Matrix Theory
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Condition of Eigenvalues for Diagonalizable Matrices

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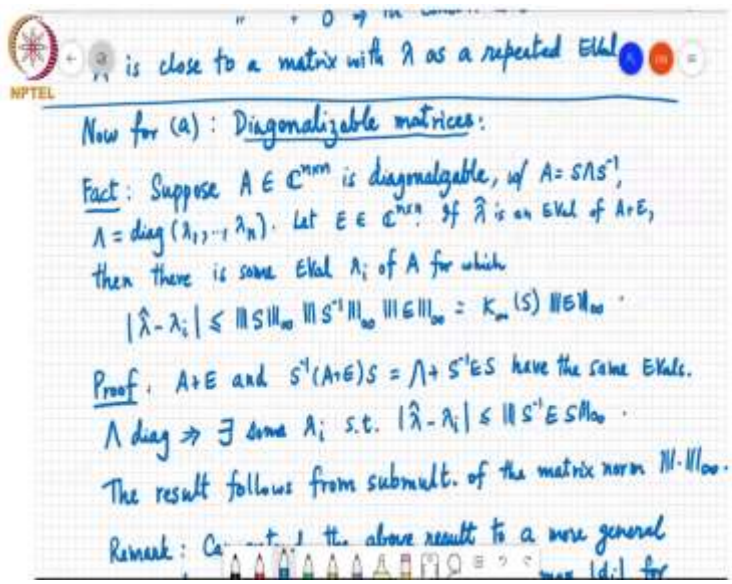

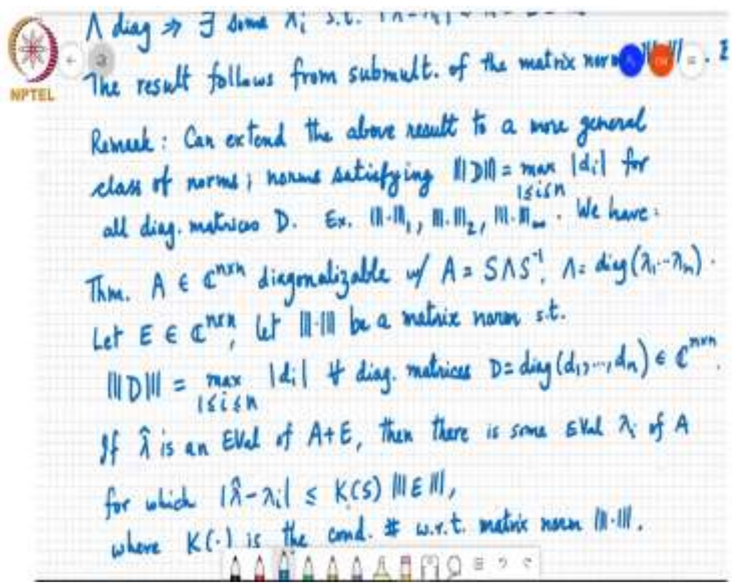

Now, let us look at what happens for diagonalizable matrices. So, first of all we have the following fact, which is that suppose A is a matrix that is diagonalizable with A equal to S lambda S inverse and lambda being a diagonal matrix and suppose E is a perturbation matrix of size n cross n.

Now, if lambda hat is an eigenvalue of A plus E, then there is some eigenvalue lambda i of A for which this difference between lambda hat minus lambda i is that most the l infinity norm of S times l infinity normal S inverse times l infinity norm of E which as we defined it earlier is k infinity of S times the l infinity norm of E.

So, k infinity of S is the condition number of S under the l infinity norm. So, we see the condition number showing up in when we try to analyse the stability of eigenvalue computations, but what matters here in the case of diagonalizable matrices is the condition number of S which is the matrix that diagonalize’s A, not the condition number of A itself.

Of course, we know that A plus E and S inverse A plus E times S have the same eigenvalues and S inverse A plus E times S is nothing but lambda plus S inverse E times S. Now, lambda is a diagonal matrix and so by the Gersgorin theorem there is some eigenvalue lambda i such that if lambda hat is an eigenvalue of A plus E then there is some lambda i such that lambda hat minus lambda i is at most S inverse ES the l infinity norm.

Now, the result follows from the sub-multiplicativity property of the matrix now that is it, we can actually extend this result to a more general class of norms, which are norms satisfying this property that the norm of D is equal to the maximum diagonal entry when the matrix D is diagonal. In some examples of such norms are the l1 norm l2 norm and l infinity norm.

So, this is the extension. So, suppose A is a diagonalizable matrix of size n cross m and A can be written as S lambda S inverse where lambda is a diagonal matrix containing the eigenvalues of A on the diagonal and let E be a perturbation matrix of size n cross n and let this norm be a matrix norm such that norm of D equals the maximum diagonal entry for all diagonal matrices.

If lambda hat is an eigenvalue of A plus e, then there is some eigenvalue lambda i f of A such that lambda hat minus lambda i magnitude is at most K of S times the norm of E this norm of E, where K is the condition number with respect to this particular matrix norm. So, let us see how to show this.

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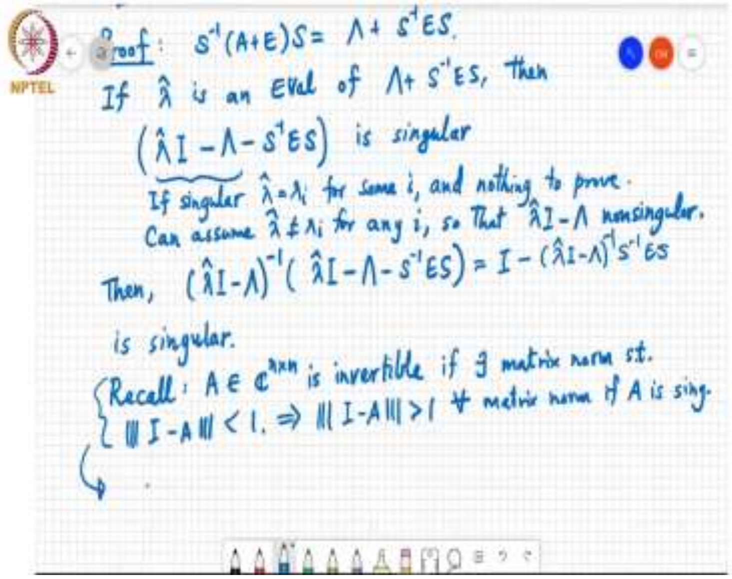

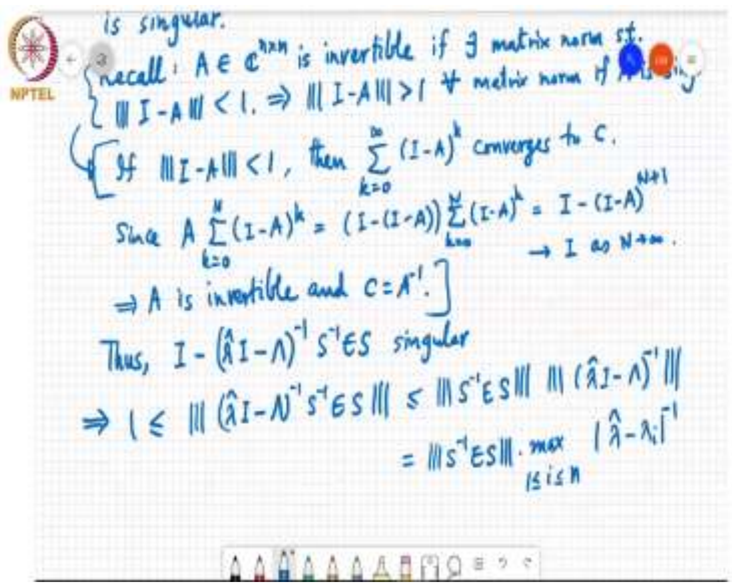

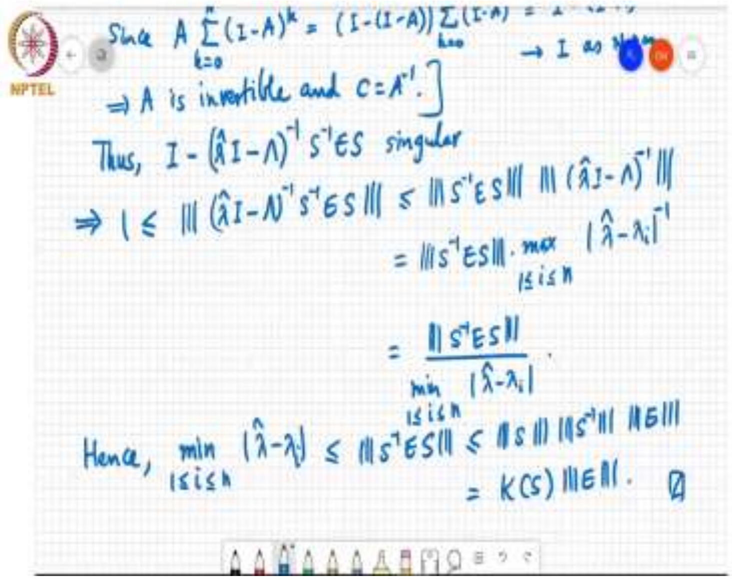

So, the starting point is the same as that to the previous result obviously S inverse A plus E times S is equal to lambda plus S inverse ES. Now, if lambda hat is an eigenvalue lambda plus S inverse ES then what we know is that if I take lambda hat times the identity matrix minus lambda minus S inverse ES, what can I say about this matrix?

Professor: Correct. So, basically eigenvalue satisfy the determinant of lambda I minus A equal to 0 and so lambda hat I minus this the determinant is equal to 0, so this matrix itself is singular. Now, if this matrix itself is singular this means that lambda hat equals lambda i for some i and then there is nothing to prove, that is this inequality will be any way satisfied, but if so I think we can safely assume that lambda hat is not equal to lambda i for any i, so that lambda hat I minus lambda is non-singular.

Then I will consider the matrix lambda hat I minus lambda inverse times lambda hat I minus lambda lambda minus S inverse ES. And then I will just expand this out, this is oops the identity matrix minus lambda hat I minus lambda inverse times S inverse ES. So, this matrix is singular. Now, recall a result we showed a long time ago, which is that A in C to the n cross n is invertible if there is a matrix norm first such norm of I minus A is less than 1.

So, what that means is that if this matrix is singular no matter which norm I considered the norm of I minus this matrix should be greater than 1, so I will just write that here implies norm of I minus A is greater than 1 for every norm for any norm A is singular. By the way, how did we show this result? Just to recall if norm of I minus A is less than 1 then we considered the (())(08:46) k equal to 0 to infinity I minus A power k this converges to C because the radius of convergence of summations z power k is 1.

Then what we do is we look at A times sigma say k equal to 0 to n I minus A power k and this is equal to we write this as I minus A times sigma k equal to 0 to n I minus A power k and when you expand this out it becomes a telescoping sum and you are left with only the first and last terms which is I minus A power n plus 1.

And this goes to the identity matrix as n goes to infinity, because the spectral radius of this is less than 1, and so or rather this matrix converges to the all 0 matrix. And so we conclude that basically this matrix whatever this converges to is the matrix A inverse. And A is, actually I should write it the other way, A is invertible and C equals A inverse. This was just (())(10:56) to recall how this goes, but now we will come back to our proof we are trying to write out.

So, we will apply this result and that is to be applied to the matrix I minus lambda hat I minus capital lambda inverse times S inverse ES. And so, I minus that matrix is so if I minus that matrix is just lambda hat minus I lambda hat, so thus I minus lambda hat I minus lambda inverse S inverse ES is singular implies I minus this matrix which is lambda hat I minus lambda inverse S inverse ES norm is greater than or equal to 1, it does not matter which norm I pick.

And so, we have that 1 is less than or equal to this norm, actually what I will do is I simplify this a bit and write it in this way, 1 is less than or equal to this which I will use sub-multiplicativity and write it as norm of S inverse ES times the norm of lambda hat I minus lambda inverse. And now I use the property that this norm it returns the largest diagonal entry whenever the matrix is diagonal and this is a diagonal matrix.

So, that this thing is actually equal to the right hand side here is equal to norm of S inverse ES times the max 1 less than or equal to I less than or equal to n more lambda hat minus lambda I inverse is the largest eigenvalue largest magnitude diagonal entry which I can also write as norm of S inverse ES (())(13:37) it by min 1 less than or equal to I less than or equal to n of mod lambda hat minus lambda I.

So, we have min 1 less than or equal to I less than or equal to n mod lambda hat minus lambda I am just taking this to the other side, so 1 is over here and that multiplied by this just gives me this is less than or equal to the norm of S inverse ES against sub-multiplicativity S S inverse norm of E which is actually equal to k of E, k of s times norm of E, which is what we wanted to show.

So, what we have done is that we have shown the importance of the condition number with respect to solving finding eigenvalues to the matrix, but there is an important difference between what we saw just now and what we saw earlier, when we were looking at the importance of the condition number in solving linear systems of equations.

So, when you are solving Ax equals B, it is a condition number of A, k of A that matter, here it is the condition number of S, k of S that matters not k of A directly. Of course, S depends on A, S is the matrix that diagonalize is this matrix A, but it is k of s that matters, not k of A directly. So, therefore, if k of S is a small number, then small changes in A lead to small changes in the eigenvalues.

But if k of S is large, then small changes in A could lead to large changes in the eigenvalues. In particular, if S is unitary, then the condition number of S is equal to 1 with respect to the spectral norm. And in this case, the eigenvalues of A are actually well conditioned, because k of S equals 1. And also recall that a matrix A can be unitarily diagonalized if and only if it is a normal matrix. So, we conclude that.

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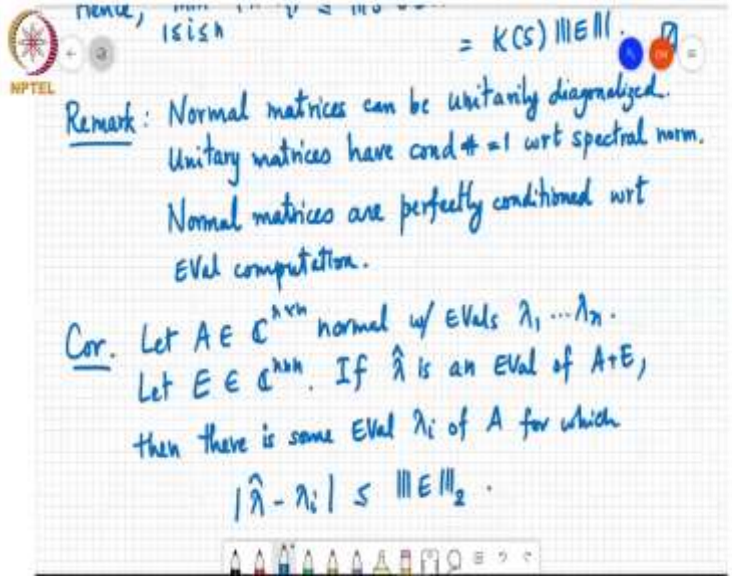

So, I will just write it this way normal matrices can be unitarily diagonalized and second point is that unitary matrices have condition number equal to 1 with respect to spectral norm, which implies that normal matrices are perfectly conditioned with respect to eigenvalue competition. So, we have the following corollary.

So, let A in C to the n cross n it is a norm matrix with eigenvalues lambda 1 up to lambda n and let E be an n cross n matrix is if lambda hat is an eigenvalue of A plus E then there is some eigenvalue lambda I of A for which lambda hat minus lambda I is less than or equal to norm E spectrum. Now, in the case where A and A plus E are both Hermitian matrices, we can actually use wiles interlacing theorem to get an even better bound. So, that is the next theorem.

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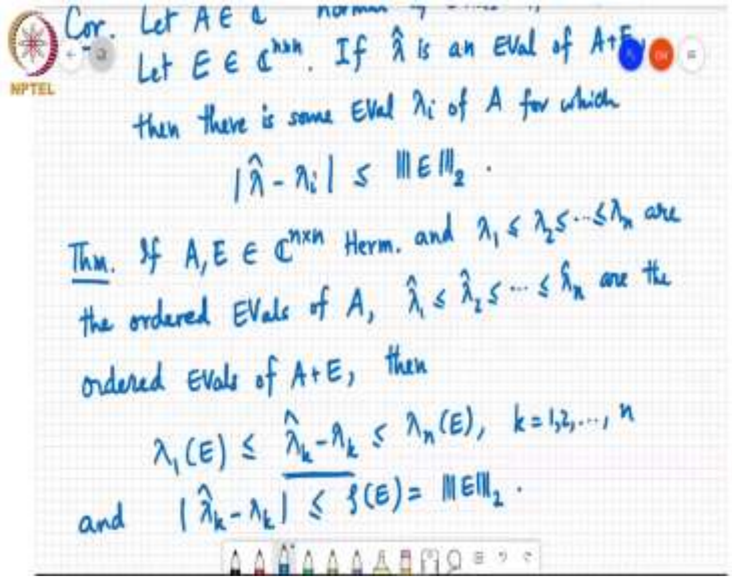

If A and E in C to the n cross n are Hermitian, Hermitian matrices is norm?

Student: Yes, sir

Professor: Yes. So, if there Hermitian are normal matrices Hermitian?

Professor: Need not be.

Professor: Need not be, correct. So, if A and E are both Hermitian and lambda 1 less than or equal to lambda 2 lambda n the ordered eigenvalues of A and lambda hat 1lambda hat 2 lambda hat n are the order eigenvalues of A plus E then lambda 1 of E less than or equal to lambda hat k minus lambda k less than or equal to lambda n of E and this is true for k equal to 1, 2, n and mod of lambda hat k minus lambda k is less than or equal to row of E spectral radius, which is equal to in this case because it is Hermitian the l2 norm of E.

So, basically this is a this is a better bound compared to the bounds w have seen earlier because it is really comparing the kth eigenvalue of A plus E with the kth eigenvalue of A. So, it is telling us which eigenvalue of A lambda hat k will be close to. So, there are a couple of different paths I can take from here and I was made to decide what I will cover in the remainder of this course, so I would like to stop here for today and I will figure out what I want to do next and continue in the in the next class.