Matrix Theory
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Partitioning the SVD

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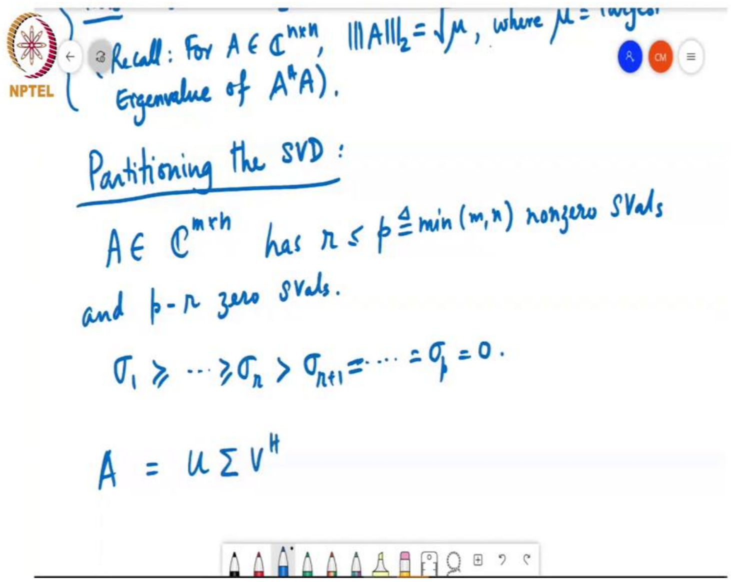

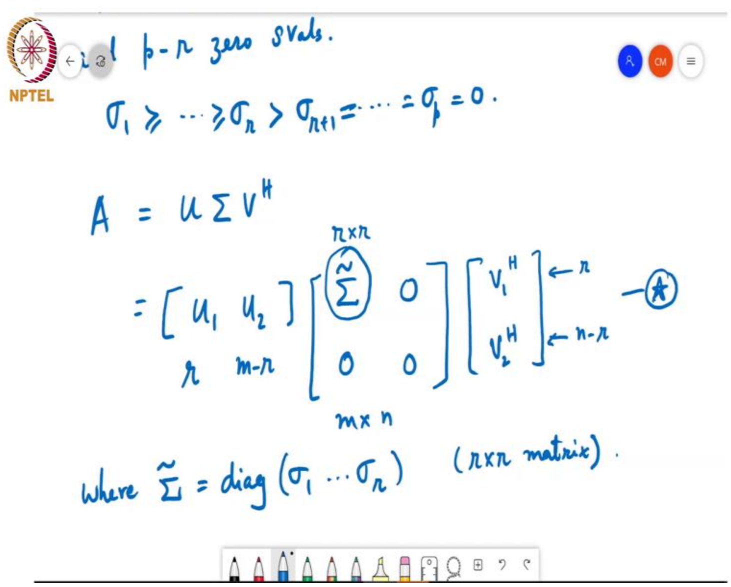

So, there is one other thing I want to make say, which is that this is about partitioning. This so this partitioning actually brings out the core structure of the matrix and what the singular value decomposition allows you to do is to partition matrices as I am going to tell you now. So, here it is.

So, again A is in C to the m by n and has so suppose this has r which is at most p defined to be min of m, n non-zero singular values and say p minus r 0 singular values. So, we will use the notation sigma 1 greater than or equal to sigma r, which is greater than sigma r plus 1 which is equal to all the other singular values sigma p equal to 0. Then I can write.

So, since u is unitary u Hermitian Av equals sigma can be written as this is equal to u sigma v Hermitian, which I can partition like this u1 u2 where this is the first r columns, this is the next m minus r columns and here I have say sigma tilde or D as I called it in this stating the theorem 0 0. This is overall an m by n matrix and this itself is r cross r and v1 v2 Hermitian and this has the first r rows and this has the next n minus r rows.

We will call this form star where sigma tilde is equal to a diagonal matrix containing sigma 1 through sigma r on the diagonal. So, this allows you to partition the singular value decomposition like this into blocks and what this partitioning does is it reveals a lot of structure in this matrix.

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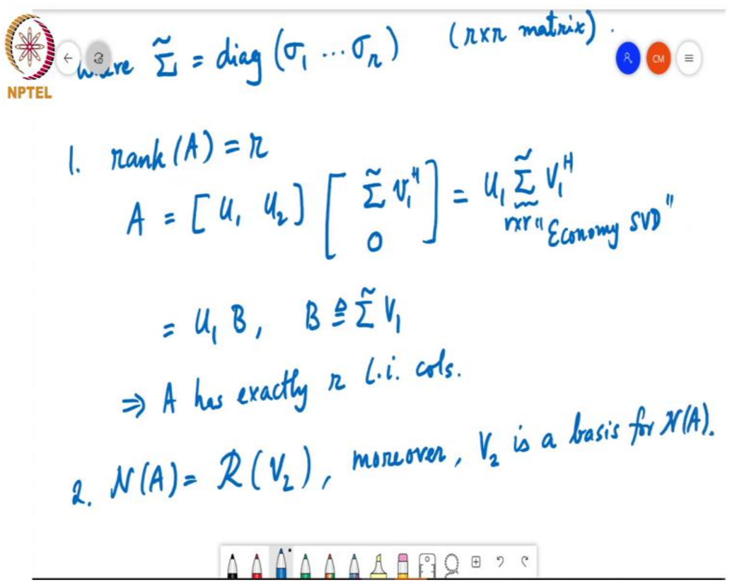

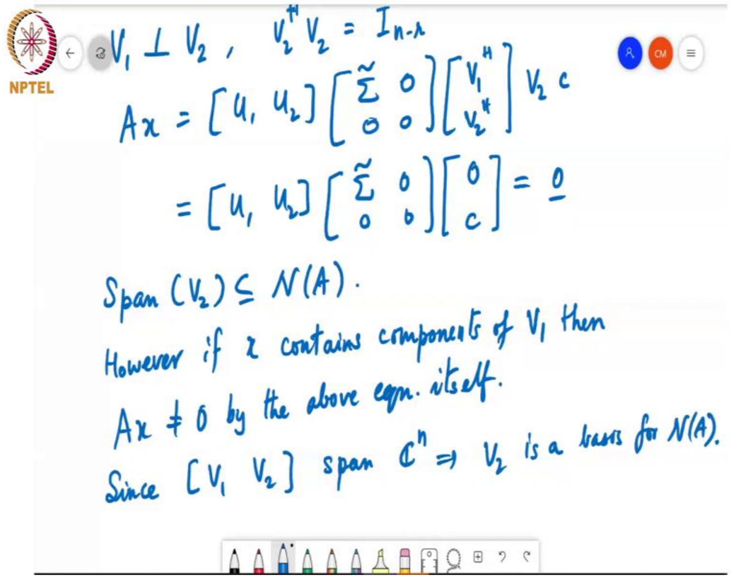

So, first of all, it shows that the rank of A is exactly equal to r and you can write A as so if I do this multiplication here, I get sigma tilde v1 Hermitian and then this whole thing multiplies 0 down here and so I can write this like this u1 u2 sigma tilde.

So, only v1 Hermitian is of size r that only multiplies this, whereas this does not this is just multiplying the, if I take a particular column here, the entries in v2 Hermitian are all multiplying 0s here, so they do not contribute to anything. So, this can be written as this times v1 Hermitian and then down here, I will have all the 0s and this itself is multiplying u2 here. So, this does not contribute anything. So, I can write this as u1 sigma tilde v1 Hermitian and this is called the economy SVD. Because this thing here is now an r cross r matrix and this is of m by r and this is r by n.

And so I can see that I see that now that A is equal to u1 times the matrix B, where B is sigma tilde times v1. So, what that means is that the ith column of A is a linear combination of the r columns of u1 with coefficients given by the ith column of B. So, that the ith column B gives the coefficients with which I should combine the columns of u1 to get the ith column of A.

So, and this sigma tilde, of course, is diagonal with non-zero entries and so it has linearly independent columns and v1 Hermitian has linearly independent rows and so sigma tilde v1 or this matrix B has linearly independent rows and so what this means is that A has exactly r linearly independent columns and that is why the rank of A equals r. The second property is that the null space of A is the same as the span of the columns of v2 and further v2 is a basis for the null space of A. So, null space of A is the set of all vectors x such that Ax equals 0 x.

Now, if x is in the span of v2, then I can write x to be equal to v2 times c where c is in some or I should be more careful n minus r. So it is in the span of v2. Now, v1 is perpendicular to v2, because the columns of v form an orthonormal set the matrix v itself is a unitary matrix. So, v1 is perpendicular to v2 and v2 Hermitian v2 is the identity matrix of size n minus r.

So, then what I have from this is that, if I take Ax, this is equal to u1, u2 times sigma tilde 0, 0, 0 times v1 Hermitian v2 Hermitian times v2 c. Now, v1 Hermitian v2 is 0, v2 Hermitian v2 is the identity matrix. But then this is only going to multiply these 0 entries here and so what I will be left with is u1 u2 times sigma tilde 0, 0, 0. And if I take this product first I get 0, v2 Hermitian v2 is the identity matrix.

So, I will get c down here and so now, if I expand this out, I will get 0 times sigma tilde 0 times c and then 0 is down here. So, this whole thing becomes equal to the 0 vector. So, of v2 is a subset of the null space of A for any vector that belongs the span of v2 satisfies Ax equals 0. So, it is a subset of the null space of A.

However, if x contains components of v1, that is to say that if the projection of x on to span of v1 is nonzero, then Ax cannot be equal to 0 by this equation itself because if it has some component of v1 then v1 Hermitian times that will be a non-zero quantity. So, I will get a non-zero quantity up here and that is multiplying a non-zero sigma tilde which is in turn multiplying a non-zero u1 by the above equation itself and v1 v2 form a complete basis for r to the for C to the n together they span C to the n which implies that v2 is basis for the null space of A.

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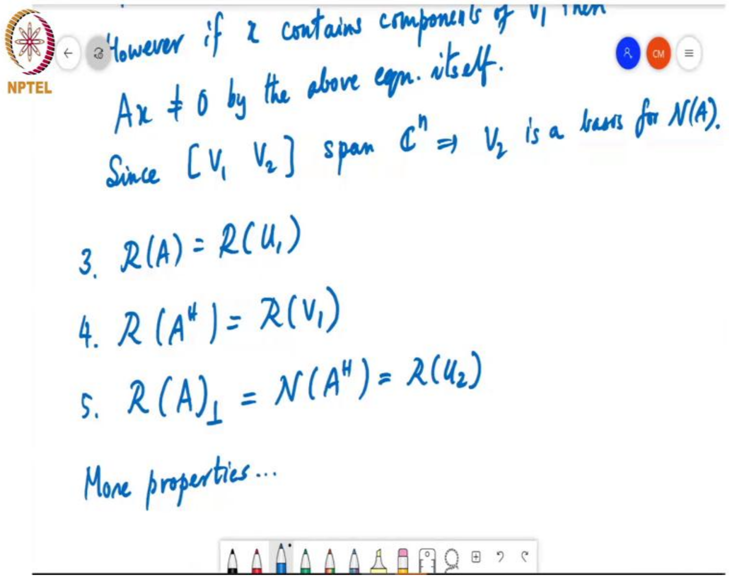

So, we can similarly say many more things like the range space of A is the range space of u1. The range space of A Hermitian is the range space of v1 and the orthogonal complement of the range space of A which is equal to the null space of A Hermitian is equal to the range space of u2 and so on and we will see some more properties. So, we are out of time, so more properties. So, we will see that in the next class. So, that is all I have for today. We will continue again on Wednesday.