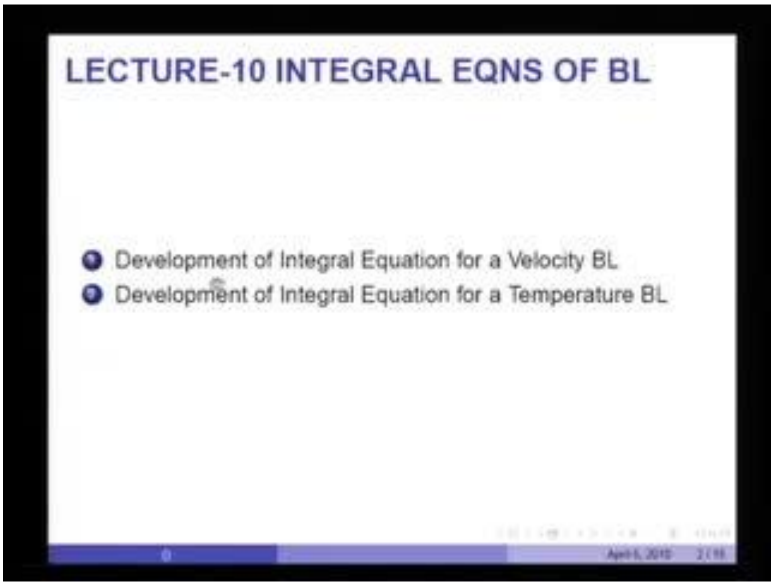

First is the development of the integral equation for a velocity boundary layer. Next is the development of integral equation for a temperature boundary layer. Solve these equations to obtain friction factor and Nusselt number for two-dimensional laminar boundary layers.

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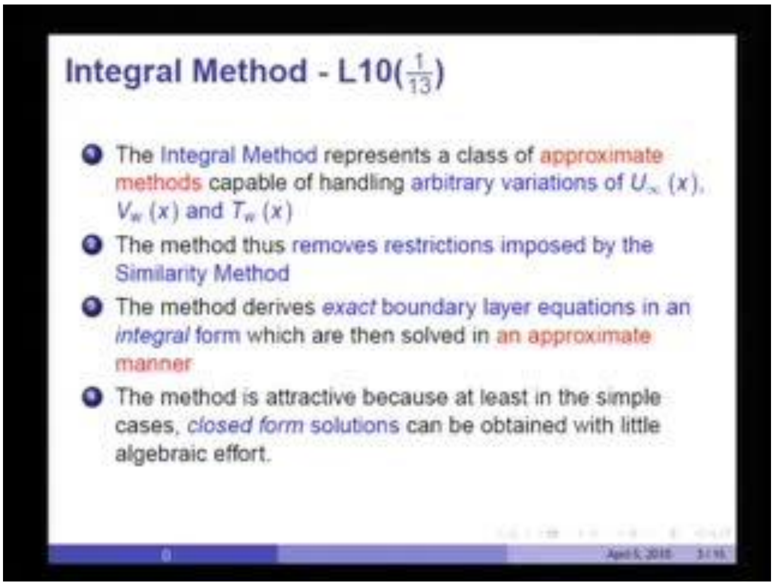

We now move to the next method, it is the integral method for solving boundary layer equations. Therefore, the task for this lecture is to derive the integral equations of a boundary layer. So, we shall first begin with the development of the equation for a temperature boundary layer.

The integral method represents a class of approximate methods capable of handling arbitrary variations of U infinity, suction or blowing velocity - V w and wall temperature variation - T w. In this respect, you will recall that similarity method permitted only certain type of variations of the free stream velocity, the wall velocity and the wall temperature variations. The method is called approximate method, not because its equations are approximate. The method actually derives exact boundary layer equations, but in integral form and not in differential. It gives them solution methodology, which is approximate and not the equations.

A very important point to remember in all further development, unlike the similarity method, this method is attractive because at least in some simple cases, closed form solutions can be obtained with little algebraic effort. At one time, people used to call the integral because you could do all the calculations by simple algebra without requiring a computer. The second advantage of the method is of course that it can now deal with any arbitrary variations of U infinity, V w and T w.

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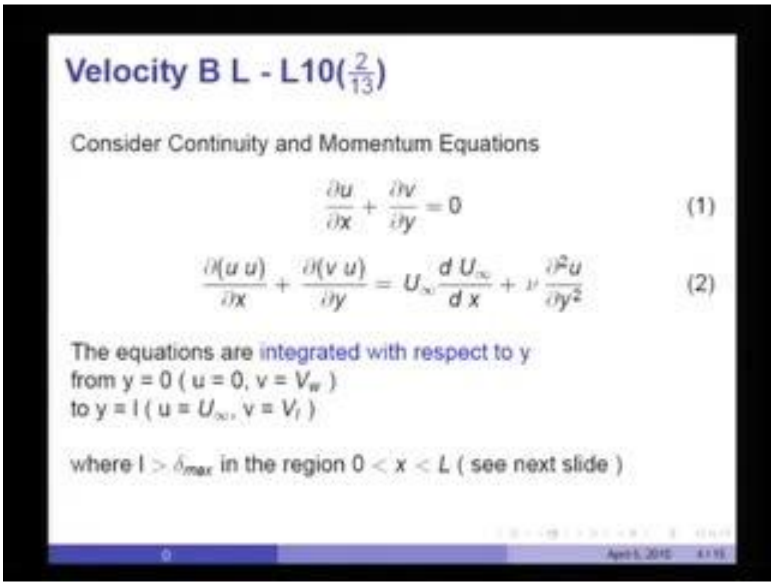

First of all, let us consider the boundary layer- d u by d x plus d v by d y equal to 0. The convection terms or the momentum transfer terms, the pressure gradient term and the viscous term. These equations are integrated term by term with respect to y of the boundary layer. From y equal to 0, where u is equal to 0 and v is equal to V w and to y equal to l, where u equals U infinity and v equals some fictitious velocity V l. l is greater than the delta max in the region 0 less than x and x less than L; you will see on the next slide.

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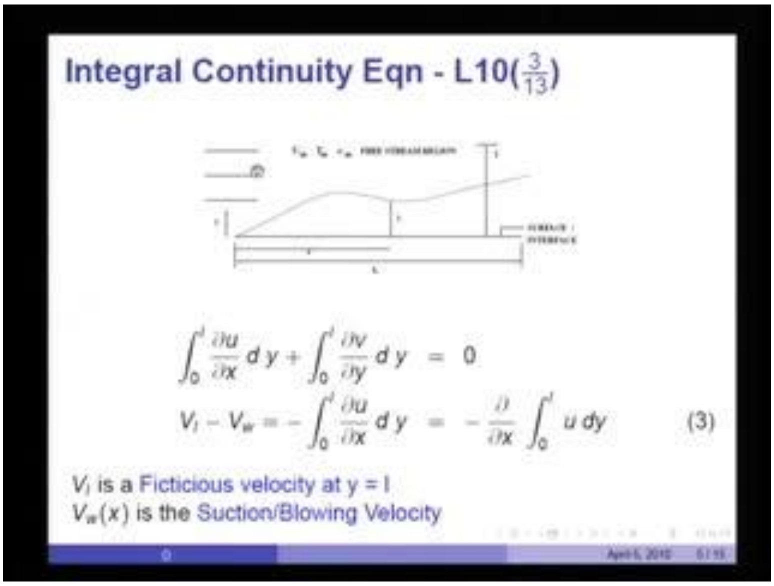

This figure shows a surface on which a boundary layer grows. Of course, a boundary layer development can be uneven. The boundary layer can grow, shrink and then grow depending on the pressure gradient that is applied. When U infinity varies arbitrarily with x, the boundary layer thickness will vary arbitrarily with x. We always choose to analyze a boundary layer over a given length L. In this length L, we make sure that we choose a dimension delta L, which is greater than delta max in this region as shown here.

Integrating continuity equation term by term, we will get 0 to l d u by d x dy. This term will simply reduce to V l minus V w and this term is taken to the right hand side as minus 0 to l du by dx dy. I can take the differential out and write it as minus d by d x of 0 to l u dy. You will see then that V l minus V w simply represents the change in the flow rate between two successive stations. V l is a fictitious velocity at l and V w (X) is the familiar suction or blowing velocity.

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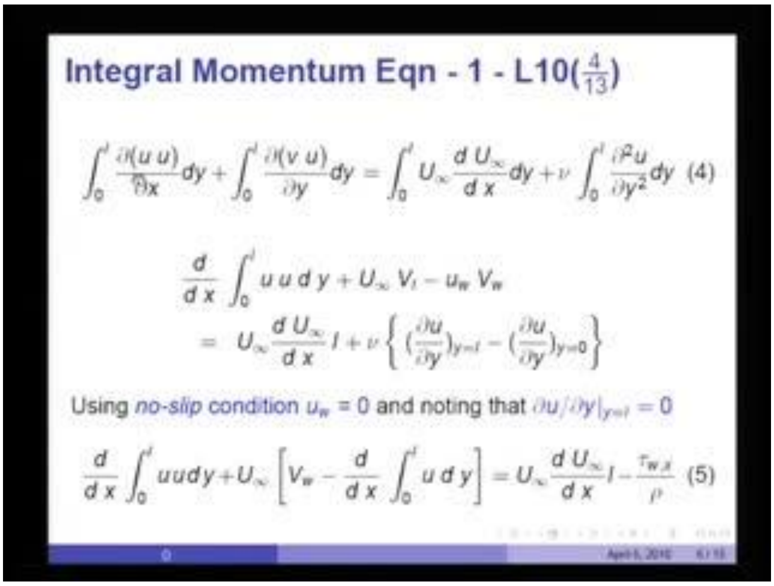

If I have to integrate momentum equation term by term, then you will see 0 to l d by dx of u u dy plus 0 to l d by dy of u v dy equal to 0 to l U infinity d U infinity by dx dy plus mu times d square u by dy square dy. On taking the differential out, this term will simply be d by dx of 0 to l u u dy plus U infinity V l minus u w V w equal to U infinity d U infinity by dx is not a function of y. Therefore, it can be taken out. Integral 0 to l dy would simply yield l, whereas this in turn would integrate to mu times du by dy at y equal to l minus du by dy at y equal to 0.

Notice, this term (Refer Slide Time: 07:37) will vanish because u w is equal to 0. So that term goes to 0 and likewise, du by dy, y at l is also equal to 0. The term that survives is mu by rho du by dy at y equal to 0. Mu into du by dy at y equal to 0, it is nothing but the shear stress divided by rho and that is what you see on the right hand side. This term is represented here and I have replaced V l into U infinity. From the previous result as shown here, V l is V w minus d by d x of u d y and of course, this term survives.

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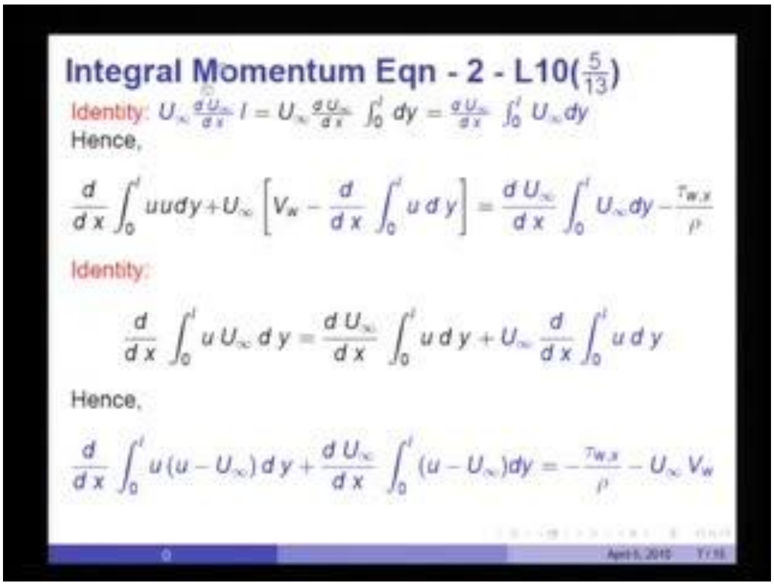

We will move further. U infinity d U infinity by dx l is nothing but U infinity d U infinity by dx into 0 to l dy. Since U infinity is not a function of y, I can absorb it inside the integral and write it as d U infinity by dx into 0 to l U infinity dy. Then, I would read the first term plus U infinity into V w minus d by dx into u dy. Now, this term is replaced by d U infinity is minus tau wall x by rho

Now, consider the identity d by dx of a product. Integral 0 to l u dy plus U infinity d by d x integral 0 to l u d y. This is precisely the term you see here; so, I am going to replace here. If I do it, then you will notice that I can write this equation in this manner, d by d x 0 to l u into u minus U infinity dy plus d U infinity by dx into 0 to l u minus U infinity into dy. This equals to minus tau wall x divided by rho and minus U infinity into V w.

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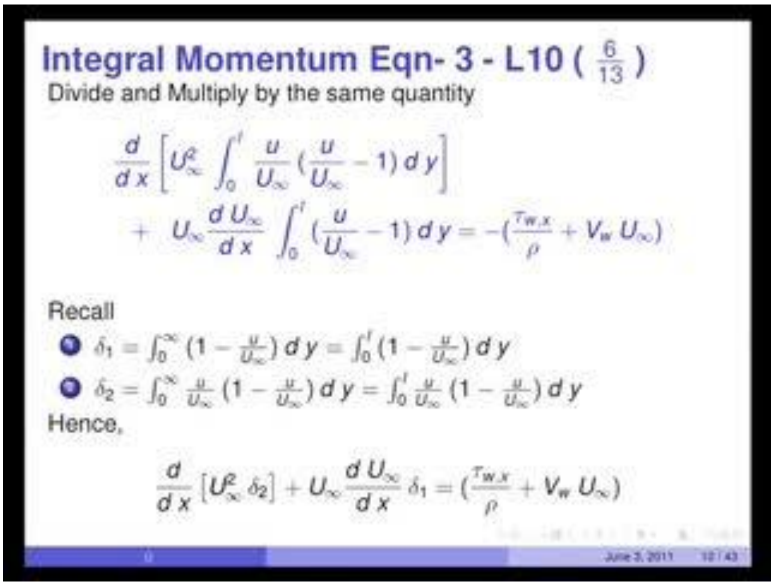

If I divide and multiply each term by the same quantity that is If I multiply this equation by U infinity square and divide by U infinity square and if I multiply this equation by U infinity and divide by U infinity, then you will notice that I can write the equation in this manner. It is written as d by dx of U infinity square u over U infinity into u over U infinity minus 1 dy plus U infinity d U infinity by dx equal to 0 to l u over U infinity minus 1 dy and all this.

Recall that this integral is nothing but our momentum thickness. This term is nothing but minus delta 1 or the displacement thickness. Therefore, this term will become minus d by dx into U infinity square delta 2. This term will become minus U infinity d U infinity by dx into delta 1. So, canceling the minus sign, which appears in each term, we would have the equation which looks like this (Refer Slide Time: 11:30) d by dx U infinity square delta 2 plus U infinity d u by d U infinity by dx delta 1 equal to the shear stress term and the suction or blowing term. I will further manipulate this equation as shown on the next slide.

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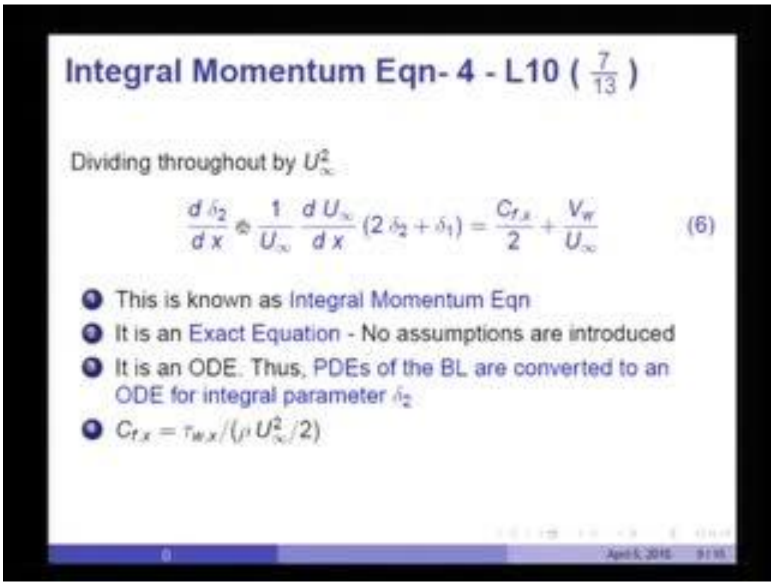

If I divide by U infinity square and open up this differential of a product as U infinity square d delta 2 by dx into delta 2 into d U infinity square by dx, then you will notice that the equation will be read as d delta 2 by dx plus 1 over U infinity d U infinity by dx into d u delta 2 plus delta 1 equal to C f x by 2 plus V w by U infinity.

So, each term now has dimension as x U infinity. U infinity has the velocity dimension. Delta 2 and delta 1 have length dimension and so does x. The terms on the right hand side completely dimensionless. It is this equation, which is known as the integral momentum equation. We have an exact equation because we have not introduced any assumptions in its derivation. We have simply integrated the partial differential equation from 0 to l because all quantities vary only with x.

The partial differential equation of the boundary layer is converted to an ODE for an integral parameter delta 2. This is very similar to what we did in similarity method, where the PDEs were converted to third order ordinary differential equation. Here, the equations are converted to a first order ordinary differential equation for an integral parameter delta 2. C fx is the coefficient of friction and that is simply defined as tau wall x over rho infinity square divided by 2. So, it is well known to you and this is the integral momentum equation.

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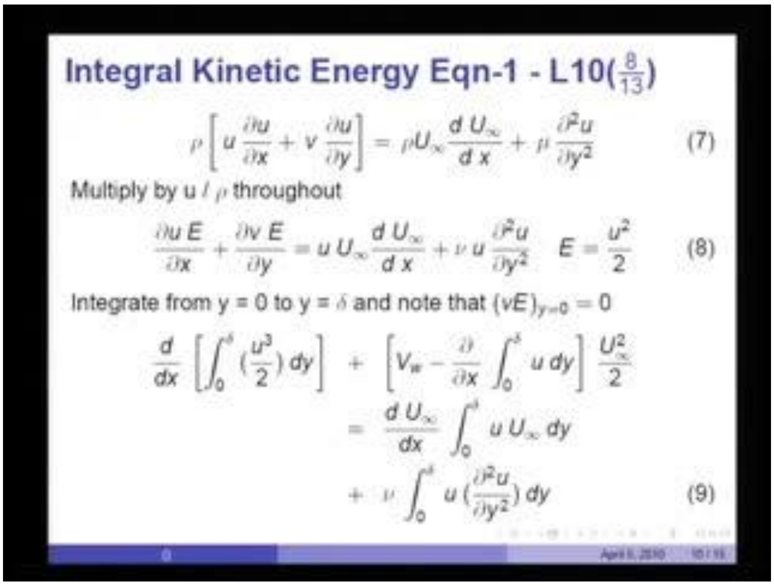

There is another variant of the equation, which is called the integral kinetic energy equation. It can be derived from the earlier momentum equation, but I will derive it from first principles. Our momentum equation reads like this, where these are the terms. If I multiply each term by u divided by rho throughout, then you will see and define E as u square by 2. You will see this term can be simply written as du E by dx, this term can be written as dv E by dy equal to U infinity d U infinity by dx plus mu into u d square u by dy square. I repeat E is the energy of the actual velocity, u square by 2.

If we integrate this equation from y equal to 0 to y equal to l or delta, where it does not really matter what we do. This term will simply become u into u square by 2, so that becomes u cube by 2 d by dx integral 0 to delta dy. This term will be v E at delta minus v E at y equal to 0, but at y equal to 0, u is equal to 0. Therefore, E is equal to 0 and that term will go up. V l into E at l or infinity is simply V w minus d by dx of 0 to delta u dy into U infinity square by 2. It would be equal to d U infinity by dx into u U infinity dy plus mu integral 0 to delta u into d square u by dy square dy. It is this last term, which I shall…because this is just in result on the next slide.

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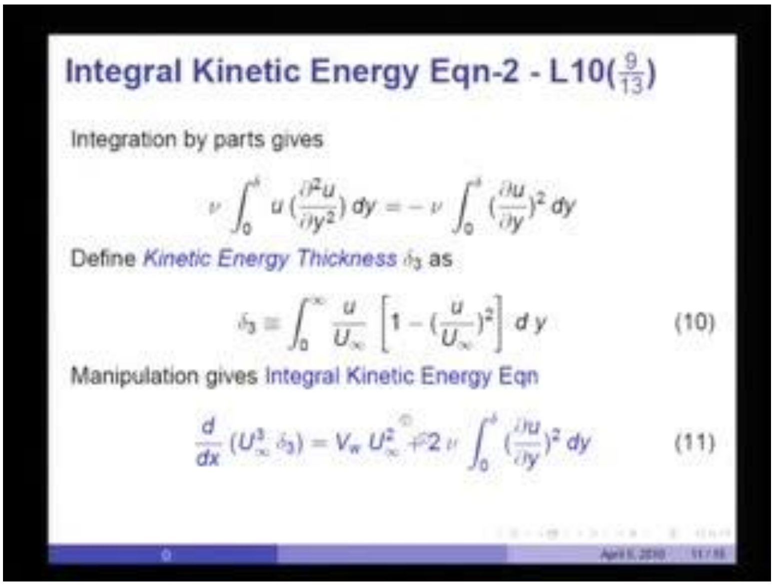

Integration by parts will give mu into 0 to delta into u d square u by dy square dy equal to minus mu into integral 0 to delta du by dy whole square. Now, in the integral kinetic energy equation, we introduce another thickness called the kinetic energy thickness. It is defined very much like the momentum thickness, except that we now have 1 minus u over U infinity square, which represents the kinetic energy deficit caused by (.) of the previous equation (Refer Slide Time: 17:11), which is shown here. You will see that this can be manipulated and read as du by dx of U infinity cube delta 3 equal to V w into U infinity square plus 2 mu du by dy whole square.

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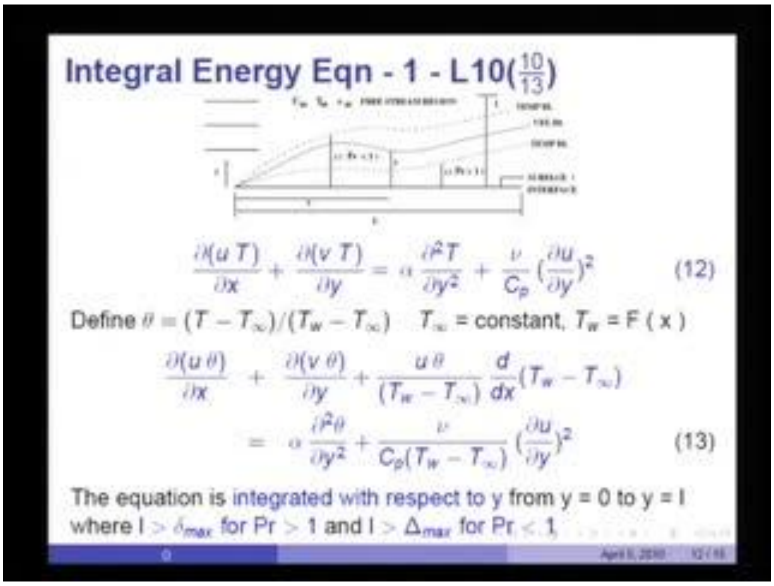

Integral kinetic energy equation is sometimes used in boundary layers analysis of boundary layers with suction and blowing. We now turn to integral energy equation. Again, consider the surface on which a boundary layer is developing. This solid line represents the velocity boundary layer. Now, the thermal boundary layer will either have a thickness greater than the velocity boundary layer thickness or smaller than velocity boundary layer thickness. You already know from your similarity solutions that when Prandtl number is greater than 1, the thermal boundary layer thickness is smaller than the velocity boundary layer thickness; whereas, if Prandtl number is less, boundary layer thickness is greater than the momentum or the velocity boundary layer thickness delta.

So, while integrating this energy equation, which includes this viscous dissipation term. We will choose l big enough, so that it is either greater than delta or it is greater than thermal boundary layer thickness, capital delta for Prandtl number less than 1. We define for convenience that theta equal to T, where T infinity is constant with x, but T wall is some function of x and I will introduce that function shortly.

This equation (Refer Slide Time: 19:21) will simply become du theta by dx. Additional term is arising out of the fact that T w is a function of x. Therefore, you will have a term called by T w minus T infinity into d by dx of T w minus T infinity. This term will become alpha delta 2 theta by dy square. This term will become mu into C p T w minus T infinity du by dy whole square. It is this equation that we shall integrate from 0 to l, where l is greater than delta max for Prandtl equal to 1, Prandtl greater than 1 and l is greater than capital delta max for Prandtl less than 1.

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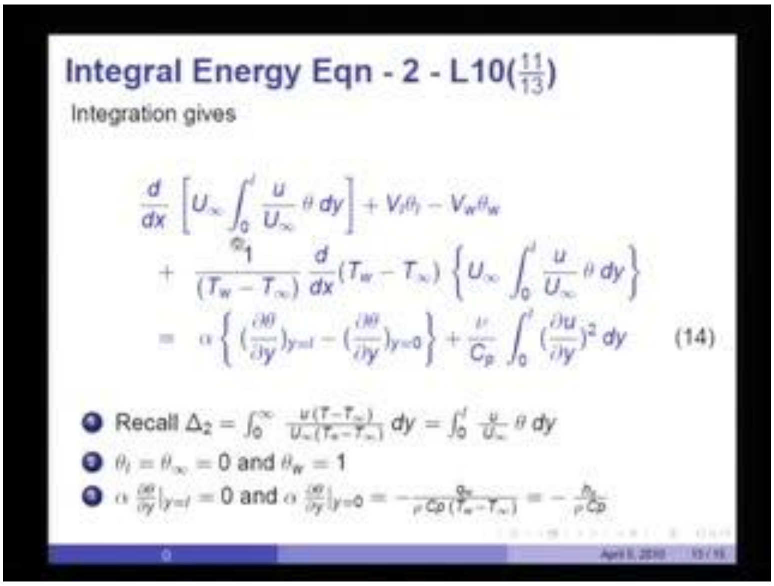

Let us do the integration on the next slide. You will notice that integration of this will simply become d by dx of integral u theta dy. If I divide and multiply by U infinity, it will be read as d by dx of U infinity from 0 to l U infinity u by U infinity theta dy.

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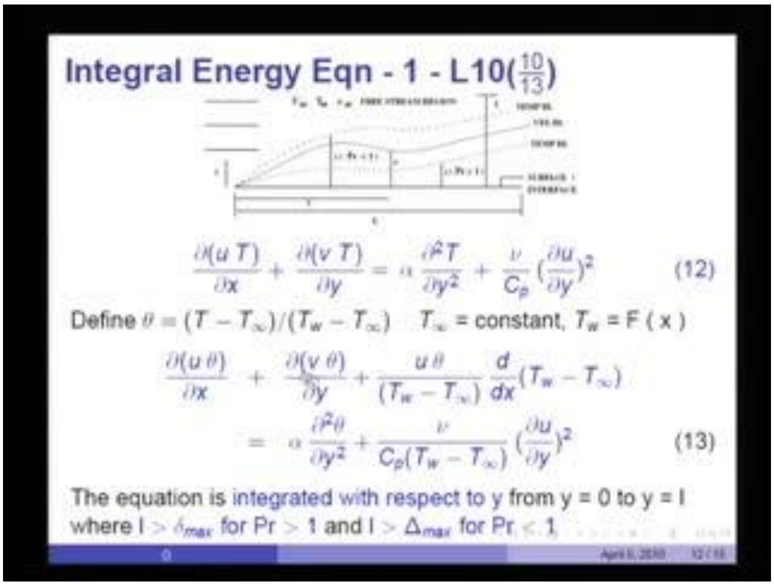

The integration of this term will simply give V l theta infinity minus V w theta w and theta w as is 1. you will see V l theta l minus V w theta w plus this term will be simply 1 over T w minus T infinity by dx of T w minus T integral u theta dy and that is what I have written and u theta I have divided by U infinity and multiplied by U infinity.

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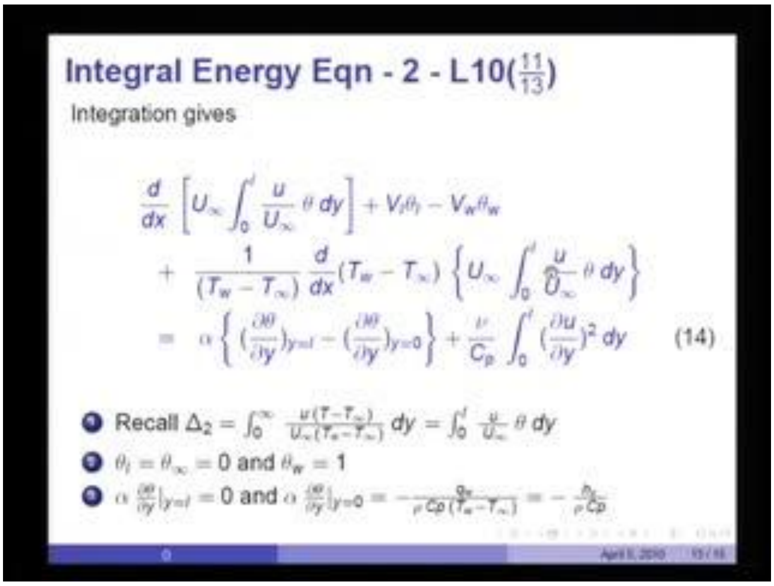

You will see V l theta l minus V w theta w plus this term (Refer Slide Time: 21:12) will be simply 1 over T w minus T infinity by dx of T w minus T into integral u theta dy and that is what I have written. I have divided by U infinity and multiplied by U infinity. The diffusion term will simply give alpha d theta by dy at l minus alpha d theta by dy at 0 plus nu whole square dy.

Now, if you recall that our enthalpy thickness delta 2 was defined as 0 to infinity U into T minus T infinity over U infinity into T 1 minus T infinity. It is nothing but 0 to l u over U infinity theta dy. In other words, this quantity is nothing but capital delta 2 into theta l because of its definition as 0.

Therefore that term vanishes, whereas theta w is equal to 1 because the temperature gradient at the infinities and at the edge of the boundary layer is 0. What this term means is that d theta by dy at y equal to 0 multiplied by alpha d theta by dy. It will be simply equal to dt dy divided by T w minus T infinity. Therefore, q w into rho C p with a negative sign and q w divided by T w minus T infinity is nothing but local heat transfer coefficient h x divided by rho into C p. So, this term (Refer Slide Time: 22:58) will be replaced by minus h x rho C p.

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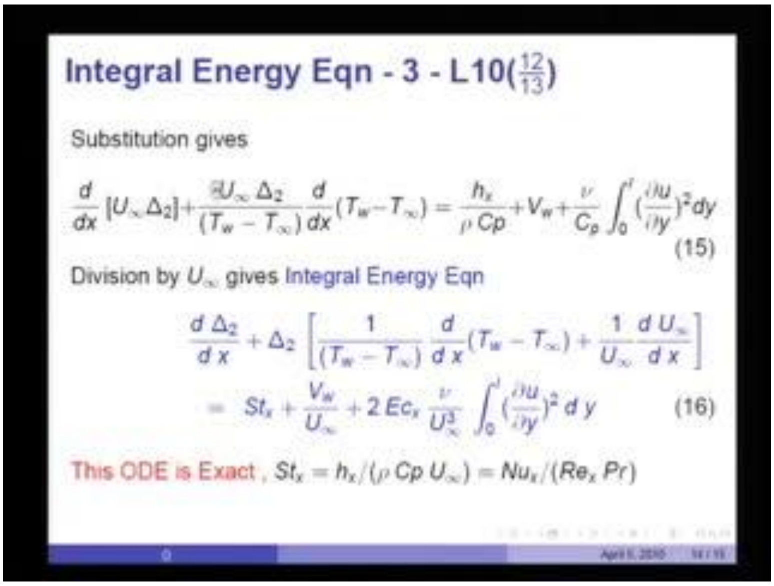

You will see the equation is now read as d by dx of U infinity delta 2 by dx plus U infinity delta 2 into T w minus T infinity into d by dx into T w minus T infinity is equal to h x divided by rho C p plus V w plus mu by C p into integral 0 to l du by dy whole square into dy.

If I divide by U infinity after opening this differential, then I would get the integral energy equation as d delta 2 by dx plus delta 2 into the wall variation term into the free stream variation or the pressure gradient term - Stanton x. It is h x divided by rho C p U infinity. I have shown here, which is nothing but Nusselt number divided by Reynolds number into Prandtl number and we have seen this before.

The wall velocity variation term, V w by U infinity and the viscous dissipation term is this. Each term here is dimensionless (Refer Slide Time: 24:24) and as you can see, this is dimensionless and this is dimensionless, delta 2 is dimensionless, this is dimensionless. Is this in dimension? It can be shown quite easily. Now, each term is dimensionless and therefore, we have an ordinary differential equation for capital delta 2 analogous to an ODE for small delta 2 in the velocity boundary layer equation.

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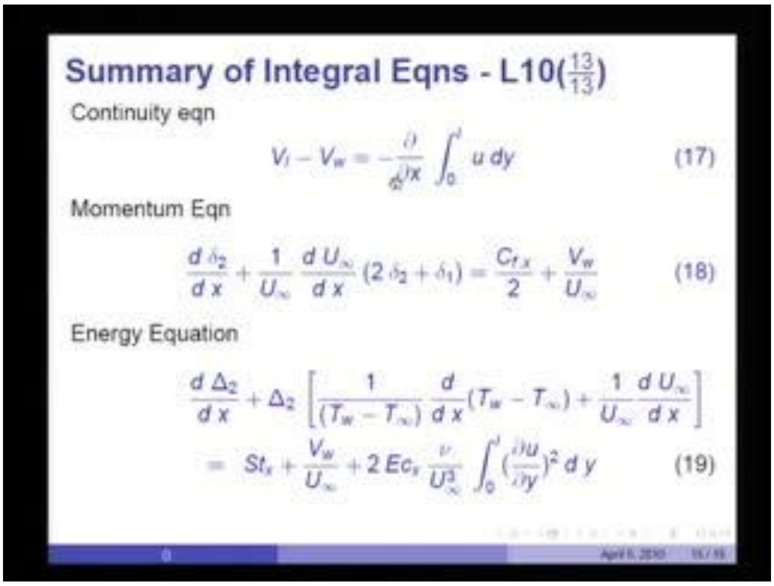

Let me summarize all the equations that we have derived. Integral continuity equation simply gives V l minus V w equal to minus d by dx of 0 to l u dy. The integral momentum equation gives variation of d delta 2 by dx and accounts for the wall velocity. The integral energy equation gives the rate of change of enthalpy thickness as delta 2 and accounts for the wall temperature variation, the free stream variation, wall velocity variation and the viscous dissipation. Our interest is always to determine Stanton x and C x. In the next lecture, I will take the solution of the velocity boundary layer equations in their integral form. Thank you.