Matrix Theory
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Location and Perturbation of Eigenvalues
Part 2: Gersgorin's Theorem

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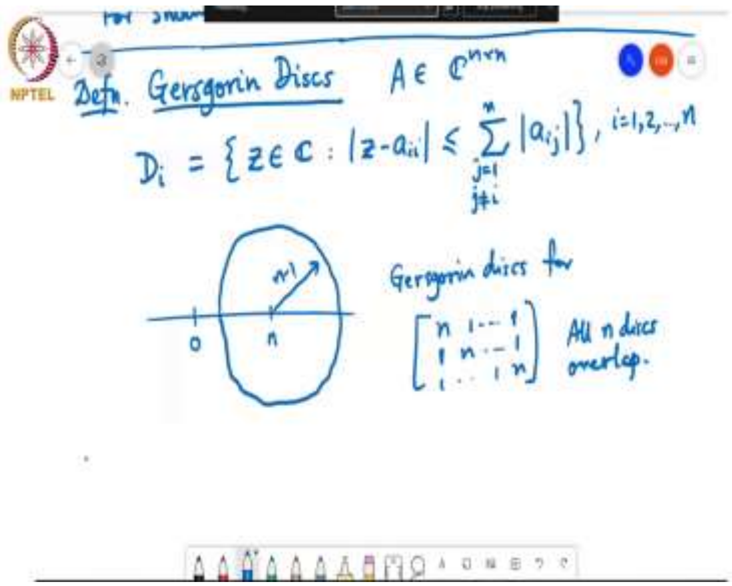

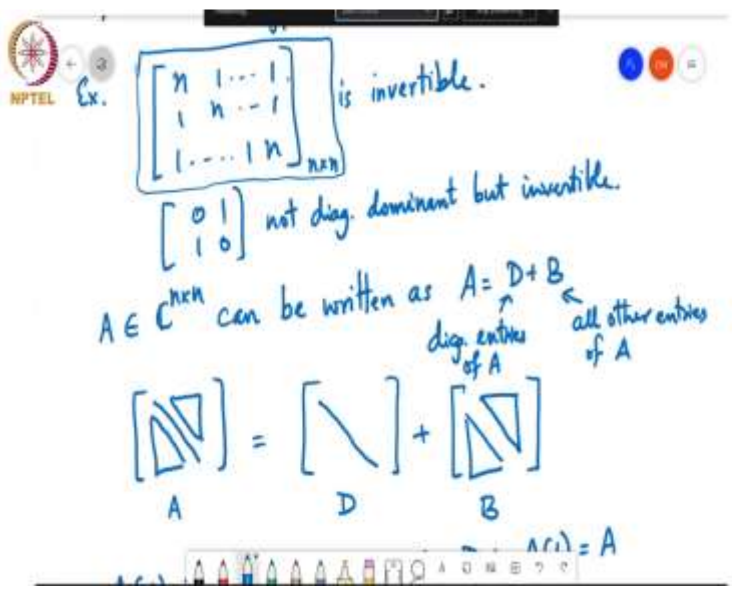

So, I need one definition, this is the notion of what is called Gersgorin discs, so A matrix in C to the n cross n and we define Di to be the set of complex numbers such that z minus aii is less than or equal to magnitude of z minus aii is less than or equal to sigma j equal to 1 to n, j naught equal to i z minus, sorry, aij i equal to 1, 2 up to n.

In other words, I am defining a circle, this is this is the radius of the circle and this is the centre of the circle and so I am defining a disc which is centred at aii and of radius equal to the sum of the magnitudes of the other entries of a in the same row, so that is how I define these discs. So, for instance in this matrix that I showed you as an example, in this matrix, the Gersgorin discs are all the same, they are centred at on the real line at n, at location n and their radius is equal to n minus 1.

So, just draw that here, so this is 0, this is n and this circle is of radius n minus 1, I do not want to draw such a big circle, so let me erase this, 0, this is n let us say this n minus 1, then I draw a circle here, all the it has n discs, every matrix of size n cross n will have n discs, but all n discs are the same, they overlap in this case. So, now let us write out the theorem.

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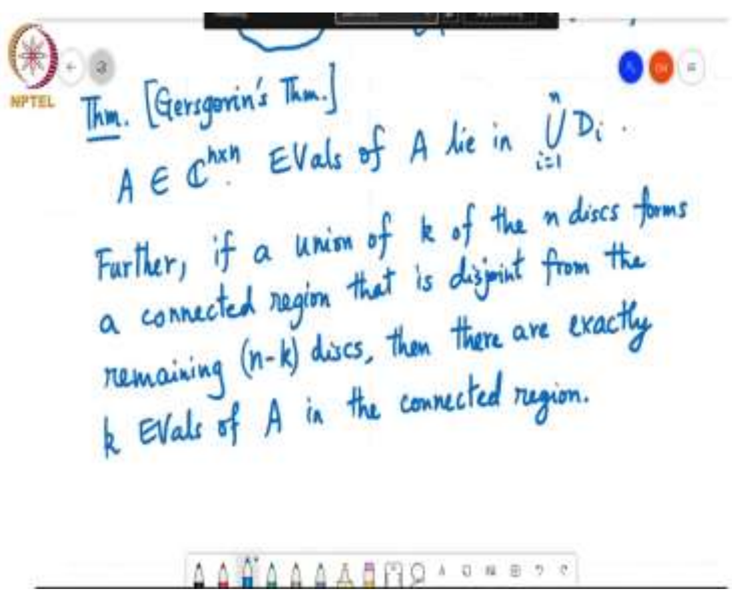

This is also one of these amazing theorems. Again, one of those completely non intuitive results, at least to me, so A is any matrix in C to the n cross n, then the eigenvalues of A lie in union i equal to 1 to n Di. So, if you take the union of these Gersgorin discs all the eigenvalues of A will lie in the union of those discs.

Further, if a union of k of the n discs forms a connected region, see these discs could overlap with each other, so if suppose k of them form a connected region that is disjoint from the remaining n minus k discs then there are exactly k eigenvalues in the connector region. So, that is what the theorem says. Any questions about the theorem?

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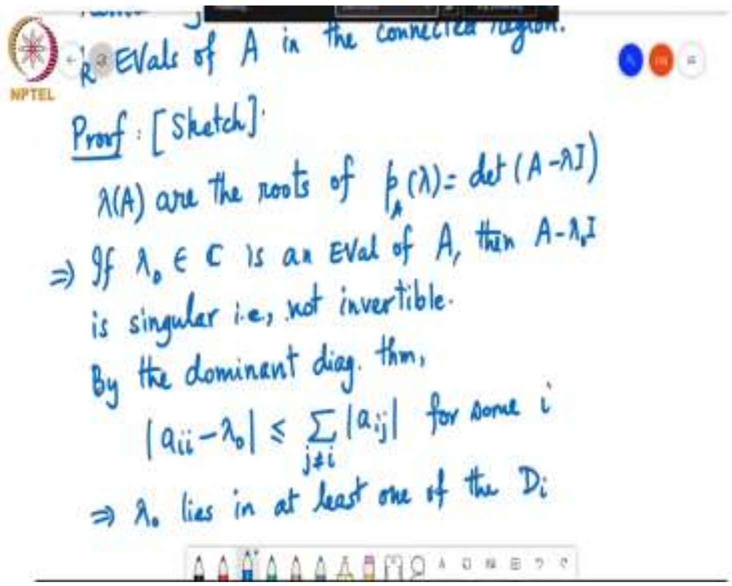

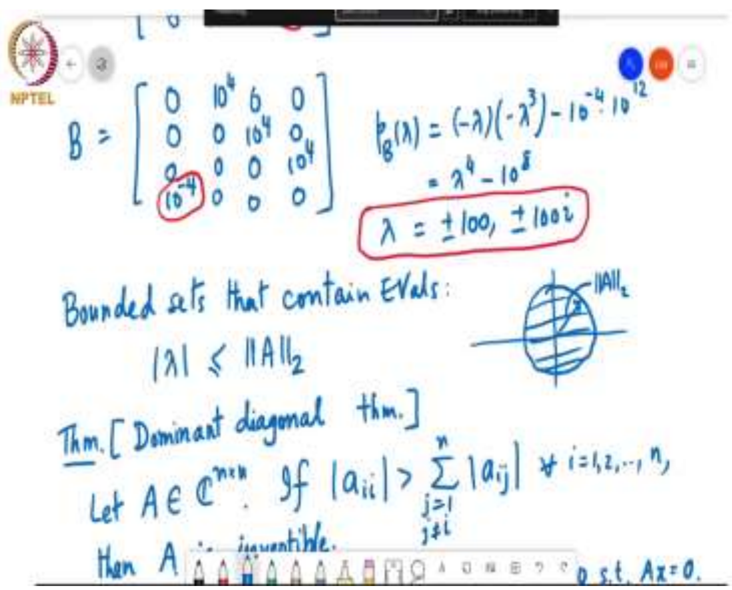

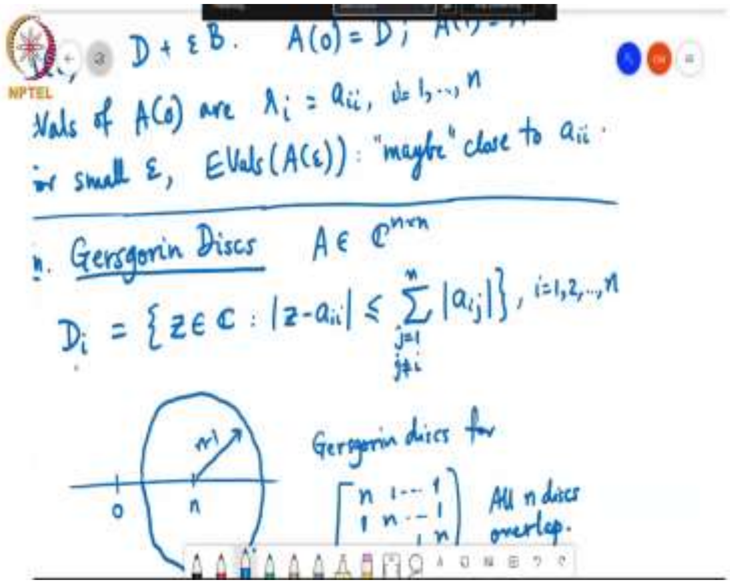

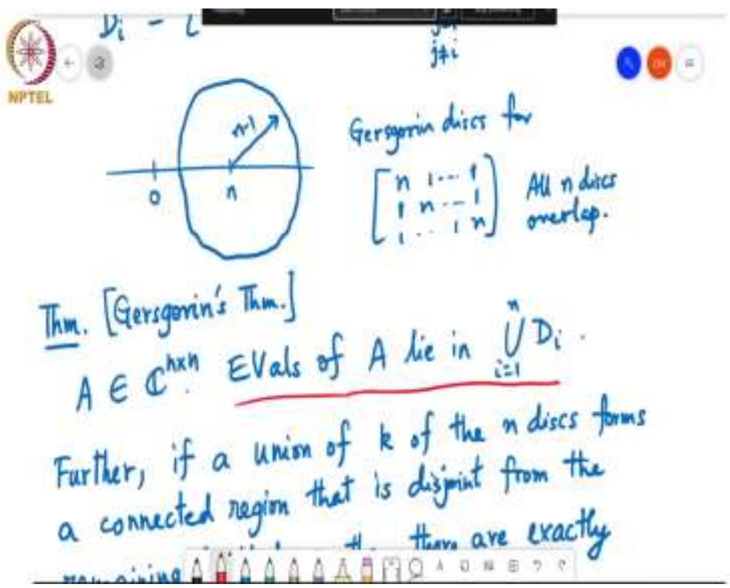

So, I will not go through the whole proof here, but I will just give you a sketch or rather just the main idea of the proof, so first of all lambda of A are the roots of the characteristic polynomial pA of lambda equal to determinant of A minus lambda i. So, what that means is that if lambda 0 which is a possibly a complex number.

Again I emphasize that we are not dealing with Hermitian symmetric matrices, so eigenvalues can be complex valued here is an eigenvalue of A, then if I look at the matrix A minus lambda 0i, this what can I say about this matrix is singular, which implies that it is not or rather it is not invertible. What that means is that the diagonal dominance condition must be violated for some index of the diagonal entries, so by the actually let me just...

So, the actual name for this theorem is dominant diagonal theorem, not diagonal domain theorem, so let me just correct that. So, by this dominant diagonal theorem it must be that the one of the diagonal entries in this violates the diagonal dominance condition that implies mod of the diagonal entry for the ith diagonal entry will be mod of aii in magnitude will be mod of aii minus lambda 0, this is less than or equal to the sum of the magnitudes of the off diagonal entries in the same row which is just mod aij, for sum i.

Which means that lambda 0 lies in at least one of the Di's, because if this is true, then lambda 0 satisfies this meaning, where is that? Here, lambda 0, if I substitute lambda 0 here, mod of lambda 0 minus aii is less than or equal to this quantity, so it is actually lying inside, so the lambda 0 as a if I replace z with lambda 0, it satisfies this condition, so lambda 0 belongs to Di.

Of course, lambda 0 may belong to other Di's as well and that is when we say that these discs are overlapping with each other, so this proves the first part of the theorem. So, we just showed this part, for the second part is where I am going to sketch the proof or rather I am just going to give you a visual proof.

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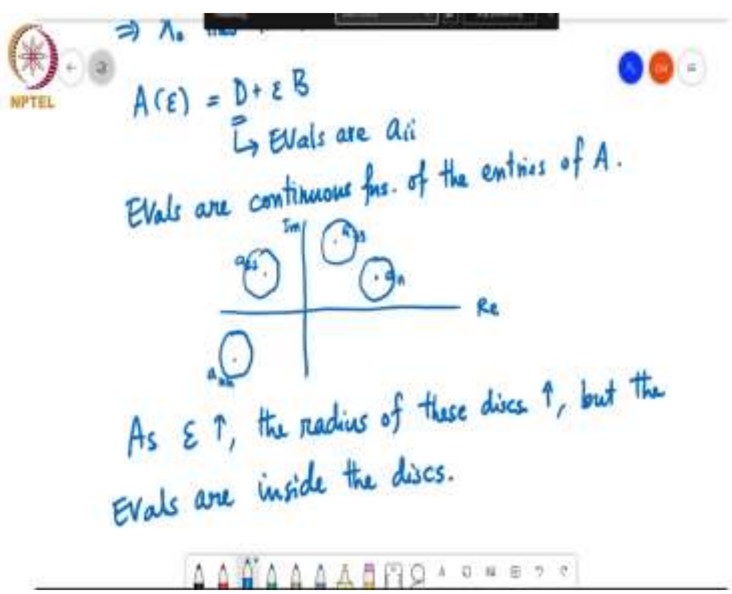

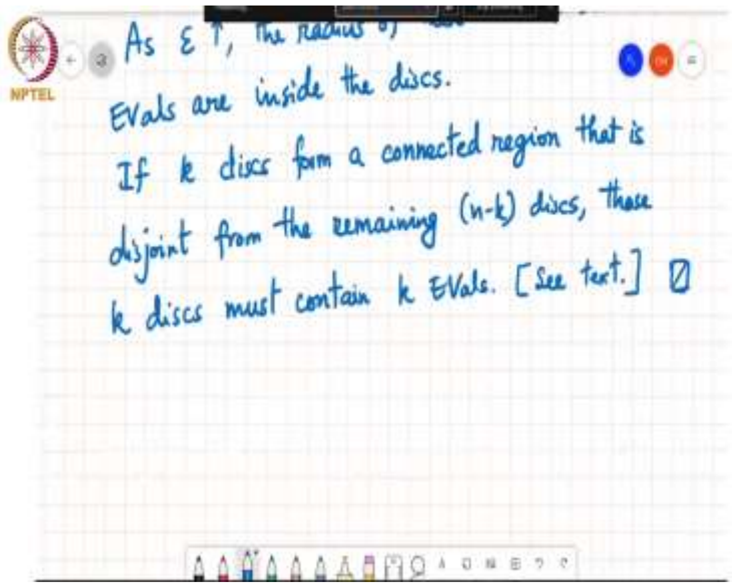

So, recall that we said we can write A of epsilon as D plus epsilon B, where D contains the diagonal entries of A and epsilon B and B contains all the off diagonal entries in A, now the eigenvalues of D are easy to locate and the eigenvalues are just aii. And now, we use the fact that the eigenvalues are continuous functions of the entries, within an epsilon neighbourhood around the eigenvalues of D.

So, in other words so for example, if I am on the two-dimensional plane, if I had say a11 over here this is the real and imaginary part, this is the two-dimensional complex plane, then a11 may be here, a22 maybe here, and a33 is here and ann is here and so basically if epsilon is very small then these radius of these Gersgorin's discs will be epsilon times summation AIG, so they will be small circles around these points those will be the Gersgorin discs and we have already seen that the eigenvalues lie inside these Gersgorin discs.

Now, as you increase epsilon, it is possible that some of these discs will end up touching each other or they end up merging and so if k of these circles, so as epsilon increases the radius of these discs increases, but the eigenvalues are always inside these discs and so if k of these circles form a connected region, which is disjoint from k circles or discs form a connected region that is disjoint from the remaining n minus k discs, then those k discs must contain k eigenvalues.

Again, this is because the eigenvalues are continuous functions of the entries and so they cannot suddenly jump to another disc, so this is the part where I am doing a bit of hand waving, it requires a little continuity argument, so you can see the text, so the full proof. So, let us see one simple example.

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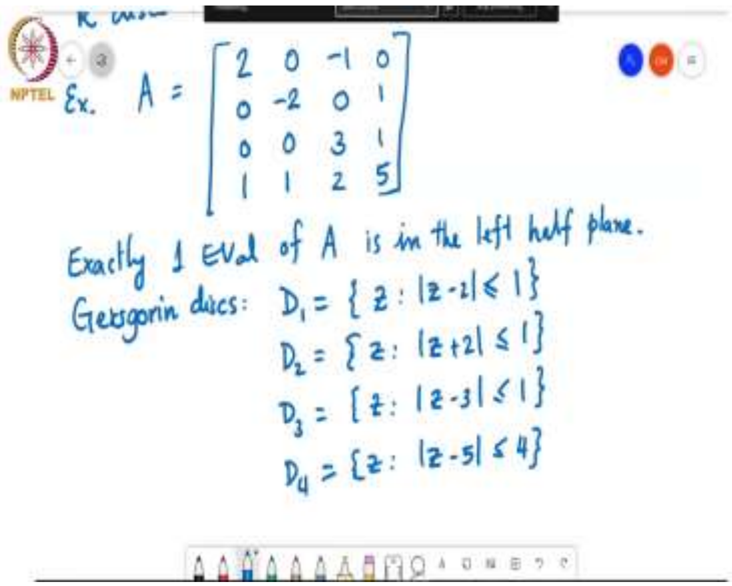

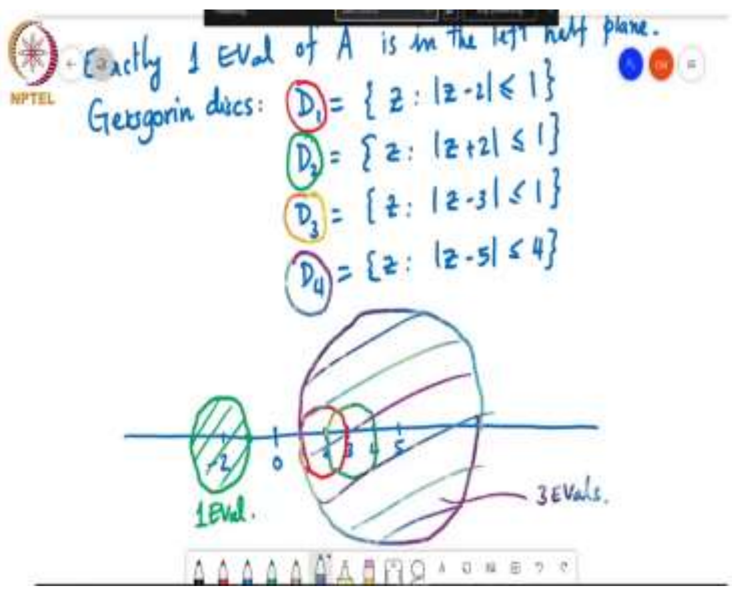

Suppose I have a matrix A, which is equal to 2 0 0 1, 0 minus 2 0 1, minus 1 0 3 2 and 0 1 1 5 does this matrix satisfy the dominant diagonal theorem?

Student: Yes, sir.

Professor: Yes. For every row when you see that the magnitude of the diagonal entry is strictly greater than the sum of the magnitudes of all other entries in the row. So, we can use this Gersgorin discs theorem to show that exactly one eigenvalue of A is in the left half plane, so what are the Gersgorin discs here? The Gersgorin discs are D1, so D1 is the Gersgorin disc corresponding to this diagonal entry, which is a set of complex numbers z, such that z minus 2 is less than or equal to the sum of all the other entries here in magnitude which is 1.

D2 is the set of z such that z minus of minus 2 which is plus 2 is less than or equal to again the sum of all the other off diagonal entries in the same row in magnitude, which is 1 D3 z minus 3 that is the third diagonal entry is less than or equal to 1 again and D4 is a set of all z, z minus 5 is less than or equal to 4. So, if I were to draw these discs on the real line, so say this is 0 and minus 1 minus 2 and this is 1 2 3 4 5.

So, the first disc is z minus 2 is less than or equal to 1, so it is the circle of radius 1 centred around 2, so I will draw it like this, I am not good at drawing circles and over here and let us say D3 this is centred at 3 and radius 1, so that will look like this. And this last one I am using funny colours here D4, this is centred at 5 and as of radius 4.

So, in fact, it is a big circle like this. So, these three circles form a connected region and this is one distinct region, so there will be one eigenvalue over here and three eigenvalues in this region. And this circle lies entirely in the left half plane and it is disjoint from the other three discs. So, that is why this has exactly one eigenvalue in the left half plane.

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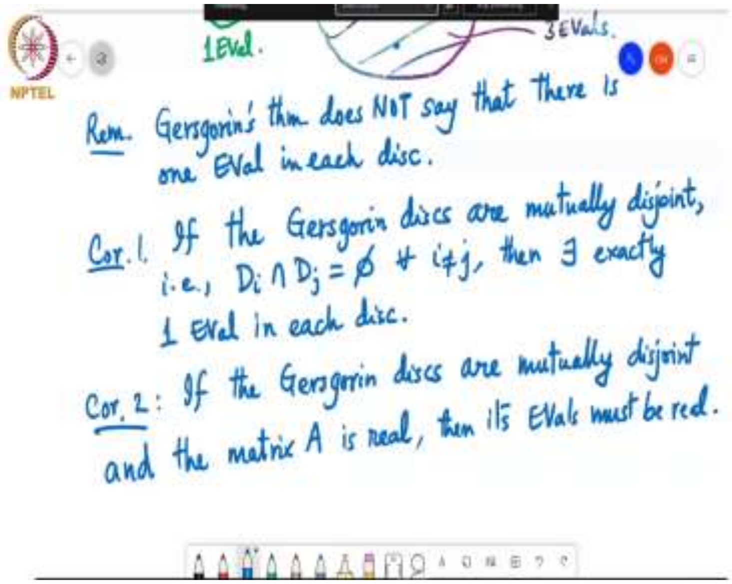

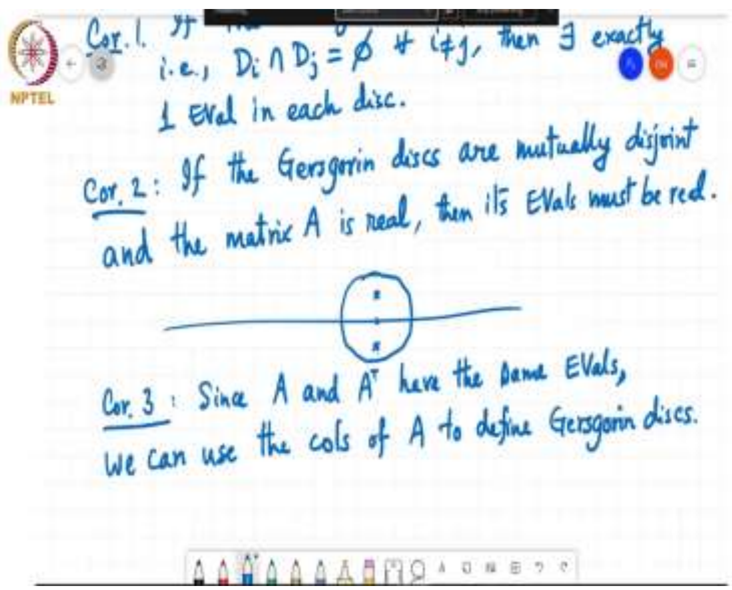

So, one remark is that Gersgorin's theorem does not say that there is one eigenvalue in each disc, this is important, this all it says is that every eigenvalue will be in the union of the discs, so pick an arbitrary eigenvalue it will sit inside the union of these discs. And the further it says that if k of the discs form a connected region which is disjoint from the remaining discs, then there are precisely k eigenvalues in that connected region.

That means that this matrix A that we wrote it could have all three of its eigenvalues over here for example, it may not be contained in these three discs. Now, but there is a corollary which is that and call it corollary one, if the Gersgorin discs mutually disjoint that is Di intersection Dj equals the empty set for all i naught equal to j, then it is true that there is exactly one eigenvalue in each disc.

In other words, when discs much it is possible that the eigenvalue passes into the other disc, it may not be in the first disc anymore, but as long as the discs remain disjoint then there will be one eigenvalue in each disc. There is another corollary, if the Gersgorin discs are mutually disjoint and the matrix A is real that is it has real valued entities, then its eigenvalues must be real. Anyone wants to reason out why this is the case. Take a guess, what can we say about the characteristic polynomial of a real valued matrix?

Student: Complex (())(26:09).

Professor: Perfect. So, the coefficients of the characteristic polynomial are real valued numbers and so the roots of the characteristic polynomial must occur in complex conjugate pairs and so each of the Gersgorin discs are mutually disjoint and they are all centred on the real line. So, if I take any particular disc like this, the roots of the characteristic polynomial, since they must occur in complex conjugate pairs, if there is a root up here, there will be a root down here.

But then if I mean there must be a root which is in this circle here, and if the root has an non-zero imaginary part, then there will be another root down here and so then it will have two roots inside this Gersgorin's disc, but then we already said in the Gersgorin disc theorem that if the discs are mutually disjoint then each disc should contain exactly one eigenvalue that is not possible, and so each disc must contain only one real valued eigenvalue.

The next corollary is that since A and A transpose have the same eigenvalues, why is this true? Why do A and A transpose have the same eigenvalues?

Student: Transpose is similar to it.

Professor: They are similar matrices, so they must have the same eigenvalues, so we can use the columns of A to define these Gersgorin discs and these could yield you know either better or additional information, there is some examples in the text you can look at, but you can see that if you define the Gersgorin discs using the rows, it will give you a set of Gersgorin disc.

If you define them using the columns that will give you a different set of good Gersgorin discs, because these matrices are not symmetric, non-necessarily symmetric and so that could that could give you some additional information about the location of the eigenvalues.

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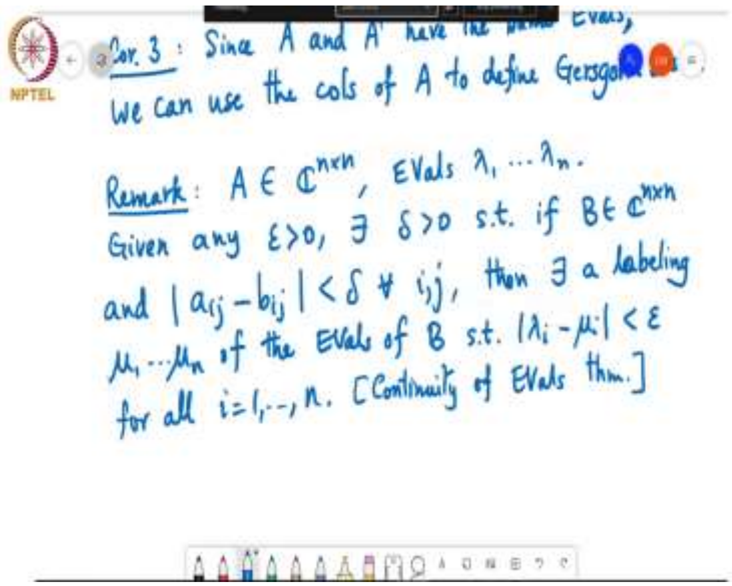

Just one more remark and this is related to this thing about the continuity of eigenvalues that I have been talking about. So, suppose A is in C to the n cross n and eigenvalues lambda 1 to lambda n, given any epsilon greater than 0 there exists a delta greater than 0 such that if B in C to the n cross n is a matrix such that mod of aij minus bij is less than delta for every ij.

So, B is a matrix that is within a delta neighbourhood of the matrix A then there exists a labelling mu1 through mu n of the eigenvalues of B such then mod of lambda i minus mu i is less than epsilon, in other words I do not have to consider the eigenvalues of B in see here the eigenvalues of B need not be even be real value and the same holds for the eigenvalues of A.

So, it is just saying that you can consider the eigenvalues of B in some order such that if you take corresponding entries of corresponding eigenvalues of A and B and subtract them and look at the magnitude that will be at most equal to the difference will be at most epsilon for all i equal to 1 to n. So, basically this is actually known as the continuity of eigenvalues theorem.

I will write it up here and it basically says that the entries of A are changed by a sufficiently small amount, then the change of the eigenvalues is also small. So, one way to see this is that the eigenvalues are the roots of the characteristic polynomial pA of lambda, now this pA of lambda depends continuously on the entries of A, the coefficients are just obtained by taking products of products and sums of the entries of A and so it depends continuously on the entries of A.

And so, this this theorem here the continuity of eigenvalue theorem is actually at consequence of another theorem which says that the roots of a polynomial with complex coefficients depend continuously upon these coefficients and this in turn is related to a famous theorem in analysis called the intermediate value theorem. So, now…

Student: Sir, how small is sufficiently small?

Professor: So, that is hard to say I mean, that depends on the matrix. So, but what it is saying is that there is a delta which is which if you choose it small enough, then as long as you take any B whose entries differ from the corresponding entries of A by at most delta, then the eigenvalues of B when considered in some particular order are such that every eigenvalue of B is within an epsilon neighbourhood of the corresponding eigenvalue of A. But how small that delta needs to be is something that depends on the matrix itself.

Student: Thank you sir. And is there any relation between delta and epsilon?

Professor: No. So, the idea is that I mean there is a relation of course in the sense that if you choose if you are starting out with an epsilon that is given to you, and corresponding to that epsilon you are choosing a delta, of course you can see that if you choose any smaller delta it will still work, but there will be a biggest possible delta such that as long as you put up by smaller than delta, the eigenvalues get better by at most epsilon.

But if you choose delta by 2 of that, or half of that, that will also work, that will satisfy this condition. So, there is a relationship, but it is more like an epsilon will determine an upper bound on how big delta can be. But again these are existence results and so it just says that there exists such a delta it does not tell you how you will find it, nor does it tell you how it depends on the matrix A.

So, now in the next class we will actually look at several interesting results that follow as a consequence of this Gersgorin disc theorem and in particular some results on the condition of eigenvalues which in turn will tell us how these eigenvalues will get perturbed when you perturb the matrix. So, we will stop here for today and continue on Wednesday.