Matrix Theory

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Summary of Rayleigh-Ritz and Courant-Fischer Theorems

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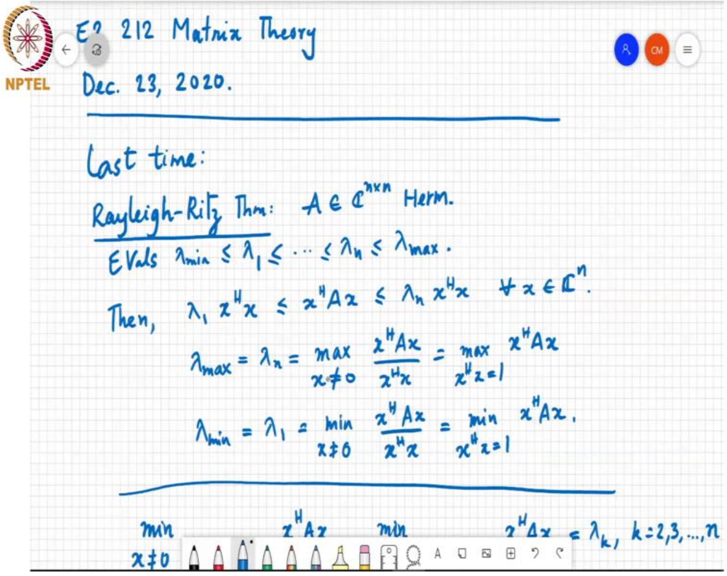

We looked at the Rayleigh-Ritz theorem, which says that if you have a Hermitian symmetric matrix A then if you order the eigenvalues. So, the Hermitian symmetric matrix has all real valued eigenvalues. So, you can order them in increasing order and so lambda 1 is the smallest eigenvalue and lambda n is the biggest eigenvalue, then lambda 1 times x Hermitian x is a lower bound on x Hermitian Ax and lambda n times x Hermitian x is an upper bound on x Hermitian Ax for any x in C to the n.

Furthermore, both the lower bound and upper bound are achievable that is, there exists an x for which this lower bound is achieved and a different x for which the lower bound is achieved and that is given by this next two statements here namely that lambda max which is equal to lambda n is the maximum value of x Hermitian Ax over x Hermitian x for x not equal to 0, which can also be written in this way.

Similarly, lambda 1 is the minimum of x Hermitian Ax over x Hermitian x for all x not equal to 0. So, this is a maximization problem and this is a minimization problem and the smallest and largest eigenvalues can be written as solutions to maximize the, here a minimization problem and here a maximization problem So, that is one thing we saw.

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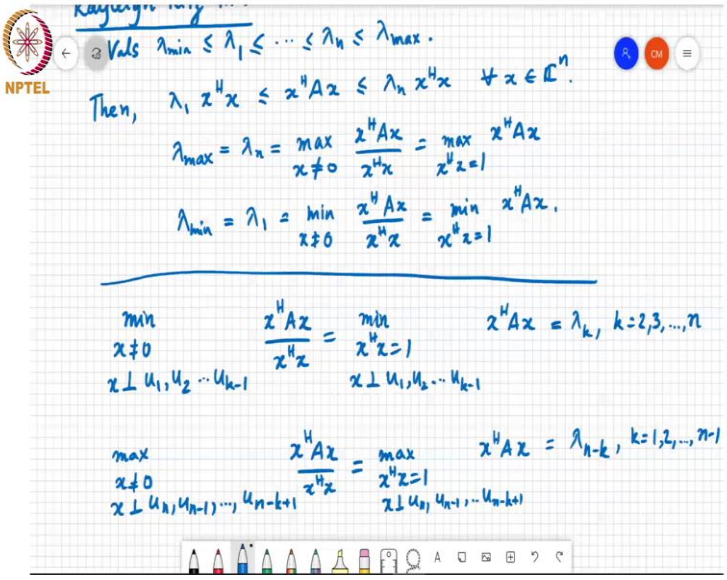

And then we also saw that by considering the fact that these eigenvalues are in increasing order if we were to maximize x Hermitian Ax over x Hermitian x subject to x being perpendicular to u1 the eigenvector corresponding to the smallest eigenvalue then in fact, if suppose you were to minimize x Hermitian Ax over x Hermitian x subject to x being non-zero and perpendicular to u1 then the solution to that optimization problem is actually lambda 2, the next largest eigenvalue.

And extending this argument we have that the minimum over all non-zero x such subject to the constraint that x should be perpendicular to the 1st k minus 1 eigen vectors that is the k minus 1 eigen vectors corresponding to the smallest k minus 1 eigenvalues of the matrix A of this objective function x Hermitian Ax over x Hermitian x that is equal to lambda k for k equal to 2, 3 up to n.

And similarly, coming down from the largest eigenvalue, if you solve a maximization problem subject to x being perpendicular to the first un, un minus 1 up to un minus k plus 1 that is the first that is k, k minus 1, k minus 1 eigen vectors corresponding to the top k minus 1 eigen vectors, eigenvalues of the matrix A of the same objective function x Hermitian Ax over x Hermitian x that is going to give you lambda n minus k for k equal to 1, 2 up to n minus 1. So, this is the this is one kind of crucial result we saw.

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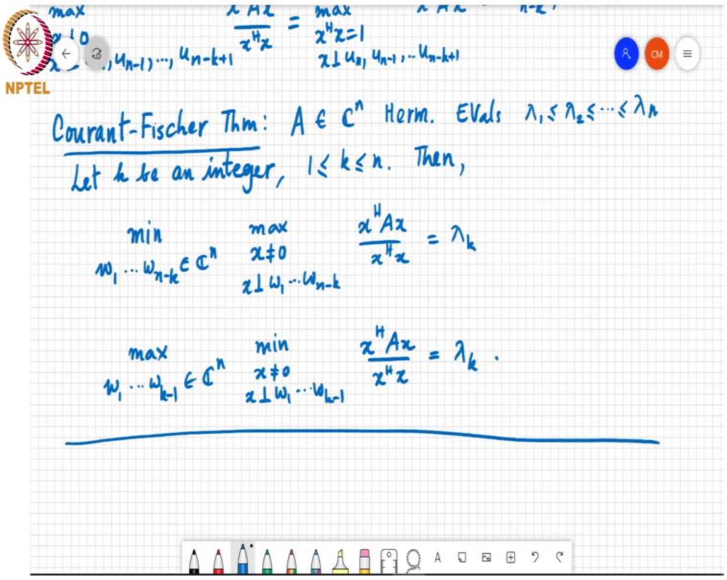

Another very crucial result which we will use many times going forward is the Courant-Fischer theorem. And, what this says? This is a min max theorem. And, what this theorem says is that, again that is the starting point or the setting is that A is a matrix of size n cross n, n is a Hermitian symmetric matrix with eigenvalues lambda 1 less than or equal to lambda 2 less than or equal to up to lambda n and let k be an integer, 1 less than or equal to k less than or equal to n.

Then there are two ways of writing lambda k. The first is a min max formulation the minimum over a set of vectors w1 up to wn minus k, C to the n, the maximum of x not equal to 0, x perpendicular to w1 through wn minus k of x Hermitian Ax over x Hermitian x is equal to lambda k.

The next formulation is a max-min formulation. The maximum of w1 through wk minus 1 of the minimum x not equal to 0, x perpendicular to w1 through wk minus 1, x Hermitian Ax cos function is a same, is also equal to lambda k and we saw the proof of this result and the proof is again like I said last time what I consider to be a somewhat clever proof there is a non-trivial step involved where we, we said that setting y1 to yk minus 1 equal to 0 where yi is equal, is defined to be what u Hermitian times, ui Hermitian times x by setting some of these variables equal to 0 you only decrease the cost function and so, that is how it, you obtain a lower bound on, on you obtain that lambda k is a lower bound on this part here.

And then you show that this lower bound is indeed achievable and therefore, they are equal. So, this I mean, as I mentioned the last time, please go over the proof of this theorem very carefully and make sure you understand it thoroughly. Because this is the, these ideas in that proof are the ones that we are going to use quite frequently in the upcoming results.