Matrix Theory
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Singular Value Definition and Some Remarks

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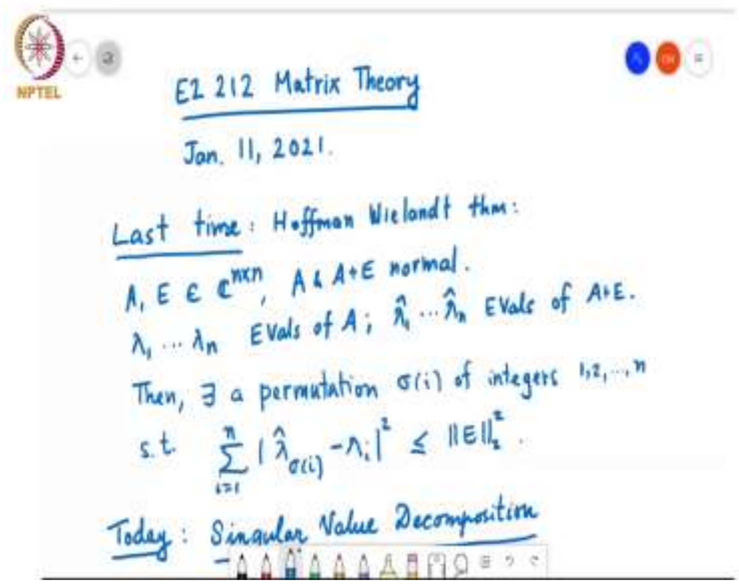

The last time we saw the Hoffman Wielandt theorem, which said that if A and E are matrices such that both A and A plus E are normal matrices and if lambda 1 through lambda n are Eigen values of A and lambda hat 1 through lambda hat n are Eigen values of A plus E then there is a permutation of the Eigen values of A plus E such that each Eigen value of A plus E is close enough to the corresponding Eigen value of A in the sense that the sum of the squares of these differences is at most the spectral norm squared of E.

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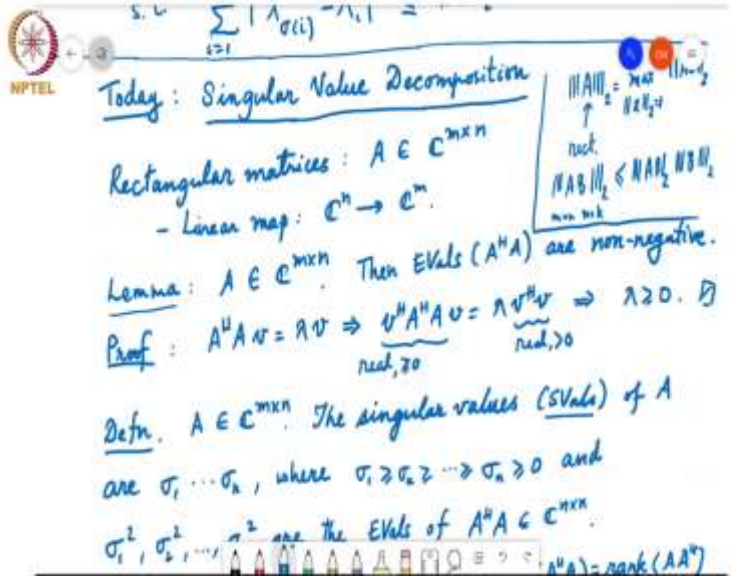

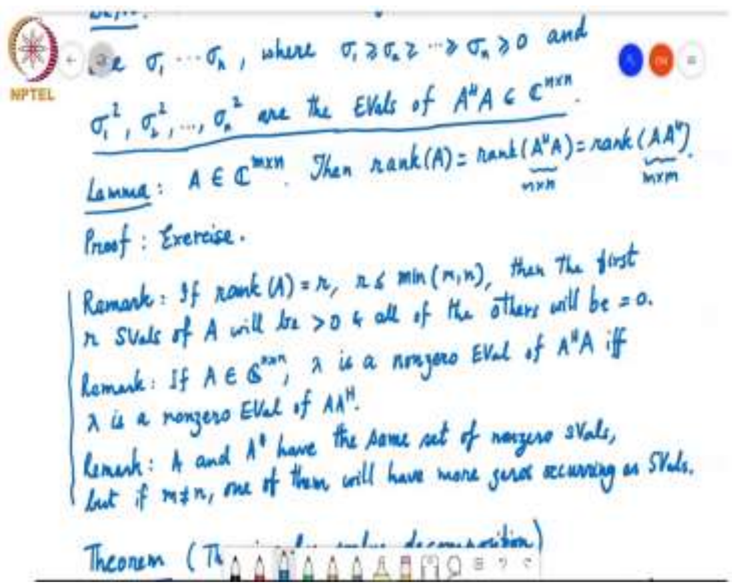

So, today we start on another topic, which is the singular value decomposition. So, far we have focused our attention on square matrices. So, now, we start discussing about rectangular matrices. Suppose, so basically these are matrices where of size m by n and m may not be equal to n. So, one can view this as a linear map from the C to the n space to the C power m space that is it maps an n dimensional vector to another m dimensional vector in the complex space.

We have a few sort of preliminary or precursor lemmas and then we will go to the main theorem. Now, there is one thing I want to mention, which is actually not here. So, remember that when we will be working with the induced spectral norm that is specifically A2 which is the max over norm x2 equals 1 norm of Ax l2. Now, we know that induce norms satisfy sub-multiplicativity.

Now, when it comes to, so, this is for square matrices we had defined this for square matrices, but we can always define something like this for rectangular matrices also. Because after all you can compute the Euclidean norm of any vector and Ax is a vector only thing is that this constraint space is over the space C to the n or x belongs to C to the n whereas, the objective is being evaluated on a vector that is sitting in C to the m.

So, it is perfectly okay to define the norm A2 to be a quantity like this even if the matrix A is rectangular. Now, we know that these induced norms satisfy the sub-multiplicativity property and that extends also to rectangular matrices. So, specifically this kind of norm it satisfies the property that AB l2 is less than or equal to A2 times B2, of course, here A and B are matrices that can be multiplied together.

So, for example, this could be m by n and this could be n by k or something, then this is still true where these norms are evaluated as given above. So, this is one property and this is easy to show it prove lies is exactly the same as the proof of the, we showed our result that said that induce norms satisfied the sub-multiplicativity property and the proof of this is exactly the same.

Now, a few other precursor lemmas that we need that so, if A is an m by n matrix, then the Eigen values of A Hermitian A are always non negative. We know this already, but anyway the proof is just one line, if A Hermitian A times B equals lambda times v simply pre multiply by v Hermitian you get with v Hermitian A Hermitian Av equals lambda times v Hermitian v but lambda is just a scalar.

And this is just the l2 norm squared of Av and this is the l2 norm squared of v and both are therefore, real and non-negative and these are Eigen vector, it is a non-zero vector. So, this is actually strictly positive. So, this means that lambda see this is real this is also real value and so lambda cannot suddenly become complex values. So, lambda is, and both are non negative. So, lambda is real valued, and it is non negative.

So, now we formally defined the singular values of a matrix. So, take rectangular matrix A and the singular values, which I will abbreviate as Svals of this matrix A denoted by sigma 1 through sigma n, where these are ordered in that sigma 1 is the largest singular value and sigma n is the smallest singular value.

So, I am indexing it by the number of columns here and sigma 1 squared through sigma n squared the Eigen values of A Hermitian A. So, this is the crucial point here, that sigma 1 squared up to sigma n squared, these are the eigenvalues of A Hermitian A and we already saw that these Eigen values are non negative, so I can take the positive square root and that gives me what the singular values are completely defines the singular values.

Now, another lemma is that, if A is of size m by n, then rank of A is the same as rank of A Hermitian A, which is the same as rank of A A Hermitian. So, A Hermitian A so if A is m by n, A Hermitian is of size n cross n and A A Hermitian is of size m cross m. So, rank of A is at most min of m, n and it is also equal to the rank of this n cross n matrix and it is also equal to the rank of this m cross m matrix I will show this, it is actually very easy to prove, first of all, rank of a matrix rank of AB is at most the rank of A and the rank of B.

So, from that, you get an inequality that says rank of A is rank of A Hermitian A is less than or equal to rank of A and then to show the reverse inequality, then, what you do is you start with A Hermitian AB equals 0 and if that is true then v Hermitian times A Hermitian Av equals 0 or in other words, Av that v Hermitian A Hermitian Av is nothing but norm of AB squared. And if that is equal to 0, it means Av must be equal to 0.

So, the null space of A Hermitian A is therefore contained in the null space of A, then you use the rank nullity theorem, you should write this out for yourself, but it is not difficult result to show. And I have a few remarks. If the rank of A is equal to r as a consequence of this, if rank of A is r then the first r singular values of A will be positive and all the other singular values will be equal to 0.

So, the rank of the matrix determines how many non-zero singular values you will have and if A is a square matrix, it is a size n cross n, then lambda is a non-zero Eigen value of A Hermitian A if and only if lambda is a nonzero Eigen value of A Hermitian A as well. And finally, if A and A Hermitian they have the same rank and they also have the same set of non-zero singular values, if m is not equal to n, then one of them will have more zeros occurring as singular values.

So, these are some small remarks and you can actually relate to these after we go through the SVD theorem.

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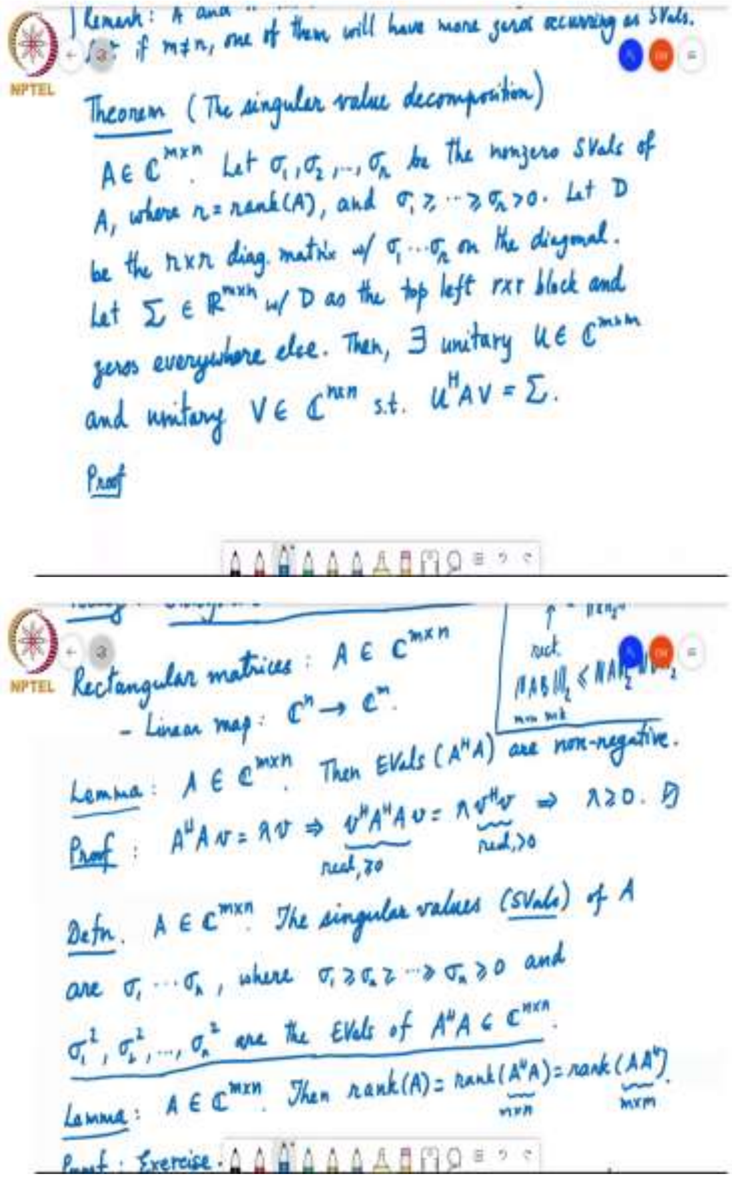

So, here is the singular value decomposition theorem. What I like about this theorem is that it applies to any rectangular matrix, there is absolutely no structural assumptions being made. So, A can be any m by n matrix and let sigma 1 through sigma r be the non-zero singular values of A where r is equal to the rank of A and suppose the singular values are ordered, so that sigma 1 is greater than or equal to et cetera greater than or equal to sigma r, which is strictly greater than 0 because there r non-zero singular values.

Let D be a D cross D sorry r cross r diagonal matrix with sigma 1through sigma r along the diagonal and zeros everywhere else and sigma is a matrix of size m by n with D as it top left r cross r block and zeros everywhere else, then you can find a unitary u of size m by m and a unitary v of size n by n such that u Hermitian Av equals sigma this diagonal matrix, but it is not a square diagonal matrix is a diagonal matrix of size m by n and it has sigma 1 through sigma r or as its top left r cross block and zeros everywhere else.

So, that is what this theorem says. So, will theorem.

Student: So, what is the difference between Eigen value and similar value?

Professor: So, you can see that here from the definition and we will discuss that a little more later. But the singular values of a matrix squared are the eigenvalues of A Hermitian A, for now, this is the relationship that we have seen. General if since we are discussing rectangular matrices, you cannot talk about Eigen values of A, Ax is not in the same space, it is of size m, whereas x is of size n.

So, you cannot write an equation like Ax equals lambda x is if a is of size m by n. But if A is a square matrix, we will see what the relationship between the singular values and the Eigen values of Ar. But for now we are discussing rectangular matrices and so we cannot write a direct relationship between I mean Eigen values are not even defined for a rectangular A.

But when I take a matrix A Hermitian A that is a square matrix and I can define Eigen values for it and the Eigen values are all non-negative and if I denote them by sigma 1 squared up to sigma n squared, then sigma 1 through sigma n will be the singular values of A that is the definition that is just directly the definition.

Student: Sir.

Professor: Go ahead please.

Student: Can we say that the singular values are the positive roots of the Eigen values of A Hermitian A.

Professor: That is right. That is exactly what we are saying here that if I denote the eigenvalues of A Hermitian A, which are all non-negative by sigma 1 squared through sigma n squared, then sigma 1 is the positive root of those Eigen values of A Hermitian A sigma 2 is the positive root positive square root of the second largest Eigen value of A Hermitian A and so on.