Matrix Theory

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Interlacing theorem I

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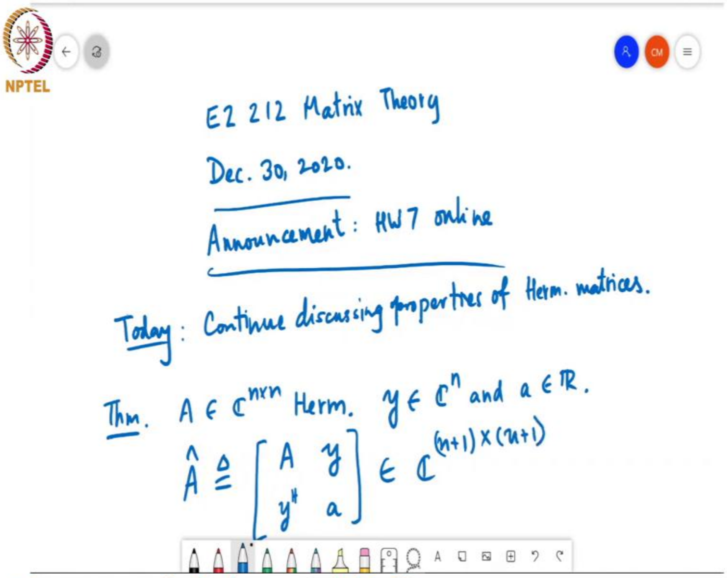

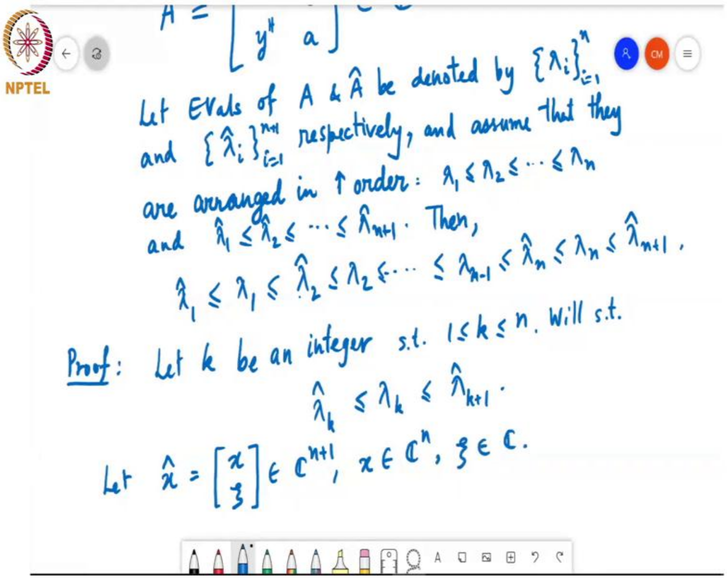

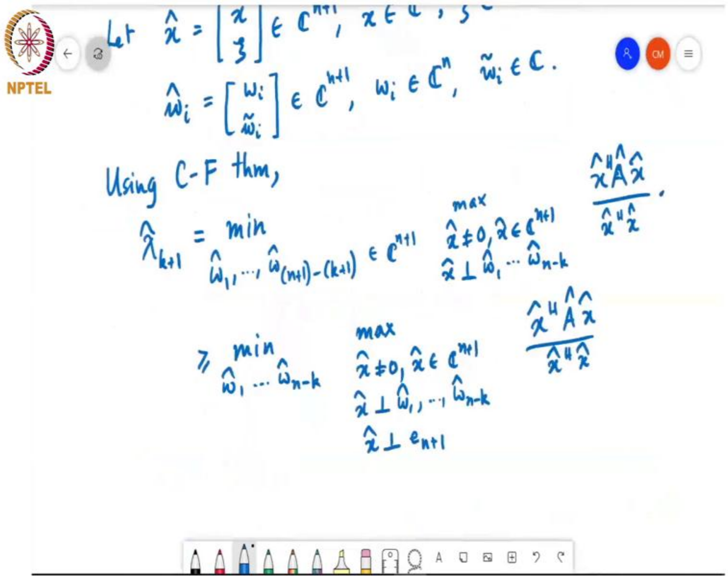

So, today, we will continue discussing properties of Hermitian symmetric matrices. In particular, we were looking at several interlacing theorems. And we at the end of the previous class, we stated the following theorem. So, A in C to the n cross n is a Hermitian symmetric matrix and y is a vector in c to the n and A is a real number.

Then, we defined A hat to be the matrix A, y, y Hermitian and a, which is an n cross n plus 1 cross n plus 1 matrix. Then as usual, we arranged the eigenvalues in increasing order. So, let the eigenvalues of A and A hat be denoted by lambda i, this is i equal to 1 to n and lambda hat i, this is I equal to 1 to n plus 1, respectively.

And assume that they are arranged in increasing order. So, that lambda 1 is less than or equal to lambda 2 less than or equal to lambda n and lambda hat 1, lambda hat 2, lambda hat n plus 1. Then, lambda hat 1 is the smallest eigenvalue, which is less than or equal to lambda 1, lambda hat 2, lambda 2, lambda n minus 1 lambda hat n, lambda n, lambda hat n plus 1. So, when you border a matrix like this with y, y Hermitian A.

The smallest eigenvalue of A, A hat is going to be less than or equal to the smallest eigenvalue of A. So, by bordering you cannot increase the smallest eigenvalue. And similarly, the largest eigenvalue of A hat is going to be bigger than or equal to the largest eigenvalue of A, so that you cannot increase, you cannot decrease the largest eigenvalue by bordering it with another matrix, another vector and scalar in the bottom right corner. So, that is what the result says we did not prove this the previous time. So, now we will write out the proof.

And it is again, sort of easy consequence of the Courant-Fischer theorem. But again, the proof is a little bit clever in my opinion, so let us see how it goes. So, let k be an integer. So, we will prove this inequalities, sequence of inequalities for some k. So, take lambda k, and show that it is less than or equal to lambda hat k. And sorry, it is greater than or equal to lambda hat k and less than or equal to lambda hat k plus 1, that is what we will show.

So, such that 1 less than or equal to k less than or equal to n. So, we will show that, lambda hat k is less than or equal to lambda k is less than or equal to lambda hat k plus 1. So, let we will define x hat to be equal to. So, keep in mind that we have to deal with matrices and vectors of both dimension n and n plus 1.

So, we will write it like this, x and zeta, this is in c to the n plus 1, where x is in c to the n and zeta is just a complex number. And similarly, we will define w hat i to be equal to wi and wi tilde, which is again in C to the n plus 1 with wi belonging to C to the n and wi tilde belonging to C. Now, we use the Courant-Fischer theorem.

So, in order to show, so we will first show this for example, so in order to show this, we want to show that lambda hat k plus 1 is greater than or equal to lambda k. And so, lambda hat k plus 1 by Courant-Fischer theorem is the minimum. So, usually in order to show lower bounds, we want to show that lambda hat k plus 1 is lower bounded by lambda k. We will use the min-max form of the Courant-Fischer theorem.

So, this is the minimum over w hat 1, all the way up to w hat n plus 1 minus k plus 1, all belonging to C to the n plus 1, of the maximum x hat not equal to 0, x hat in C to the n plus 1. And x hat perpendicular to all these other vectors, w hat 1, all the way up to w hat. So, n plus 1 minus k plus 1 is n minus k. And the objective function is the same x hat Hermitian A x hat divided by x hat Hermitian x hat.

Now, this is greater than or equal to, so this is where the slightly clever part of the proof comes in. So, the min over the same thing, w hat 1 up to w hat n minus k, the maximum of x hat not equal to 0, x hat in C to the n plus 1 and x hat perpendicular to w 1 hat up to w n minus k, I will throw in an extra constraint, x hat perpendicular to e n plus 1, this is the last column of the n plus 1 cross n plus 1 identity matrix.

So, if I throw in an extra constraint here, the maximum achieved by the same objective function x hat Hermitian Ax hat divided by x hat Hermitian x hat, if I throw in an extra constraint, I may not be able to achieve as higher maximum as this. And so, whatever this will achieve, and on top of that I am minimizing with respect to these w hats. So, by inserting an extra constraint over here, this quantity ends up becoming at most this quantity, so, this is a lower bound.

Now, these x hats are all perpendicular to e n plus 1, which means that the last coordinate of x hat is always equal to 0. And so, if I account for that, while searching over this space of x hat belonging to C to the n plus 1, the only components of x hat that are still under my control are the first n components, the n plus 1th component does not matter.

And similarly, if I say x hat should be perpendicular to all these vectors, it is sufficient to search over vectors, where the first n components of x hat are orthogonal to the first n components of w hat 1, w hat 2, up to w hat n minus k. So, I can simply reduce this down to an n, n length vector related constraint. And further, if I have made the last entry of x hat equal to 0, this is sorry, this is A hat here.

I mean, that is the application of the Courant-Fischer theorem is not A I mean, it cannot multiply with A because this is an n plus 1 length vector anyway. And further, if the last entry of x hat is equal to 0, if you go back to the definition of A hat, and see if you multiply by this by a vector x hat Hermitian, and by right multiply by x hat, but the last entry of x hat equals 0, that will kill all these components. So, that is the same as taking x, which is the first n entries and doing x Hermitian Ax.

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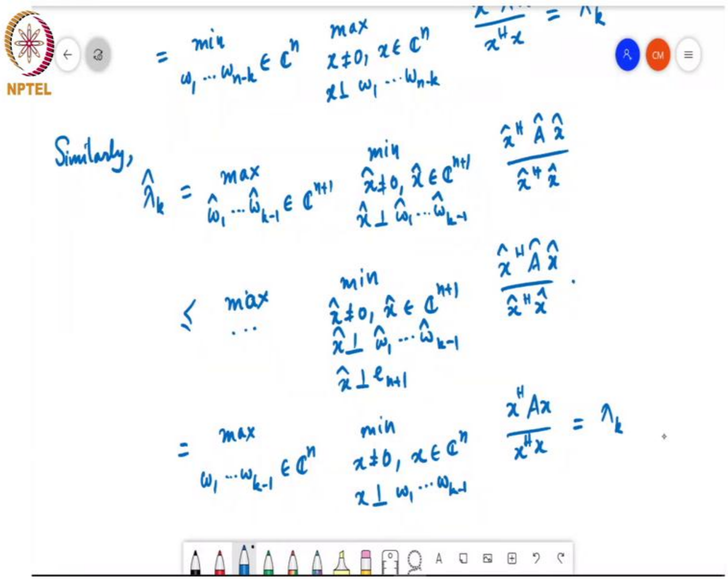

So, I can rewrite this, whatever is over here, I can rewrite all of this as exactly equal to the minimization of w 1 up to w n minus k in C to the n. So, these were in C to the n plus 1 of the maximum over x not equal to 0, x in C to the n, x perpendicular to w 1 through w n minus k. And I do not need this constraint now, because I have already accounted for the fact that the last entry of x equals 0 in writing out these constraints, so is this just x Hermitian Ax over x Hermitian x.

And by the Courant-Fischer theorem, this is exactly equal to lambda k. So that proves the first half of the result. And similarly, if I take lambda hat k, and now I need to show that this is less than or equal to lambda k, so I will use my starting point will be the max-min version of the Courant-Fischer theorem.

So, max goes over w hat 1 through w hat k minus 1 in C to the n plus 1, the minimum over x hat not equal to 0, x hat in C to the n plus 1, and x hat perpendicular to all these vectors, w hat 1 through w hat k minus 1, x hat Hermitian A hat x hat divided by x hat Hermitian x hat. This is just Courant-Fischer theorem for lambda hat k. And this is less than or equal to, I will use the same trick as before.

So, this is the max over the same thing, the min over x hat not equal to 0, x hat in C to the n plus 1, x hat perpendicular to w 1 hat through w hat k minus 1. And now I throw in one extra constraint, which is x hat is perpendicular to e n plus 1. And now, I am throwing in this extra constraint on a minimization problem. So, the minimization problem, x hat Hermitian A hat x hat divided by x hat Hermitian x hat.

So, now with this minimization problem, I may not be able to achieve as lower minimum as I would do here, because this is a more constrained problem. And that is why this inequality goes like this as less than or equal to. And now that I have made x hat perpendicular to e n plus 1 that means the last entry of x hat equals 0. So, I can then rewrite this as, this is the maximum over n if the last entry of x hat equals 0, this can be reduced to an orthogonality between x and w 1 through w k minus 1.

So, then, I can do a maximization over w 1 up to w k minus 1 in c to the n and the minimum over x not equal to 0 x in C to the n and x perpendicular to w 1 through w k minus 1 x Hermitian Ax over x Hermitian x, which is exactly equal to lambda k by the Courant-Fischer theorem for the matrix A and finding lambda k. So that is, that is the proof. Now.