

Matrix Theory
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Lecture 55

Solving pivoted system and LDM decomposition

(Refer Slide Time: 00:14)

The image shows two screenshots of a Microsoft Whiteboard. The top screenshot is titled "Solving the pivoted system:" and contains the following handwritten notes:

$$Ax = b$$

$$P^T L U \Pi^T x = b.$$

Set $y = U \Pi^T x$
 $z = \Pi^T x$

Then solve:
 $Ly = Pb$
 $Uz = y$

On the right side, the following equations are written:

$$LU = PA\Pi$$

$$PP^T = P^T P = I$$

$$\Rightarrow P^T L U \Pi^T = A$$

The bottom screenshot shows the same content as the top one, but with an additional equation at the bottom:

$$x = \Pi z$$

So, then the one small point is, so in order to solve the pivoted system. So, recall that goal was to solve $Ax = b$ in some computationally efficient manner. So, what we will do is, so if we have $LU = PA\Pi$ then since $PP^T = P^T P = I$, this is a property of permutation matrices, is the identity matrix that a similar thing holds for Π . That means that if I pre multiply by P^T , sorry $P^T L U \Pi^T$ is equal to this matrix A .

And so, I will use that here, $P^T L u P^T x = b$. And so, what we will do is, we will set $y = u P^T x$, and $z = P^T x$, then we solve $Ly = Pb$. So, I am multiplying this by P , then we saw $uz = y$. And finally, we compute $x = Pz$.

So, basically if rows of A are pivoted, then the corresponding entries of b must be exchanged. So, that is what Pb does. If columns of A are pivoted, then the corresponding elements of x have to be exchanged. That is what $P^T z$ does. So, this is how you solve the pivoted system. Now, the last point that we have not discussed is, what happens if A is rank deficient.

(Refer Slide Time: 02:45)

What if A is rank deficient?
Pivot elem $\rightarrow 0$.
If $\text{rank}(A) = r < n$,
At the beginning of the $(r+1)^{\text{th}}$ step, will have
 $A^{(r)}(r+1:n, r+1:n) = 0$
 $P_k = \pi_k = M_k = I$ for $k = r+1, \dots, n$
 $\Rightarrow$ Can terminate GE after r steps.
Get $PA^T = \begin{bmatrix} L_{11}^{(r \times r)} & 0 \\ L_{21} & I_{n-r} \end{bmatrix} \begin{bmatrix} U_{11}^{(r \times r)} & U_{12} \\ 0 & 0 \end{bmatrix}$
Dimensions: $(n-r) \times r$ for L_{21} , $(n-r) \times r$ for the bottom-left zero block, and $(n-r) \times (n-r)$ for the bottom-right zero block.

So, the point is that the pivot elements are guaranteed to be non-zero only if A is not rank deficient. If A is rank deficient, then the pivot element will go to 0 at some point. And so, if rank of A equals R , which is less than n , then at the beginning of the r plus 1th step, will have A of rows r plus 1th through n , and columns r plus 1 through, n will become the all 0 matrix.

And so, then, we can simply set $P_k = \pi_k = M_k = I$ for k equal to r plus 1 all the way up to n . So, once we see that this sub-matrix becomes the all 0 matrix, we do not have to do any more Gaussian Elimination steps. So, can terminate Gaussian Elimination after r steps, and we will get PA^T equals matrix that is a LU , but it is L_{11} which is going to be r cross r , 0, L_{21} which is of size n minus r cross r , and then I_{n-r} minus r times U_{11} which is again of size r cross r , U_{12} , 0 which is of size n minus r cross r and then 0 which is a size n minus r cross n minus r .

So, this is the kind of form you will get. So, in fact, this can be used to find the rank of a matrix. But in practice, you will need to have a threshold on the entries of this matrix to decide when you are going to call it the all 0 matrix. So, this is the, I think these things will become very easy to, I mean very clear to you if you were to try to do these Gaussian Elimination steps using rank deficient matrix. So, pick your favourite 3 cross 3 or 4 cross 4 matrix, make it rank deficient.

So, for example, write down three arbitrary columns and then write a fourth column which is the sum of those three columns or something. Now, it becomes a rank deficient matrix and then try to execute the Lu decomposition on it with pivoting and you will see the, see that they will be after r steps, this submatrix will go to 0 and so you can stop the Gaussian Elimination there.

So, what we just discussed about pivoting is called the full pivoting algorithm, it gives you maximal numerical stability because at each stage it is pulling the biggest magnitude entry in A_{kk} of r minus 1, sorry, A_{22} of k minus 1 and putting that to be the top left entry. Now, but then that is also maximally numerically expensive to compute because you have to compare at least. So, A_{kk} , sorry A_{22} of k minus 1 has n minus k plus 1 is of size n minus k plus 1 cross n minus k plus 1 so that many entries need to be compared in magnitude.

And you need to find the biggest entry, find its location, and then do a row and column exchange. So, as far as numerical computation is concerned, this increases the computational complexity of the Gaussian Elimination. However, I mean so people also sometimes do partial pivoting where for example, you may only perform row operations. In that case, you will have the T matrices as discussed above, but you will not have the P matrices, you will not do the column exchanges, this can reduce the number of comparisons required. And of course, there are cases where pivoting is not required.

So, for example, in what are called diagonally dominant matrices, matrix is diagonally dominant if the magnitude of the diagonal entry is bigger than the sum of the magnitudes of all the off-diagonal entries in the same row or the same column. Then such matrices are called diagonally dominant matrices. And for such matrices, pivoting or row and column exchanges are not required.

Or in other words, even if you did that, it would just return P_k equals the identity matrix and P_{k+1} equals the identity matrix each time, so you do not have to do those comparisons. So, in some cases, pivoting is not required. So, this is kind of an area in itself on how to efficiently

compute the Lu decomposition, there is a lot one can say. But for the purposes of this course, I think this is for you to get an idea of how it is in work, how it is done, and what it is, is sufficient. There are a few other useful modifications of this Lu decomposition that one often considers, so I will allude I mean highlight them a bit here.

(Refer Slide Time: 09:58)

Modifications to the LU decomposition

LDM (or LDM^T) decomposition
 If no zero pivots are encountered in GE
 Then $\exists$ unit Δ L & M and diagonal D s.t.
 $A = LDM^T$.

Since $A = LU \exists$. Let $U = DM^T$, where $d_{ii} = u_{ii}$
 (Each row of $M^T = (\text{row of } U) / u_{ii}$).

$Ax = b \Rightarrow LDM^T x = b$

1. Let $y = DM^T x$, solve $Ly = b$
2. Let $z = M^T x$, solve $Dz = y$

2. Let $z = M^T x$, solve $Dz = y$

3. Solve $M^T x = z$, obtain x .

2. **LDL^T decomp. for a symmetric matrix.**
Result: If $A \in \mathbb{R}^{n \times n}$ is symmetric & nonsingular,
 then $L = M$.

$A = LDM^T \Rightarrow \underbrace{M^{-1} A M^{-T}}_{\text{Symmetric}} = \underbrace{M^{-1} L D}_{\Delta} \Rightarrow \text{Diagonal.}$

D diag, nonsing
 $M^{-1} L D$ is diagonal $\Rightarrow M^{-1} L$ diagonal
 But M (hence M^{-1}) and L are unit Δ

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Result: If $A \in \mathbb{R}^{n \times n}$ is symmetric & nonsingular, then $L = M$.

$$A = L D M^T \Rightarrow \underbrace{M^{-1} A M^{-T}}_{\text{Symmetric}} = \underbrace{M^{-1} L D}_{\Delta} \Rightarrow \text{Diagonal.}$$

D diag, nonsing
 $M^{-1} L D$ is diagonal $\Rightarrow M^{-1} L$ diagonal
 But M (hence M^{-1}) and L are unit Δ
 $\Rightarrow M^{-1} L$ is unit Δ
 The only unit Δ that is also diag is I
 $\Rightarrow M^{-1} L = I$ or $M = L$. $\square$

Modifications to the Lu decomposition. So, the first is what is called the LDM decomposition. It is also often called the LDM transpose decomposition. This, can be done if there are no 0 pivots that are encountered in the Gaussian Elimination process. Because if you encounter a zero pivot, it does not mean you can stop there, you can stop there only if this entire matrix A_r of r plus 1 through n and r plus 1 through n , that sub-matrix of A_r is all zeros.

But if the pivot element is 0, then you have to do some row and column exchanges to bring a non-zero entry to the r plus 1 comma r plus 1 entry and then you can proceed with the Gaussian Elimination. But if no-zero pivots are encountered in the Gaussian Elimination process, then there is a unit lower triangular L and M , both are unit lower triangular. That is why we write it like this as M transpose and diagonal D such that A is equal to LDM transpose.

So, essentially, what we are doing here is, since A is equal to Lu , what we can do is this, this exists, so we let u is equal to D times M transpose where d_{ii} equals u_{ii} . What this is doing is, each row of M transpose is equal to the corresponding row of u divided by this element u_{ii} . So, it is just extracting the diagonal entries and pulling them out, and writing it is D times M transpose.

So, this gives you a decomposition of this form, where L and M are both unit lower triangular. So, then, if you were to again if you had to solve a Ax equals b , then you have LDM transpose x equals b . So, what we will do is, we will let y be this part here, DM transpose x and solve Ly equals b , then let z equals this part here, M transpose x and solve Dz equals y .

This is a diagonal matrix, so this is easy to do. And third step is to solve $M^T x = z$ and obtain x . So, and in particular, if the matrix is symmetric, then you can actually have a decomposition of the form LDL transpose decomposition for symmetric matrix. So, the idea is that if $A \in \mathbb{R}^{n \times n}$ is symmetric and non-singular, then L equals M . So, LDM transpose becomes LDL transpose.

So, this is not hard to see. So, since $A = LDM^T$ that means that $M^{-1} A M^{-T} = L D$. I just pre-multiplied by M^{-1} and post multiplied by M^{-T} and these are unit lower triangular matrices. So, their inverses always exist. So, this is $M^{-1} L D$. Now, M is lower triangular, so M^{-1} is also unit lower triangular.

So, $M^{-1} L D$, D is diagonal. And so, this whole thing is still lower triangular. And A is symmetric, so $M^{-1} A M^{-T}$ is symmetric. Now, the only way a symmetric matrix equals a lower triangular matrix is if it is a diagonal matrix. So, $M^{-1} A M^{-T}$ is diagonal and if you call it some D' then you can take M times that, and then M^T multiply on the right and you see that A is equal to LDL transpose, (())(16:50) that form.

So, basically D is diagonal and non-singular. And so, we just showed that $M^{-1} L D$ is diagonal which implies $M^{-1} L$ is also diagonal. But M , hence M^{-1} and L are unit lower triangular, which implies that $M^{-1} L$ is unit lower triangular. But if it is lower triangular, the only unit lower triangular matrix, unit lower triangular matrix that is also diagonal is the identity matrix. So, that means that $M^{-1} L$ equals the identity or M equals L .

So, basically the decomposition is of the form LDL transpose. We stop here for today. And the next class I will continue with another closely related decomposition called Cholesky decomposition and that is applicable to positive definite matrices, positive definite Hermitian matrices, and that will lead our way into a detailed discussion of Hermitian symmetric matrices and positive definite matrices, which will occupy several weeks of further lectures.