Matrix Theory
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Eigenvalues: Majorization theorem and proof

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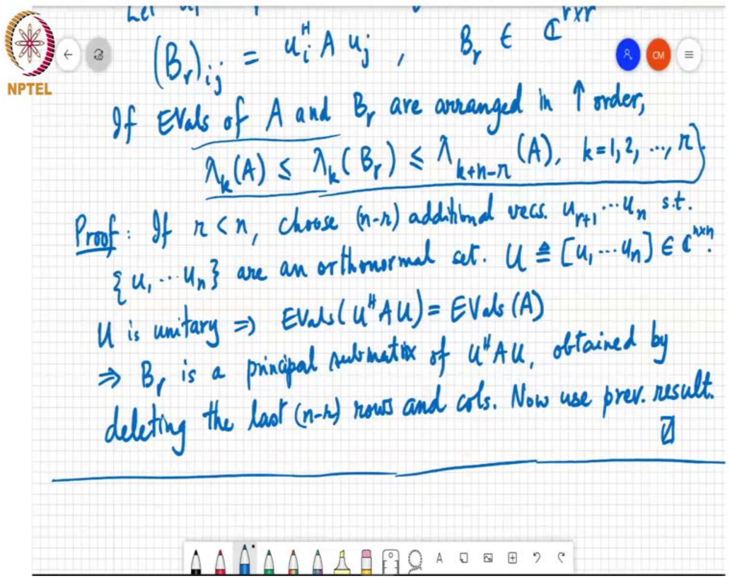

Some of the eigenvalues. So, there is a close connection or appears to be a close connection between the diagonal entries of a matrix and the eigenvalues. What is the precise relationship between these two? Or can we be a bit more specific, other than saying that they are both real valued and they have the same sum, can we be more specific about the relationship between these two sets of real numbers and this notion is what is called majorization. So, we will define this notion now.

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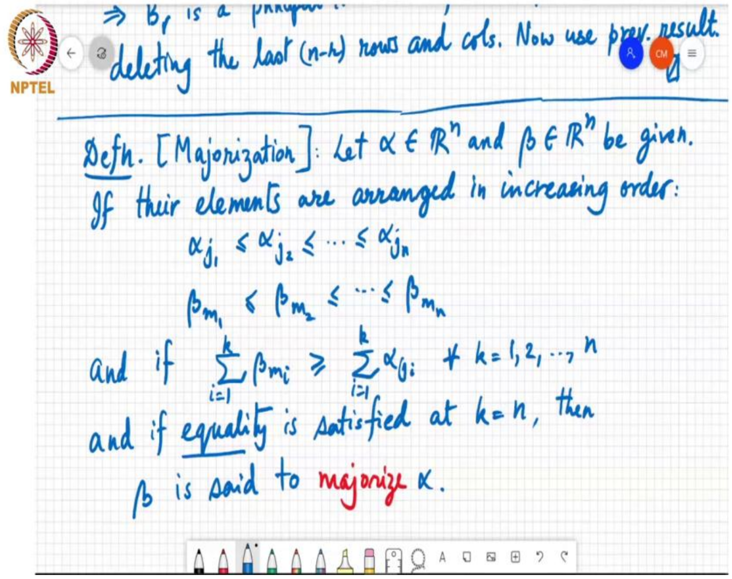

By the way, the word majorization also appears in the context of optimization, but that is a different notion do not get confused this is a notion of majorization that we are defining here in linear algebra. So, let alpha in R to the n and Beta in R to the n be given. If their elements are arranged in increasing order meaning that alpha j1 less than or equal to alpha j2 less than or equal to less than or equal to alpha jn.

So, these are the indices where the j1 is the index in alpha for which the corresponding entry in alpha has the smallest value and j n is the index for an entry in alpha where the corresponding entry of alpha has the largest value and similarly beta. So, beta say m1 less than or equal to beta m2 less than or equal to beta mn and if the summation i equal to1 to k beta mi is greater than or equal to sigma i equal to 1 to k alpha ji for every k equal to 1, 2, up to n and if equality is satisfied at k equal to n, then beta is said to majorize alpha.

The vector beta majorizes is alpha is in the sense that it is, we say that beta is kind of greater than or equal to alpha, if the sum of the k smallest entries in beta is greater than or equal to the sum of the k smallest entries in alpha. And this holds for k equal to 1, 2 all the way up to n minus 1, and the sum of the entries are equal.

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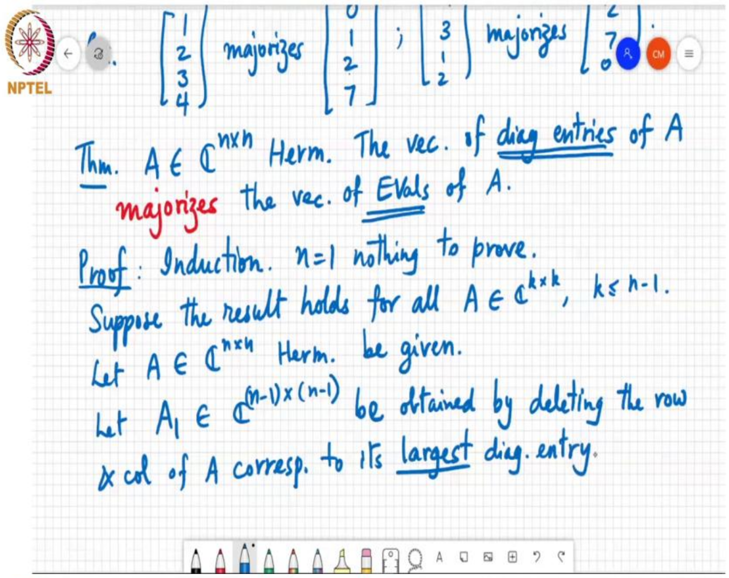

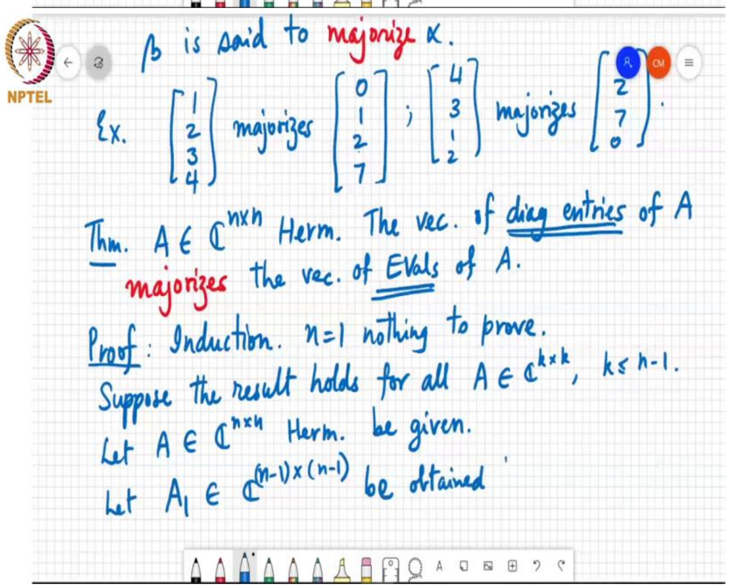

So, for example, this vector 1, 2, 3, 4 majorizes 0, 1, 2, 7. So, this number is bigger than this number, 1 plus 2 is 3 is bigger than 0 plus 1, 1 plus 2 plus 3 is 6 is greater than 0 plus 1 plus 2, which is 3. But the sum of all these guys is 10. And the sum of all these guys is also equal to 10.

So, equality is met when you add up all the entries together. But I do not need these to have been arranged in increasing order like this. If I had written it like this 4, 3, 1, 2 this majorizes. And I do not need this to have been written in increasing order 1, 2, 7, 0. I can write it like this also. So, these two vectors, but of course, it is possible that two vectors that have the same sum still do not majorize each other.

This is a very special structure not all vectors can be ordered like this. So, this is the notion of majorization. So, we have the following result. So, A is in C to the n cross n Hermitian. Then the vector of diagonal entries of A majorizes the vector of eigenvalues of A. That means that if I take the smallest diagonal entry of A, that will still be greater than or equal to the smallest eigenvalue of A, if I take the sum of the two smallest diagonal entries of A that will be greater than or equal to the sum of the two smallest eigenvalues of A and so on. So, let us quickly proof this.

So, you can see this is very interesting and again what I consider a very counter intuitive result. That you will be able to find such a very interesting relationship between the diagonal entries of A matrix and the eigenvalues of A matrix. So, the proof goes by induction. So, induction meaning will look at the size of the matrix and use induction over the size of the matrix. When I take n equals 1 the if you take a scalar that is equal to the diagonal entry is also equal to the eigenvalue and so there is nothing to proof.

Now suppose this result holds for all matrices of size k cross k and k going up to n minus 1. I know we need to show that the result holds for k equal to n. Now, we need to show that this result holds for n. So, A be a n cross n matrix. We are given matrix. Now, consider the matrix A1 which is obtained by deleting a row and column and for A1 this result holds by our induction hypothesis, how do I get A1.

I get it by be obtained by deleting the row and column. Corresponding to its largest diagonal. So, find out which is the largest diagonal entry of A and that row and column I delete and I call the matrix A1. So, now if you are (())(09:46), you are already seeing how this proof will go. So now the A1 is obtained by deleting a row and column and so we will use the interlacing result.

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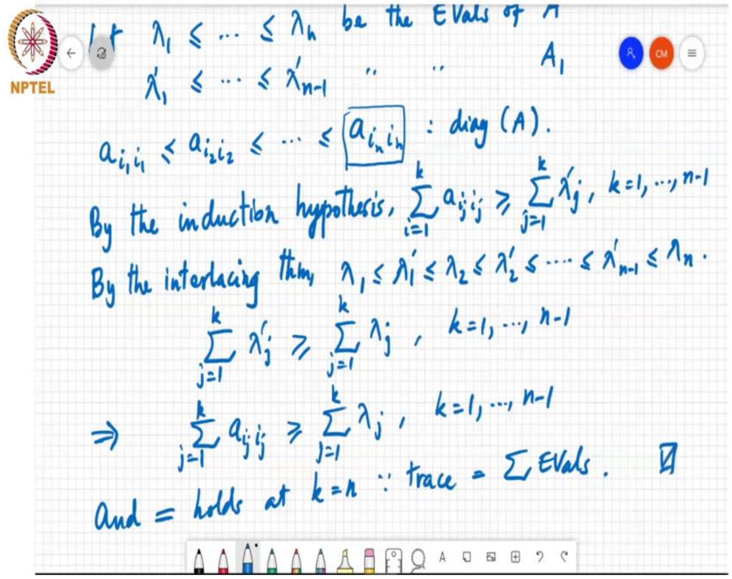

So let lambda 1 less than or equal to less than or equal to lambda n be the eigenvalues of A and lambda dash 1 less than or equal to lambda dash n minus 1 be the eigenvalues of A1. Now, by the induction hypothesis. First of all, let us say a i1 i1 less than or equal to a i2 i2 less than or equal to etcetera less than or equal to a in, be the diagonal entries of A.

We obtained A1 by deleting the row the in inth, row and column of the matrix A. Now, by the induction hypothesis summation i equal to 1 to k aij ij. So, the diagonal entries arranged in increasing order of the matrix A is the same as the diagonal entries of the matrix A1 arranged in increasing order. So, I can write aij ij here.

This is greater than or equal to sigma j equal to 1 to k lambda dash j and this is true for k equal to 1 through n minus 1 this is just directly from the induction hypothesis. Now, from the interlacing theorem, we have that lambda 1 is less than or equal to lambda 1 dash is less than or equal to lambda 2 less than or equal to lambda 2 dash less than or equal to etcetera up to lambda dash n minus 1 is less than or equal to lambda n.

So, that means that if I am taking the first k guys here. Instead, if I add a lambda 1 through lambda k, I will get a each of these lambda 1 dash is greater than or equal to lambda 1 lambda 2 dash is greater than or equal to lambda 2 and so on. So, I can write sigma j equal to 1 to k lambda dash j is greater than or equal to sigma j equal to 1 to k lambda j and this is true for k equal to 1 all the way up to n minus 1.

And so, this inequality continues to hold with lambda j this summation j equal to 1 to k lambda j sitting here. So, sigma j equal to 1 to k aij ij is greater than or equal to sigma j equal to 1 to k, lambda j. And this is true for k equal to 1 through n minus 1. But equality certainly holds because the trace equals the sum of the eigenvalues. So, that is it.

So, the what we have shown is that the vector of diagonal entries of a matrix A majorizes the vector eigenvalues of a matrix A and this, we use this interlacing property. It is an essential ingredient in showing such results.

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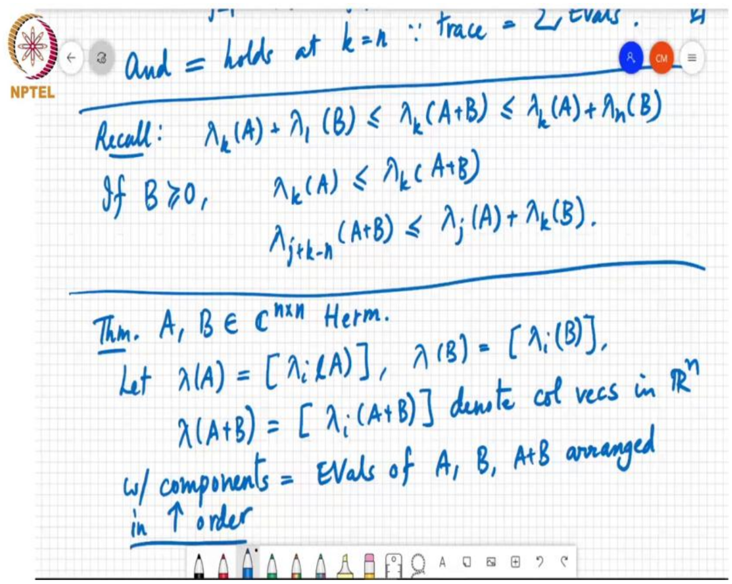

Now, majorization is actually very useful in expressing the relationship between the eigenvalues of the sum of a matrix and the individual components. So, for example, if you recall, we have seen results like lambda k of A plus lambda 1 of B is less than or equal to lambda k of A plus B is less than or equal to lambda k of A plus lambda n of B.

So, the kth eigenvalue of the matrix A plus B is at least equal to the k eigenvalue of A plus the smallest eigenvalue of B and at most equal to the kth eigenvalue of A plus the largest eigenvalue of B and if B is positive semi definite then this is non-negative, so, we have lambda k of A is less than or equal to lambda k of A plus B and you also have that lambda j plus k minus n of A plus B is less than or equal to lambda J of A plus lambda k of B.

So, these are some results that we have seen earlier. So, in this context we have two more results that am just going to state because we do not have time to do the proofs right now, but these are results that talk about majorization type relationships between eigenvalues of the summands to the eigenvalues of the sum of the of two matrices. So, the first result is the following.

So, A, B are n cross n Hermitian symmetric matrices So, let lambda of A be a vector and its entries are lambda i of A and lambda of B be the another vector whose entries are lambda i of B. Similarly, lambda of A plus B is a vector with lambda i of A plus B as its entries and so, these denote column vectors in R to the n with components equal to the eigenvalues of A B and A plus B arranged in increasing order. This is very important. They are arranged in increasing order.

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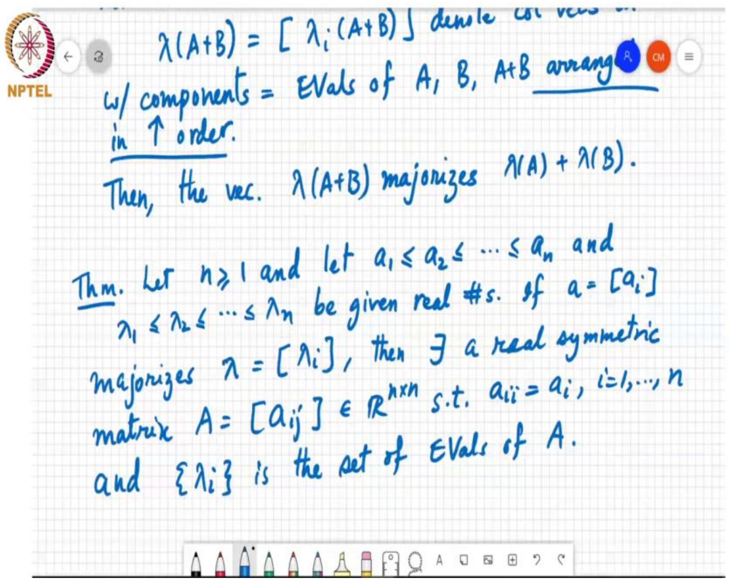

Then the vector lambda of A plus B majorizes lambda of A plus lambda of B and so, that is one result. So, it talks about more precise relationship between the eigenvalues of A plus B and the eigenvalues of A and the eigenvalues of B, but what you have to do is to arrange the eigenvalues of A and B in increasing order and then add them together then this vector of eigenvalues of A arranged in increasing order majorizes the vector of lambda of A plus lambda B then we also have the following Converse result.

So, let n be at least equal to 1 and let a1 less than or equal to a2 less than or equal to up to an and lambda 1 less than or equal to lambda 2 lambda n. So, imagine that these are some diagonal entries and these are some eigenvalues and suppose that this vector majorizes vector lambda which has lambda i as its entries then there exists a real symmetric matrix A equal to aij being its entries in R to the n cross n such that aii equal to ai, i equal to 1 to n. So, it has ai as its diagonal entries. And lambda i is the set of eigenvalues of A.

So, given a set of numbers real numbers, which where one set of real numbers, majorizes the other set of real numbers then you can find a matrix A such that the vector A are the contains, the vector A forms diagonal entries of this matrix and the vector lambda forms the eigenvalues of this matrix. So, there is always such a matrix that you can find. So, that is all I wanted to say today. We will continue on Monday.