Matric Theory
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Fundamental theorem of linear algebra

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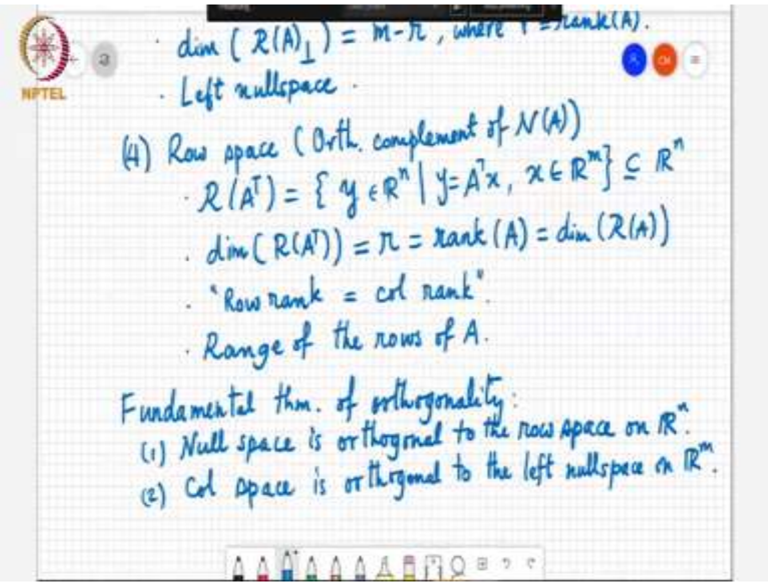

So, this is a fundamental theorem of linear algebra.

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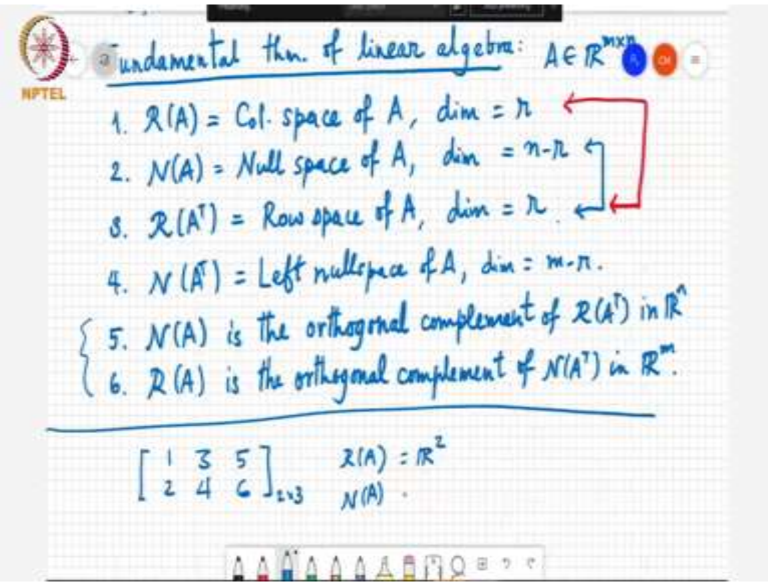

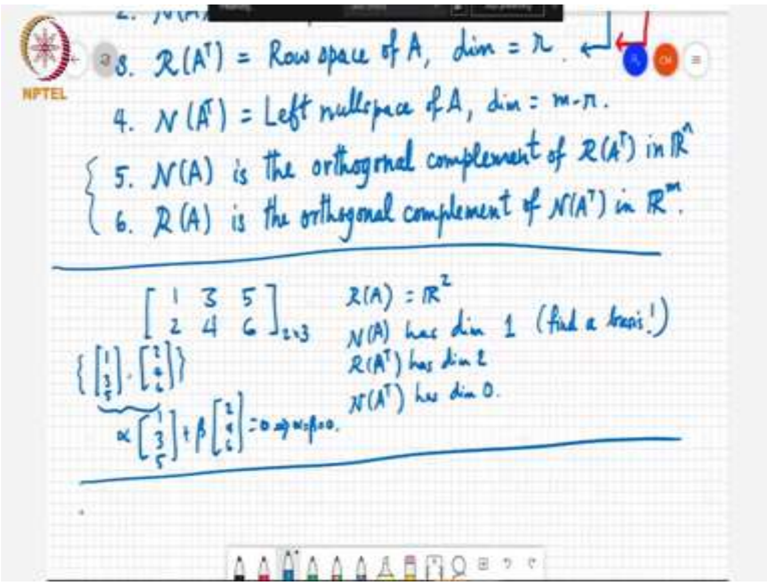

So, R of A is the column space of A and it has a dimension equal to R. So, again A is in M by N, the null space of A and it has a dimension equal to N minus R. The column space of A transpose also known as the row space of A and it has a dimension R, then the fourth subspace is the null space of A transpose, the left null space of A and it has dimension M minus R. And the fifth point is that is these two points I already mentioned above.

Let me put it this way; is the orthogonal complement of R of A transpose in R to the N and R of A is the orthogonal complement of N of A transpose in R to the N; this is called the fundamental theorem of linear algebra. So, now the next thing I want to discuss is about the rank.

Student: Can you can you please summarize what each statement of this fundamental theorem means, just a glance.

Professor: See the first four statements are more definition, R of A is defined to be the column space of A and think of this dimension of the column space of A equal to R as basically a statement that is saying that there are R linearly independent vectors in the columns of A and the null space of A are the set of vectors that map to 0 and its dimension is N minus R and that happens because the null space of A is the orthogonal complement of R of A transpose.

So, these two together should always add up to the dimension of R to the N, which is the total dimension of the space of which these two are orthogonal complement subspaces. And the row space R of A transpose is essentially the range space of the transpose of A, which means you have exchanged the rows and columns and the point is that the R of A transpose and R of

A always have the same dimension, I put this in red; so this is another very important point, which is something that I said will come, we will actually show this later on, but this is another point which is in no way intuitively obvious to me.

But regardless of which matrix you pick the row space and the column space have the same dimension or the row rank and the column rank are always equal. And what the next statement is saying is that N of A transpose is the left null space, so it is the null space defined on A transpose and again because R of A is the orthogonal complement of N of A transpose, the dimension of the left null space and the dimension of R of A must add up to M.

So, if the dimension of R of A was R, the dimension of N of A transpose must always be M minus R and these two are basically again coming from the definitions of how you define N of R, it is a set of all vectors that are orthogonal to rows of A and therefore it is the orthogonal complement of R of A transpose and R of A is the orthogonal complement of N of A transpose.

Student: So, sir can you take a simple example of 2 by 3 matrix like, 1, 2, 3, 4, 5, 6 and maybe just tell what they all represent like, take 1, 2, 3, 4, 5, 6.

Professor: See, if you, I mean that is actually not a good example; too many numerical examples, but since you requested I will quickly look at this particular example. So, you can see that the columns of this matrix, there are two columns that are linearly independent and of course, since they are in two dimensions the third column will be linearly dependent on these two columns, so this is a 2 cross 3 matrix.

So, the R of A, it is the column space of A and in this case it is actually equal to R squared, because these two columns together can already span the entire R square and so the null space of A which is the set of all vectors that map to 0, it has dimension N minus R which is 3 minus 2, which is 1, and you can see that if I take the vector, so you have to actually work it out, but…

Student: Okay, sir, I got it. I think I got it.

Professors: It has dimension 1, so there is you can find the basis.

Student: Sir, we are doing a couple of these problems in the problem session tomorrow, so it will be more obvious tomorrow. You can really continue with the class right now.

Professor: So similarly yeah just for the sake of completeness R of A transpose, now this is, if I take a transpose, it is these vectors, 1, 3, 5, the span of these vectors 2, 4, and 6 and these are linearly independent vectors, so it has dimension 2 and these two vectors are a basis, so you can just directly take these two vectors as the basis for R of A transpose and N of A transpose has dimension to 0.

So, if I want to find a vector which when multiplied by these two vectors, so that is if I do alpha times 1, 3, 5 plus beta times 2, 4, 6 and I set that equal to 0, this is only possible if alpha equals beta equals 0.

Student: Yes, sir fine, I got it.

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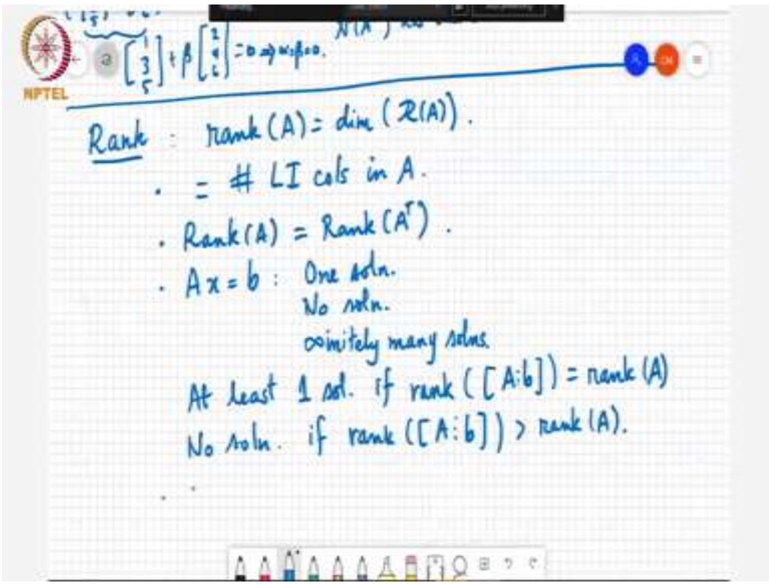

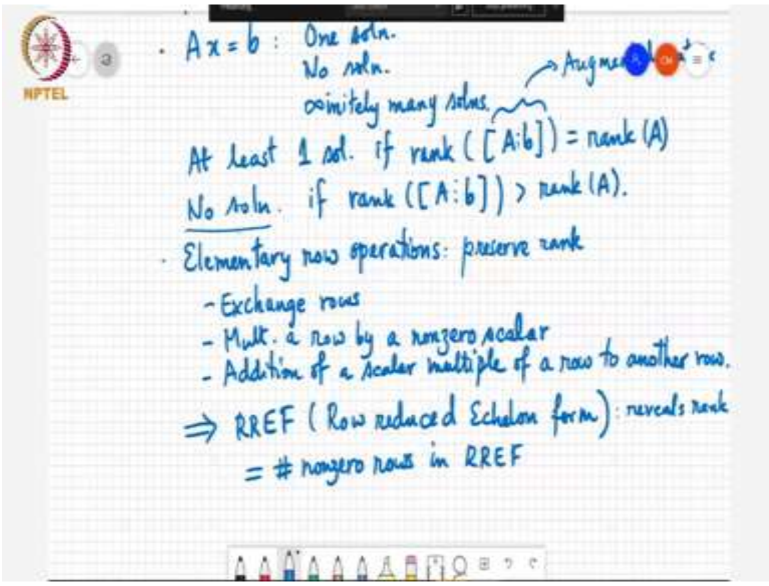

Professor: So, the next thing I want to talk about, there is not much time left but at least I can put down the definition. So, rank, so the rank of the matrix we already defined that it is the, the dimension of the range space of the matrix and is equal to the number of linearly independent columns in A and the remarkable fact that I mentioned earlier is that rank of A equals the rank of A transpose. This is what we often refer to as row rank equals the column rank.

So, related to the rank is the property that if you take the system of linear equations Ax equals b, then it can have either one solution or no solution or infinitely many solutions these are the only three possibilities. So, it can never have two solutions for example. And it has at least one solution if the rank of the augmented matrix equals the rank of A.

So, basically when this happens it means that b is in the column space of A because adding b is not changing the rank of A and so this is at least one solution. And it has no solution if rank of A, b is greater than rank of A. How do we find the rank of A? It is through these things called elementary row operations. So, I will not go through these in detail.

Again I am assuming that you have seen how to compute the rank of a matrix in your undergraduate program and so you know how to do these, but maybe in the homework I will give you an example, couple of matrices, where you can go over the motions of computing the rank by doing the row reduction and just refresh your memory on how it is done.

But these elementary row operations have the property that they preserve the rank, they do not change the rank, so that is the reason why (())(11:25) rank, then that tells you what the rank of the matrix is, the rank of the original matrix is because none of the operations you did on the matrix changed its rank. So, what are these elementary row operations? You can exchange rows; you can scale a row by a nonzero scalar.

If you multiply a row by 0 you may change the rank, so you are not allowed to multiply by 0, but you are allowed to multiply by any nonzero scalar, addition of a scalar multiple a scalar multiple of a row to another row, three elementary row operations which will result in what is known as RF or the row reduced echelon form.

Student: Sir?

Professors: Yeah!

Student: Yeah, so in the first statement rank of A, b equals to rank of A, it means like even though you are adding one more vector to the A making it, so your dimension is not getting changed ultimately, so it means that vector whatever b you have added, it is independent of either of the vectors in the A, is it mean that?

Professor: Say that again.

Student: So, when you are making A, b the augmented matrix you are adding the column matrix of b like you are adding the vector b to the vector A, like you are adding it…

Professor: Yes.

Student: So, that means when it is equal to rank of A, it means the dimension is not getting changed, so...

Professor: The dimension of the column space is not changing, yes.

Student: So that means the vector b is independent of like, it is a dependent on one of the vectors of A.

Professor: It is not linearly independent of the columns of A, if b was linearly independent of the columns of A then appending b will definitely increase the rank by 1 and so that is the no solution case.

Student: So that, is you are adding one more extra dimension after adding base, the second point.

Professor: Yes, so that means that because the rank increased it means that the point b cannot be reached by just taking linear combinations of columns of A and that is the reason why Ax equals b will never have a solution.

Student: Right, sir.

Professor: So, this row reduced echelon form, it reveals the rank. It is equal to the number of non-zero rows in the row reduced echelon form. So, we are out of time. In the next class we will discuss further about this rank and related properties and so that concludes what I wanted to say today. Are there any other questions?

Student: Sir?

Professors: Yeah!

Student: Sir I wanted to know, like you said matrix is a linear transform, so in any linear transform we are only interested in the range of the transform which is here the column space of A, so what additional information does the left null space of A gives that we are so interested in finding it or solving it?

Student: So, it is a good question. So, the point is that, so of the four subspaces the null space of a matrix is the orthogonal complement of the row space of the matrix and the column space of the matrix is the orthogonal complement of the left null space of the matrix. So, in some sense if you know what N of A is, you completely know what R of A transpose is.

And so knowing N of A there is no real additional information you are getting about R, by knowing R of A transpose, you know that they are the orthogonal complements of each other, but basically making these connections is the core of how you look at mathematics is you try to ask what are the relationships between these.

So, the way to think about it is that you start with a matrix and you can define these four subspaces and you ask are they related in some way and you find that these are the relationships between them and then you realize that if I know the null space of a matrix, then I already know exactly what the row space of the matrix is because it is just the orthogonal complement.

So, it is a long winded answer to your question, but it is just a connection between these subspaces.