Matrix Theory
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Lecture 58
Properties of Hermitian matrices

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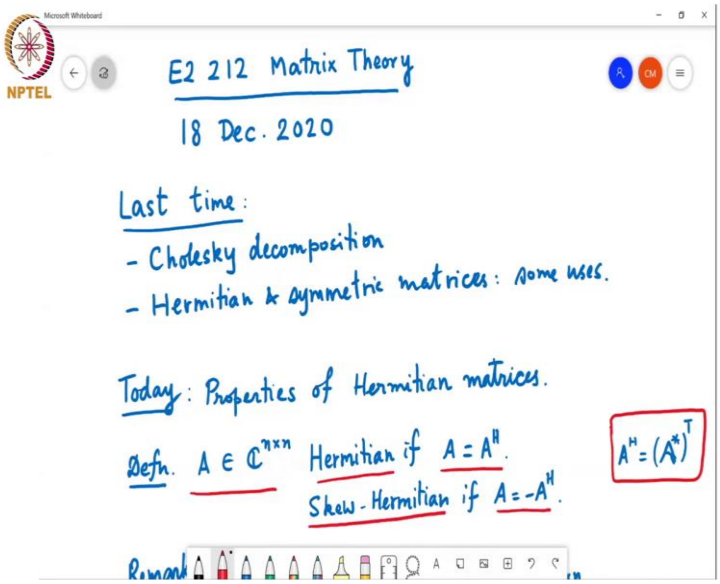

So, good afternoon. So, last time, we were looking at the Cholesky decomposition, and we closed out the discussion, summarize the chapter, and then started discussing about Hermitian and symmetric matrices. And I alluded 2-3 applications where Hermitian matrices are useful, or they arise in such applications. The first was in computing the Hessian of matrix, the second was in the quadratic form, and the third was in graph theory.

So, we will continue the discussion about Hermitian and symmetric matrices. This is a fairly long chapter in Horn and Johnson. So, we will go through that chapter in some detail. So, we begin with the basic definition that matrix H, matrix A of size n cross n is said to be Hermitian. If A equals A Hermitian, the Hermitian is nothing but the conjugate transpose of the matrix. We say that the matrix is Skew-Hermitian if A equals minus A Hermitian. So, that is the basic definition.

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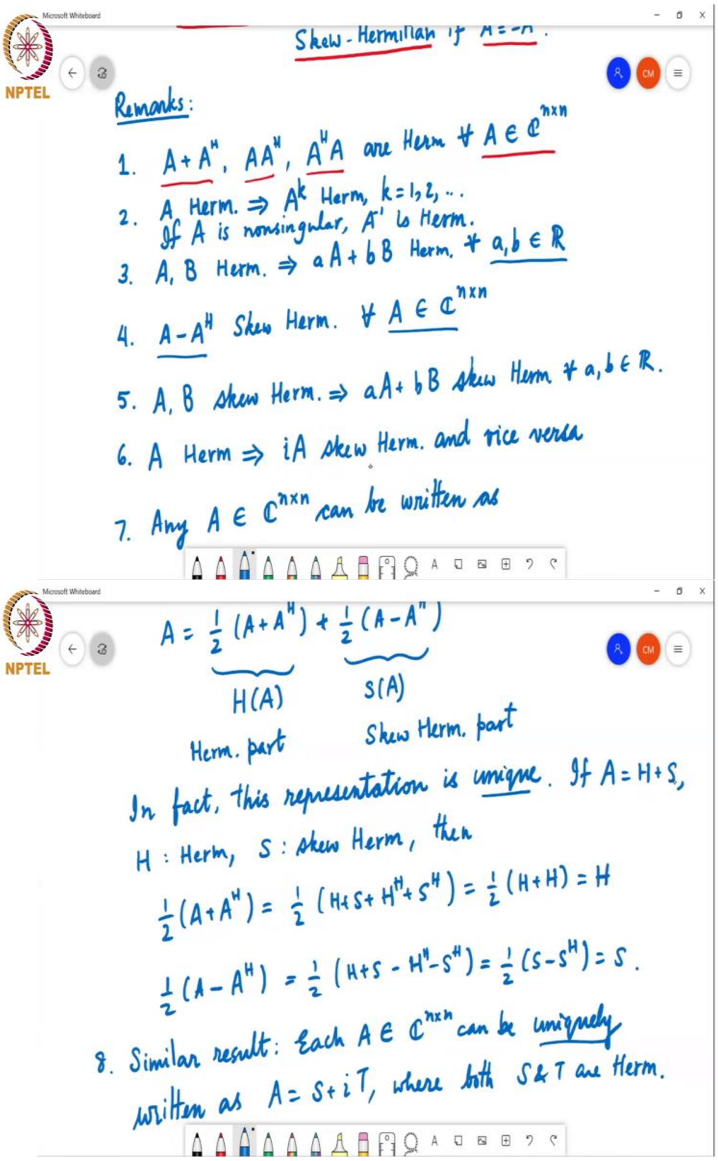

Now, some very immediate and obvious facts are like this. For any matrix A, if I consider the matrix A plus A Hermitian or AA Hermitian or A Hermitian A, they are all Hermitian matrices, just take the conjugate transpose of this you will get the matrix back. Now, if A is Hermitian, then A power K is A Hermitian matrix for any integer power. And in fact, if A is non-singular, A inverse is also Hermitian.

Then, if A and B are two Hermitian symmetric matrices, then their linear combination with real valued coefficients aA plus bB is also A Hermitian symmetric matrix. Of course, if I take complex valued coefficients here, then it may not remain Hermitian after the linear combination. And for any matrix A, if I consider the difference between A and A Hermitian, that is going to be Skew-Hermitian, because if I take the Hermitian of this, that becomes A Hermitian minus A which is minus of A minus A Hermitian.

These are very useful properties; we will see that in a second. And similarly, if you take two Skew-Hermitian matrices, then their linear combination with real valued coefficients is always skew Hermitian. And if a is A Hermitian matrix, then iA is a Skew-Hermitian matrix and vice versa, that is an A is a Skew-Hermitian matrix and iA will be a Hermitian matrix.

Any matrix A, whether its Hermitian or not can be written as the sum of two matrices, the first matrix being one half A plus A Hermitian, the other being one half A minus A Hermitian. So, if I expand this out, I get half A plus half A which is equal to A and half A Hermitian minus half A Hermitian, which goes to 0. So, this is equal to A and this first part here is H of A this is a Hermitian symmetric matrix, this second part I call it S of A, and that is a Skew-Hermitian matrix.

So, this is called the Hermitian part of A and this is called the Skew-Hermitian part of A and this representation is unique. In other words, if I write A to be H plus S, where H is a Hermitian matrix and S is a Skew-Hermitian matrix, then we have that half A plus A Hermitian equals H because A is H plus S plus H Hermitian plus S Hermitian and S equals minus S Hermitian. So, these two terms cancel and H equals H Hermitian. So, that becomes half H plus H, which is equal to H.

And similarly, if I take half A minus A Hermitian, then I will get S. So, there is a one-to-one correspondence between A and the two matrices H and S, I cannot write A in any other way as the sum of a Hermitian symmetric matrix in the Skew-Hermitian matrix. Similar result is that each A or any A in C to the n cross n can be uniquely written as A equal to S plus iT, where both S and T are Hermitian that is almost trivial from this because if S is, if I can write A as H plus S and S is Skew-Hermitian, I can write this S as i squared minus i squared times S. And i, that will be the same as i times iS, and iS is going to be Hermitian symmetric because S is Skew-Hermitian, that is something we just saw. So, that is exactly what is here.

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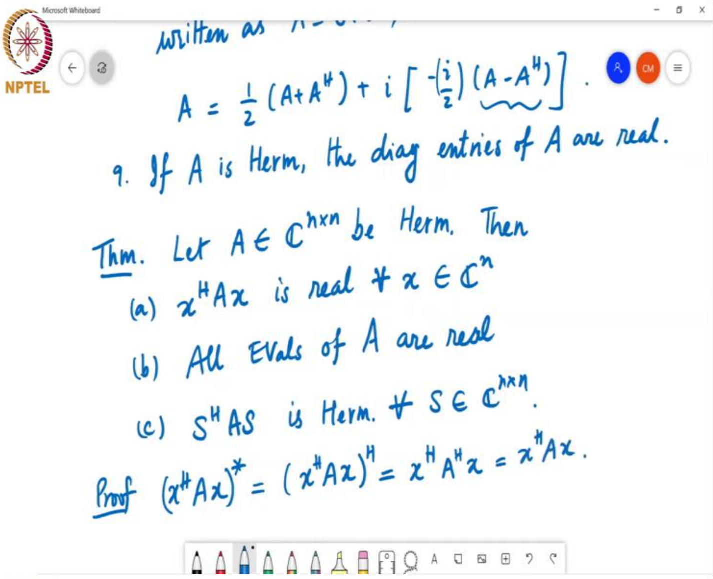

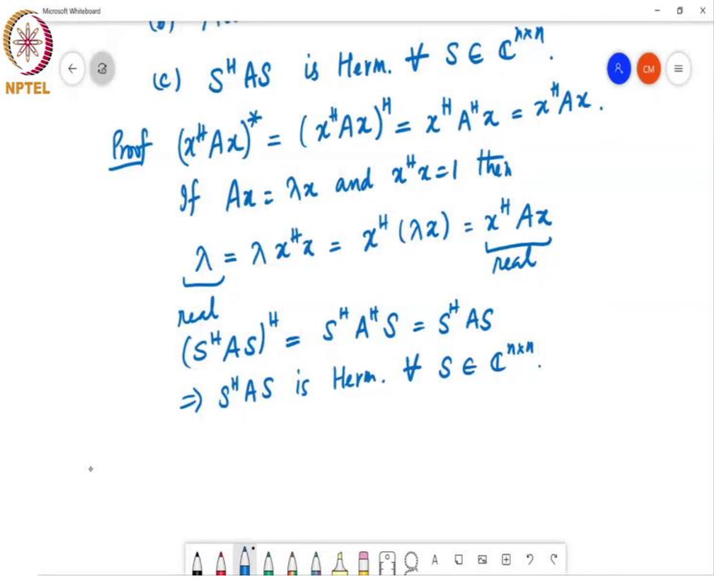

So, I can write A as half A plus Hermitian plus i times minus i over 2 times A minus A Hermitian, this is a Skew-Hermitian part of A with the factor half and then when I multiply that by i, I get a Hermitian symmetric matrix and then there is i times sitting here. So, this coefficient is square root of minus 1, but S and T may not be real valued matrices, but the point is both S and T are Hermitian symmetric matrices.

And then we have that, if A is Hermitian then the diagonal entries of A are real because when I take the conjugate transpose the diagonal entry stay away there, it says A equals A Hermitian the diagonal entries cannot be complex valued, because otherwise cannot have a non-zero imaginary part because otherwise the diagonal entries will not match. So, basically in a Hermitian symmetric matrix, all the entries above the diagonal are complex conjugates of the reflected entries below the diagonal.

And so, if A is Hermitian then we can fully characterize or we can fully specify this matrix A by using n real numbers for the n diagonals and n times n minus 1 over 2 complex numbers for all the off-diagonal entries. So, although A has n squared entries, it has A Hermitian symmetric matrix has n plus n real valued numbers and n into n minus 1 by 2 complex valued numbers that completely specify the matrix.

Now, so basically Hermitian symmetric matrices are too complex matrices as real numbers are too complex numbers. So, here is one result that kind of makes this point. So, let A be a Hermitian symmetric matrix, then so a, x Hermitian Ax is real for every x in C to the n. All eigenvalues are real. C S Hermitian AS is Hermitian for every S in C to the n cross n. These are pretty obvious facts I mean.

So, for example, if you want to show A, if I take x Hermitian Ax complex conjugate that is the same as taking its conjugate transpose because this is after all a scalar, x Hermitian Ax Hermitian and Hermitian of the product of vectors and matrices works exactly the same way as taking the transpose but with the extra conjugate in there. And so, this can be x Hermitian.

So, this is x Hermitian coming over on the other side, A Hermitian x this is x Hermitian, Hermitian which is the same as x coming on the right side. And since A equals A Hermitian this is equal to x Hermitian Ax. And similarly, if Ax equals lambda x, and I will take a unit norm eigenvector. So, x Hermitian x equals 1, then lambda which is equal to lambda times x Hermitian x, because x Hermitian x equals 1 is equal to I will write that as x Hermitian times lambda x, because lambda is a scalar and lambda x is the same as Ax.

So, that is equal to x Hermitian Ax. And this, we just showed that this is real valued. And since, this is real value, lambda is real value. So, all the eigenvalues of Hermitian symmetric matrix are real value. And finally, S Hermitian AS, if I take its Hermitian, then I get S Hermitian A Hermitian S, which is equal to S Hermitian AS. So, that means that S Hermitian AS is Hermitian for every S.

So, I guess these are pretty obvious facts, but they turn out to be very useful facts for later. But to take a Hermitian matrix and you conclude x Hermitian Ax that is always real value. So, for instance, normally, if I take a quantity like x Hermitian Ax, that would be complex value, and I cannot order those values. So, because you cannot order complex numbers. And so, if I had to say minimize something like x Hermitian Ax, that is a tough problem if A is an arbitrary matrix, but if I said A is a Hermitian symmetric matrix, x Hermitian Ax is always real valued.

So, that is a perfectly valid thing to try to minimize or maximize. And similarly, all the eigenvalues of A being real means that I can order the eigenvalues, I can ask which is the smallest eigenvalue, which is the largest eigenvalue and so on. And we will see that S Hermitian AS is being Hermitian symmetric also will be very useful for us later. The thing that you should think about now is whether the converse is true.

So, suppose x Hermitian Ax, you computed x Hermitian Ax and found that it is real for all x and C to the n. Does it mean that A must be a Hermitian symmetric matrix? Likewise, suppose you found all the eigenvalues of A and you found that they are all real valued. Does it mean that is that Hermitian symmetric matrix, and so on. But we will come to that in a few minutes.

So, when, if I consider, so, I mentioned that this is a theorem that illustrates somehow that Hermitian symmetric matrices are too complex valued, complex valued matrices as real numbers are too complex numbers. So, how is that it is because suppose I take n equals 1. So, what is this property saying it is saying that, so n equals 1 means A is just some Hermitian symmetric complex number.

And if that is the case, it means that A must be a real valued number because its conjugate transpose, which is its complex conjugate is equal to A itself. So, it means A must be a real valued number. And if A is a real valued number, x Hermitian Ax is going to be this real valued number A times mod x squared. And that is real for every x, for every complex valued C, complex valued x.

And if it is a 1 cross 1 matrix, whatever that A is, that is an eigenvalue of the matrix. And so, it just means that A is real. And this being Hermitian is the same as saying mod x squared times A is real for every S being a complex number. So, I asked about the converse of these points here is the result about that.

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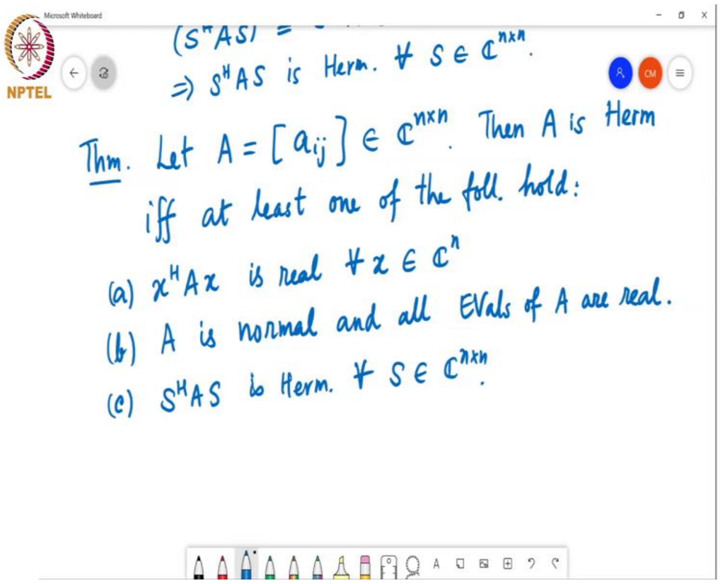

So, I will write the aij being its entries in this form. Because we will need these entries to prove the result. Then A is Hermitian if and only if at least one of the following holds. A is normal and all eigenvalues of A are real. c, S Hermitian AS is Hermitian for every S in C to the n cross n. So, that is not a normal matrix. And remember that all Hermitian matrices are also normal because if a matrix satisfies AA Hermitian equal A Hermitian A, then it is a normal matrix, but for Hermitian symmetric matrix A Hermitian equals A.

So, AA Hermitian equals A square which is equal to A Hermitian A. So, all properties of normal matrices for example, that eigenvectors corresponding to distinct eigenvalues are orthogonal that there is a complete set of eigenvectors that the matrix is uniquely diagonalizable. All of them hold for Hermitian symmetric matrices also. And we will use these properties extensively in the coming results, in the results that we are going to discuss.

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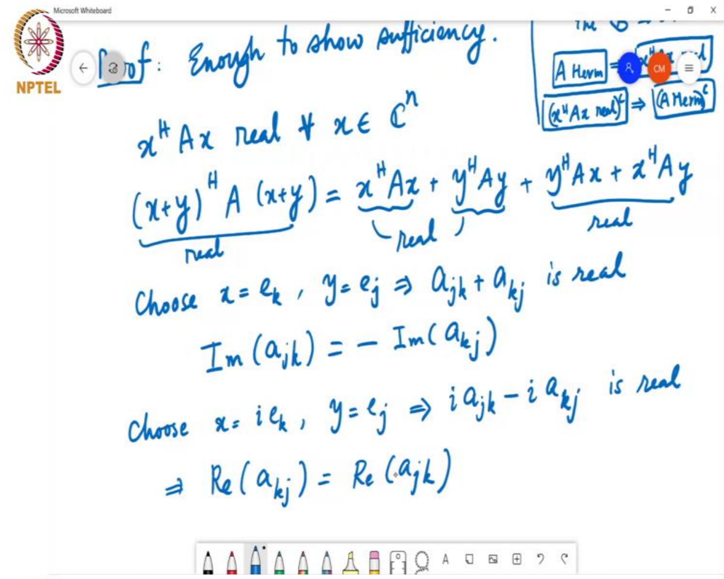

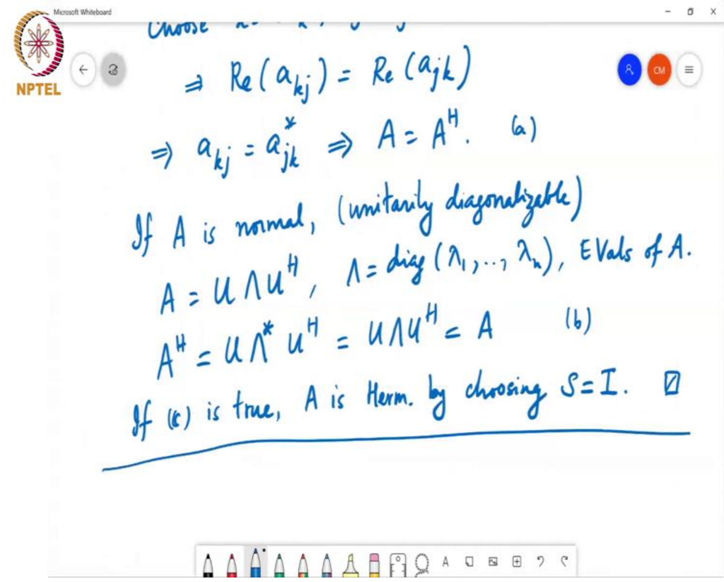

So, now proof. So, when I say that A is normal, I am already assuming a lot about A. But in addition to its being normal if the fact that if it is true that all the eigenvalues of A are real, then it is Hermitian symmetric. So, Hermitian symmetric matrices are one special case of normal matrices. And if I assume that the eigenvalues are real, then this matrix A is Hermitian symmetric.

So, let us see how to show this result. Now, the necessity is what we showed in the previous result. So, it is now enough to show sufficiency. So, in other words, so, let me actually maybe explain this, this is basic logic. So, the statement of the theorem says that A Hermitian if and only if one of these conditions hold, which means that we need to show that A Hermitian if this condition holds.

And then we call that the necessary condition. And we also need to show that A Hermitian only if this condition holds, which is that this condition is sufficient for A to be Hermitian. We have already showed that if A is Hermitian then this is true, that is the sufficiency result. Because again, basic logic is that if A, statement A, so statement A implies a statement B, then the complement of statement B implies that is not B implies not A.

And so, what we showed is that A being Hermitian implies x Hermitian Ax is real. Which means that, if not B is that x Hermitian Ax real for all x, this complement which means that there is some x for which x Hermitian Ax is not real implies not A, which is. So, that is, so, the only if part is already shown by the previous result. So, we need to, it suffices to show sufficiency.

Namely, that if this condition holds then A is Hermitian, if this condition holds then A is Hermitian, if this condition holds then A is Hermitian. So, if x Hermitian Ax is real for every x in C to the n then if I consider x plus y Hermitian A x plus y. Now, this is also a real value because of this condition here this if I expand it out, I get x Hermitian Ax plus y Hermitian Ay plus y Hermitian Ax plus x Hermitian Ay.

Now, x Hermitian Ax is real because again, because of this assumption again and y Hermitian Ay is also real. So, now, this is real and this is real and so this must be real also because, if this was a complex number there is no way that this equality would be satisfied. So, we know now that this quantity like y Hermitian Ax plus x Hermitian Ay is always real valued regardless of which x and y I choose.

So, I can choose some in, some I can cleverly choose x and y and see what happens. So, if I choose x equal to ek, ek'th column of the n cross n identity matrix and y equal to ej, then what, y Hermitian Ax will do is that x Ax will pick out the k'th column of A and y Hermitian times Ax will pick out the j'th entry of the k'th column. So, j'th row and k'th column.

So, that means that ajk plus similarly, this will pick out akj, this is real. So, if this is real that means that if I consider the imaginary part of ajk, this is the negative of the imaginary part of akj, the imaginary parts must cancel otherwise this would not be real. Next choose x equal to i times ek and y equal to ej then what we have is that if I consider y Hermitian Ax that will give me i times ajk and x Hermitian Ay, this i will become A minus i, when I take the Hermitian here.

So, I get minus i akj and this is again real. Now, if this is real, then it means that the real parts of these two must cancel each other and there is a negative sign. So, this means that the real part of akj equals the real part of ajk. So, if you look at, if you now look at ajk and akj they are real parts are equal and the imaginary parts are the negative of each other. So, that means that akj is equal to ajk complex conjugate and this is exactly the same as saying A equals A Hermitian.

So, that proofs part a, (())(15:18). Now, if A is normal then it is unitarily diagonalizable, you have seen that result already all normal matrices are even unitarily diagonalizable. Which means that I can write A as U lambda U Hermitian, where lambda is a diagonal matrix containing lambda 1 to lambda n along the diagonals, which are the eigenvalues of A and U is a unitary matrix.

So, now, in general, basically A Hermitian would be equal to U lambda, so, this is the Hermitian of this and then lambda Hermitian but I can write it as lambda star because lambda is a diagonal matrix times U Hermitian. But if we say that the eigenvalues are all real value lambda star equals lambda. And so, this is equal to U lambda U Hermitian but that is equal to A from here itself.

So, A is Hermitian symmetric. So, this shows b and similarly, if it is true that S Hermitian AS is Hermitian for all S in C to the n cross n, then if c is true, A is Hermitian by choosing what, is the identity matrix. So, if I just substitute, I mean this is A Hermitian, A is a Hermitian for every S, I do not. So, it suffices to say that, when S equals I, then I get A must be equal to A Hermitian. So, that is it. So, that is this prove.

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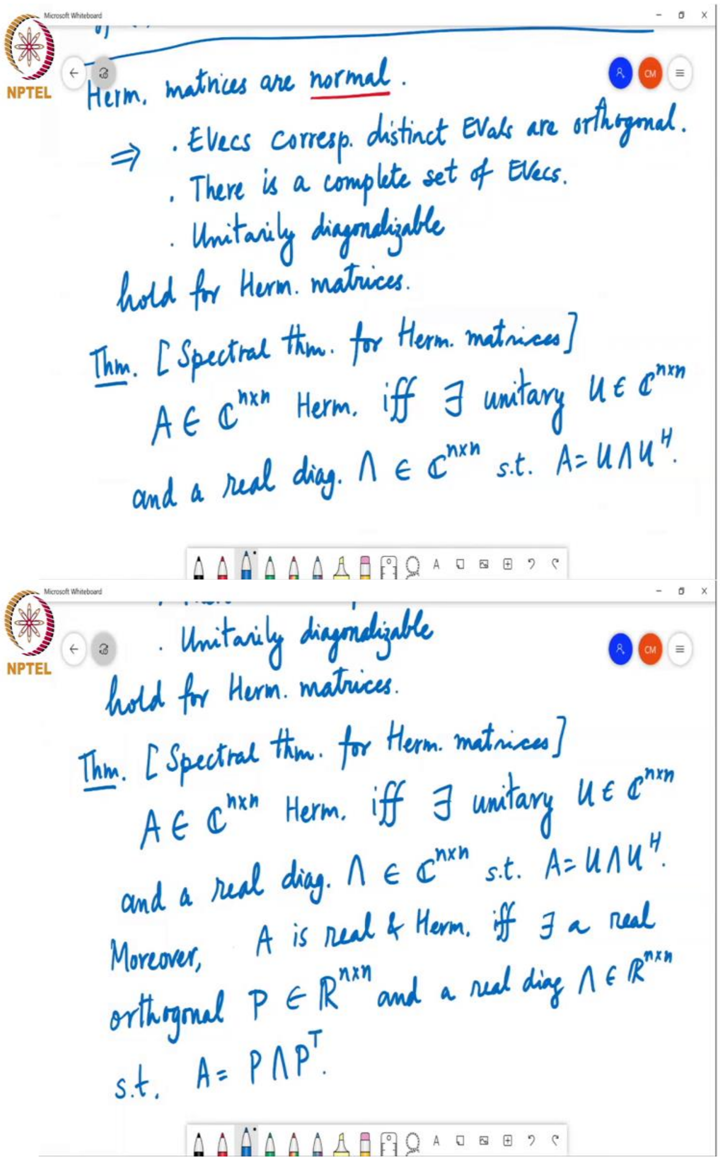

This the point I made about Hermitian matrices being normal, this is an important point. So, I want to write it here, Hermitian matrices are normal. So, that means that all properties of normal matrices apply to Hermitian symmetric matrices. So, for example, eigenvectors corresponding to distinct eigenvalues are orthogonal. And there is a complete set of eigenvectors. That is a geometric multiplicity of every eigenvalue equals its algebraic multiplicity.

So, Hermitian matrix can never be defective it is unitary diagonalizable. So, all these hold for Hermitian matrices. So, one result we have seen earlier was the spectral theorem for Hermitian matrices, which was a specialization of the result for normal matrices. So, the result says that A C to the n cross n is Hermitian if and only if there exists unitary u and a real diagonal lambda.

So, I could write this as real to the C n cross n such that A equals u lambda u Hermitian. So, moreover if A is real and Hermitian but if it is real value, it just means it is symmetric. So, if it is real and symmetric, put it this way, A is real and symmetric if and only if there exists a real orthogonal matrix P. Again, I could write it as real. So, actually I will write it as and a real diagonal lambda such that A equals P lambda P transpose.