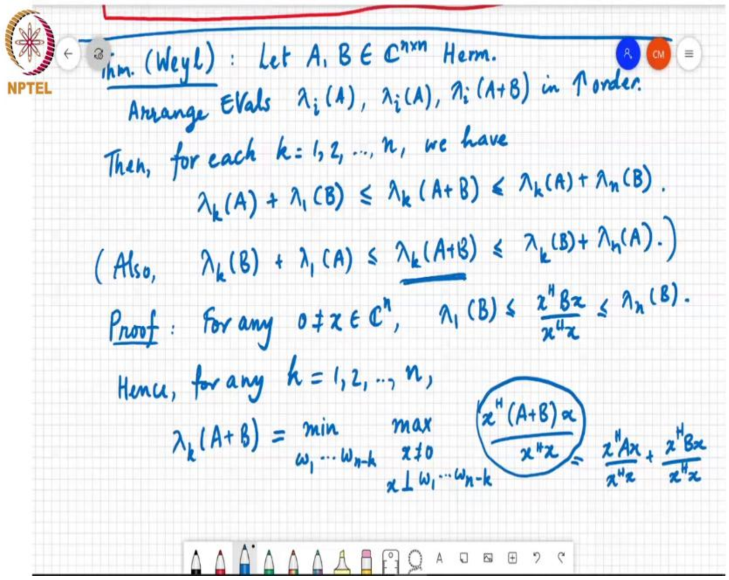
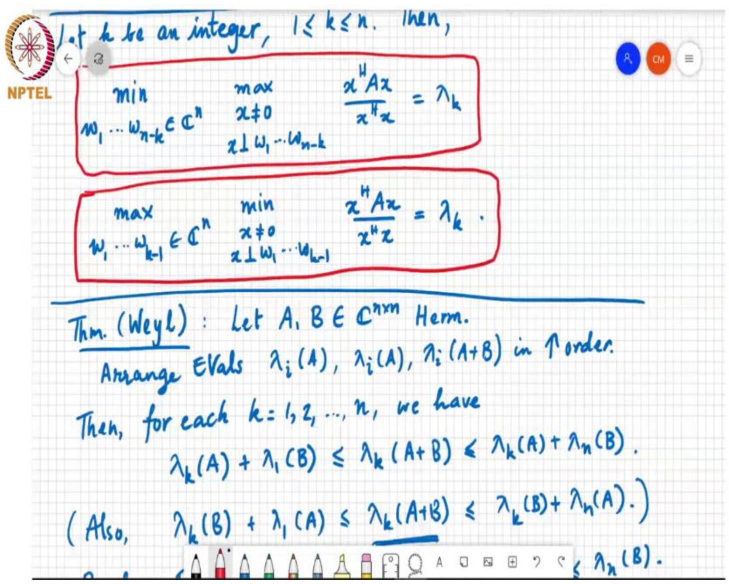
Matrix Theory
Professor Chandra R Murthy
Department of Electrical Communication Engineering
Indian Institute of Science, Bangalore
Weyl's theorem

(Refer Slide Time: 00:13)





This, theorem, the Courant-Fisher theorem is very powerful. And so we will discuss many results that come out as a consequence of this theorem. The 1st is a result which relates the eigenvalues of A plus B with those of A or B. This is a theorem due to somebody called Weyl, so it is called Weyl’s theorem.

So, let A, B, C to the n cross n be Hermitian symmetric matrices. Then, as usual we will arrange the eigenvalues in increasing order. So, I am introducing notation here. So, lambda i of A is the i'th largest eigenvalue of A. So, lambda 1 of A is the smallest eigenvalue lambda 2 of A is the 2nd, is a next bigger eigenvalue and so on. And lambda i of A plus B is the i'th eigenvalue of A plus B when the eigenvalues are arranged in increasing order.

Then, for each k equal to 1 2 up to n, we have, lambda k of A plus lambda 1 of B is less than or equal to lambda k of their sum. And which is in turn less than or equal to lambda k of A plus lambda n of B. Now, when you look at this result, I think, at least some of you will see that the result is obvious, the k’th eigenvalue of A plus B can is going to be at least lambda k of A plus lambda 1 of B, and at most lambda k of A plus lambda n of B.

So, for example, if D was the identity matrix, then what you see from this result is that the eigenvalues of A plus the identity matrix, actually you know this already that all the eigenvalues of A will get shifted up by 1. And therefore, this result is also saying that lambda k of A plus the identity matrix is at least equal to lambda k of A plus 1 and at most equal to lambda k of A plus 1. In other words, lambda k of A plus B is equal to lambda k of A plus 1. So, let us see, let us just write out how this thing is proved.

Student: Sir?

Professor: Yeah.

Student: Sir, A and B are can be interchanged here?

Professor Chandra R. Murthy: Yes, of course. So, there is nothing special about B here, B is not in any way different from A, A and B are both Hermitian symmetric matrices. So, in fact, before I proceed, I will write that also. So, you can also say k of B plus lambda 1 of A is less than or equal to lambda k of A plus B lambda k of B plus lambda n of A.

So, basically if you want to obtain bounds for lambda k of A plus B, you can choose the min of these two quantities lambda k of A plus lambda 1 of B, lambda k of B plus lambda 1 of A, sorry you can choose the max of these two that will still be a lower bound on lambda k of A plus B and you can choose the min of these two and that will also be an upper bound on lambda k of A plus B.

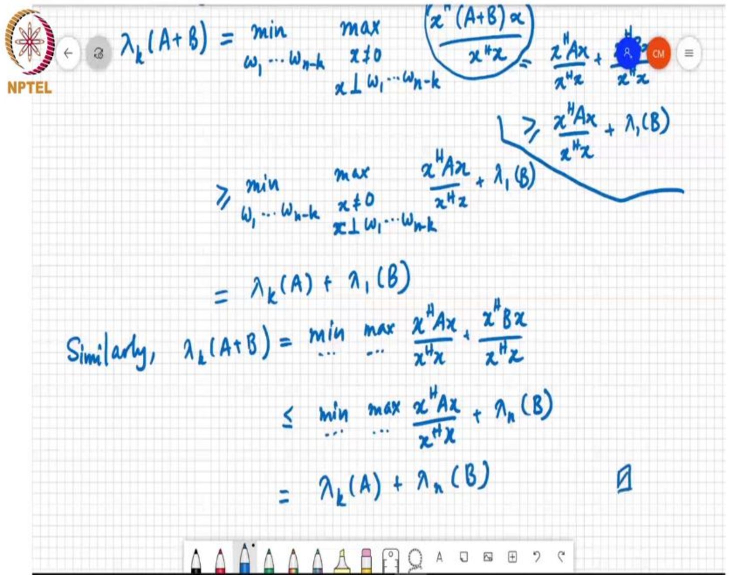
So, proof. So, we know that for any 0 not equal to x in c to the n, by the Rayleigh-Ritz theorem, lambda 1 of B is less than or equal to x Hermitian Bx over x Hermitian x is less than or equal to lambda n of B. So, as a consequence for any k 1, 2, up to n, if I look at lambda k of A plus B, by Courant-Fisher theorem, this is equal to the min over w 1 through w n minus k.

So, I am using the min-max formulation. So, you should pay attention to this because you will see that for some results, we will use this min-max formulation. And for some other results, we will start from the max-min formulation. And it is a very interesting exercise to see if you can prove the same result by starting from the max-min formulation instead of the min-max formulation.

In some cases, it will turn out that proof kind of works out the same way. But in some other cases, it will turn out that the proof going one way is much easier than the proof going the other way. So, max over x not equal to 0, x perpendicular to all these vectors of x Hermitian times A plus B times x over x Hermitian x.

So, this is just good Courant-Fisher theorem, just written out for this case. And this itself I can write as x Hermitian Ax plus x Hermitian Bx over x Hermitian x. So, I can, I will just, instead of writing a whole step, I will just say that this is equal to x Hermitian Ax over x Hermitian x plus x Hermitian Bx over x Hermitian x. Now, this x Hermitian Bx over x Hermitian x is at least equal to this. So, if I replace the x Hermitian Bx by lambda 1 of B, I am only decreasing the value of whatever this thing is.

(Refer Slide Time: 8:02)

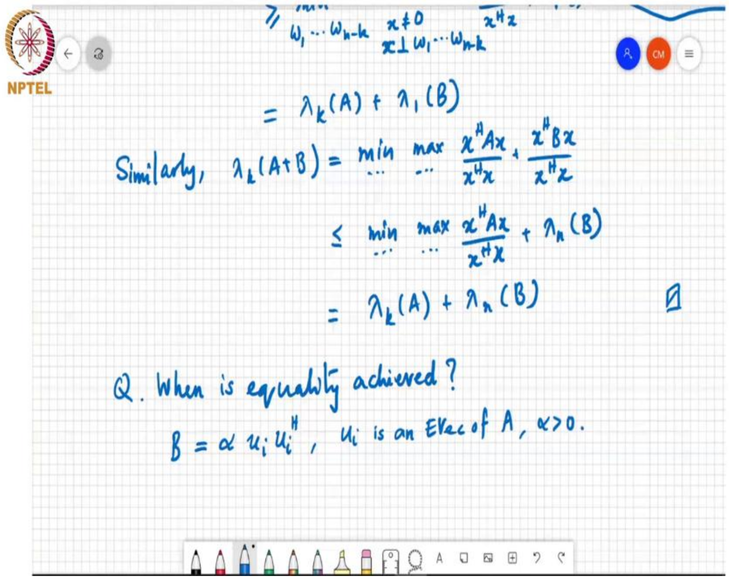


So, this is greater than or equal to, I will, in fact, I will write that here itself, so that it is clear. This is always for any x not equal to 0, this is greater than or equal to x Hermitian Ax over x Hermitian x plus lambda 1 of B. And so, that means that this lambda k of A plus B is greater than or equal to, because I replace this by its lower bound, min over w 1 through w n minus k max x not equal to 0, x perpendicular to w 1 through w n minus k x Hermitian Ax over x Hermitian x plus lambda 1 of B.

And this part now does not depend on x anymore, but this part by Courant-Fisher theorem, this is directly lambda k of A. And so, this is exactly equal to lambda k of A plus lambda 1of B. And in a similar way, instead of replacing this by lambda 1, I could replace it by lambda n, and then I get an upper bound. And then I substitute lambda n here. And this is still equal to lambda of A.

So, I get the upper bound that lambda k of A plus B is equal to the min over something of the max over something it is all these things that I am not writing again and again, x Hermitian Ax over x Hermitian x plus x Hermitian Bx over x Hermitian x and this is less than or equal to this min over all these things, the max over all these things x Hermitian Ax over x Hermitian x plus lambda n of B and this is just lambda k, that proves the result.

(Refer Slide Time: 10:32)



So, some small thing that you can think about is, when would equality be obtained in the bounds of the Weyl's theorem.

Student: x Hermitian Bx by x Hermitian x equal to lambda n B and lambda 1 B.

Professor: So, x is a variable of optimization here. So…

Student: All lambdas are equal?

Professor: You should think about it. So, for example, suppose B was equal to some alpha times, I will write it as ui ui Hermition, where ui is an eigenvector of A. So, suppose it was like this, then yeah. So, by choosing B, B of this form, and alpha being some positive number, you can actually attain these bounds. You should, I do not want to give you the whole answer here. But do think about it, but this is the form of the B that will attain these bounds.