Matrix Theory
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Lecture 51
Polynomials and Matrices

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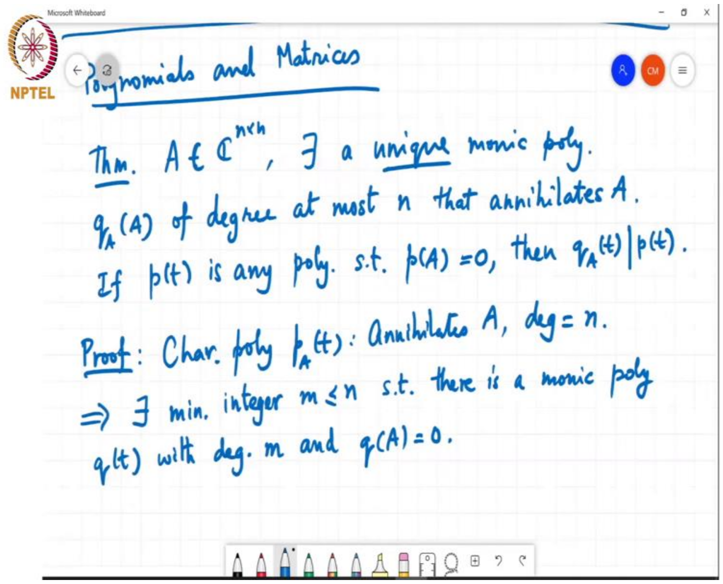

The next thing I want to talk about that is polynomials and matrices. So, basically given a polynomial p of t I can always compute or define something called a where all the t's are replaced by A and the constant term is replace by that constant term times directly matrix. And we have already seen that there is a there is a strong connection between polynomials and matrices for example the Cayley Hamilton theorem which says that for every matrix A there is a characteristic polynomial which annihilates A that is P of A equal to 0.

But there could be polynomials of lower degrees degree n minus 1 or n minus 2 etc. That also annihilates a. So, one related question you can ask is what is the minimal degree polynomial that annihilates A. Now, for any polynomial that annihilates A you can always scale that polynomial and make the leading coefficient that is the coefficient of the highest power of that polynomial or in that polynomial equal to 1. And such polynomials are called Monic polynomials.

Of course, note that since they the highest power has the leading coefficient equal to 1 a monic polynomial cannot be an identically 0 polynomial. Of course, if you take the identically 0 polynomial that annihilates any matrix but that will saying much. So, we are not interested in that polynomial. So, we want to find the lowest degree monic polynomial that annihilates A. So, that is given by the following theorem.

So, given matrix A in C to the n cross n there exists a unique monic polynomial qA of A of degree at most n that annihilates A. That is qA of A is equal to 0. So, for the given such a polynomial if p of t is any polynomial that such that p of A equals 0 then qA of t divides p of t. So, that is the theorem.

So, let us see how to show this. So, qA of A is a polynomial annihilates A and so suppose pA of t is a so the characteristics polynomial pA of t is a it annihilates A this is a Cayley Hamilton theorem. And it has a degree equal to n. So, that means that the there exists a minimum integer m such that there is monic polynomial q of t with degree m and q of A equals to 0.

So, we have already found something that is of degree n so the minimum degree polynomial that annihilates A call that q, q of t and this has some some degree m which is going to be at most equal to n and it satisfies the condition that it annihilates A that is q of A equals to 0. So, that basically degree of q of t is less than or equal to the degree of p of t.

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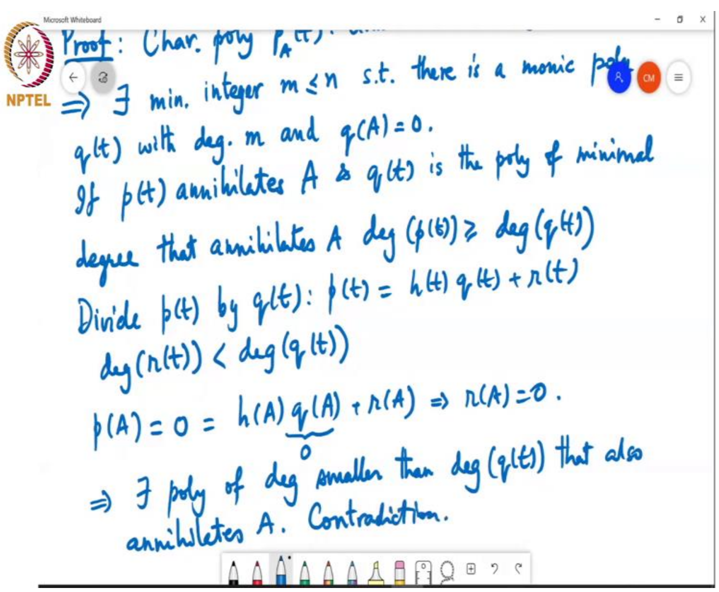

Professor: Now, pA of t, now suppose p of t

Student: Sir.

Professor: Yes.

Student: Sir, here (())(05:25)

Professor: Different, so annihilates A and say q of t now q of t we have defined it to be the minimal the polynomial or the minimal degree that annihilates A then certainly it must be true that the degree of p of t is greater than or equal to degree of q of t. Because, q of t is after all the polynomial of minimal degree that annihilates A.

So now, what we will do is we divide p of t by q of t you can divide polynomials you have seen this in your high school the algorithm called as Euclid's algorithm you can use and what this does when you do this division is that you can you will end up with p of t is equal to some h to t times q of t plus some r of t. So, this is the quotient polynomial this is the remainder polynomial. And by since this is the remainder polynomial only property that needs to satisfy is that degree of r of t is strictly less than the degree of q of t .

Because, if there is some extra power less than r of t than you could take out one more factor into this h of t. So, now since p of A is another polynomial that annihilates A so p of A equals 0 and this is equal to h of A times q of A plus r of A and q of A is already equals to 0 and so that implies r of A equals 0. So, what we have done is we now found that polynomial of smaller degree than q of t that annihilates A. So, this is a contradiction. Because, we started out by assuming that a the q of t is the minimal degree polynomial that annihilates A.

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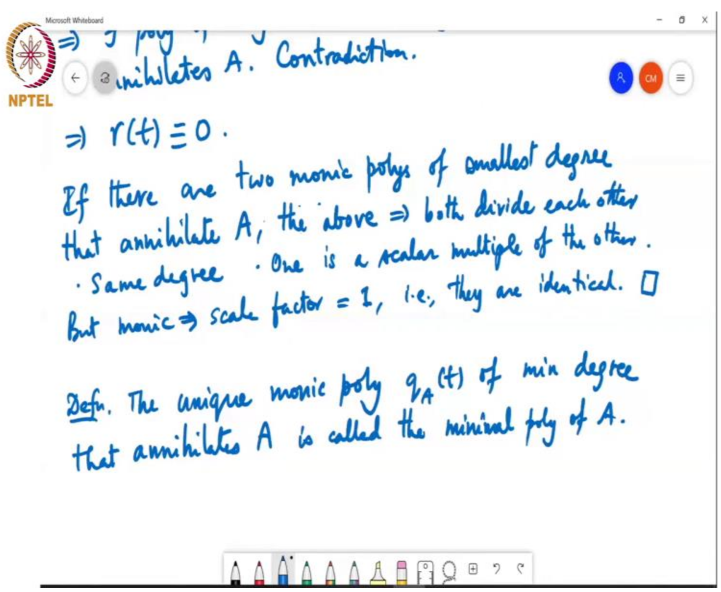

So, that means that r of t must be must be the all 0 or just the 0 polynomial. In other words it means that q of t divides p of t for if p of t any other polynomial that annihilates t. Now, if there are 2 monic polynomials of the smallest degree we have to show the uniqueness so if there are 2 monic polynomials of smallest degree that annihilates A then this same argument here implies that both these polynomial divide each other.

And if they divide each other then it means that they it means that they have the same degree and it also means that one is a scalar multiple of the other. But they are monic that means that their leading coefficient is equal to 1. So, if there are there are just a scalar multiple of each other and the leading coefficient is equal to 1 and both of them then scale factor is just equal to 1. And so, they are identical. So, this establishes the uniqueness.

So, basically the so that that is basically the definition. So, the unique monic polynomial qA of t of minimum degree that annihilates A is called the minimal polynomial of A. So basically, these minimal polynomials have lots of very nice properties I will just talk about 1 here but I think you can look at it in textbook for many more interesting results around minimal polynomials. 11.45

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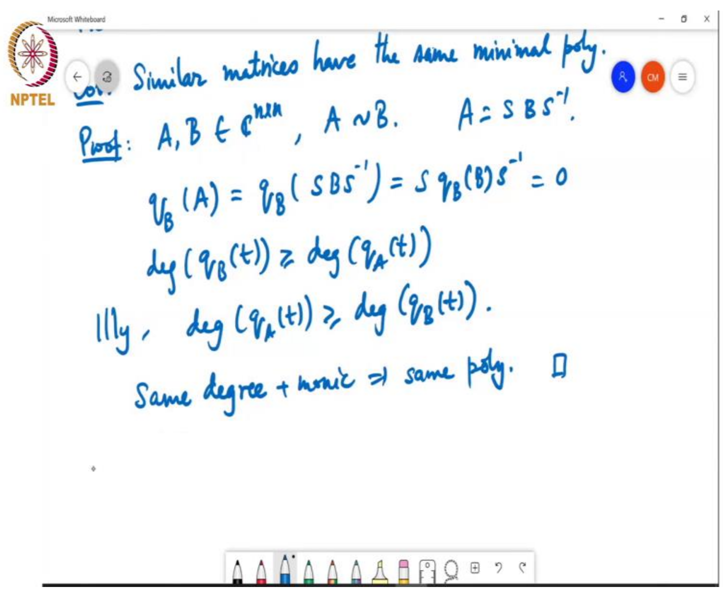

So, here is the corollary. Similar matrices have the same minimal polynomial. So, this is very easy to see so A and B are again n cross n matrices and A is similar to B. That means I can write A as S B S inverse by using some invertible matrix S so if I take q if I compute qB of A so qB of A, qB of t is b minimal polynomial of the matrix b and if I compute qB of A that is equal to qB of S B S inverse and this is equal to S qB of B times S inverse and qB of B equals to 0. Because, qB is a polynomial of minimal degree that annihilates this so this is equal to 0. So, that implies that so, qB of A is 0 that means the degree of qB of t is greater than or equal to the degree of qA of t.

Because qA of t is a polynomial of minimal degree that annihilates A and qB of t some other polynomial that all annihilates A. So, degree of qB of t must be at least equal to the degree of qA of t. Now, there is nothing special about A or B in this argument so you can simply repeat the arguments exchanging A and B and similarly, say that degree of qA of t is greater than or equal to the degree of qB of t. And so that means that they have the same degree and adding the fact that they are monic implies that they have the same polynomial. That is the same argument we made about uniqueness of the previous proof.

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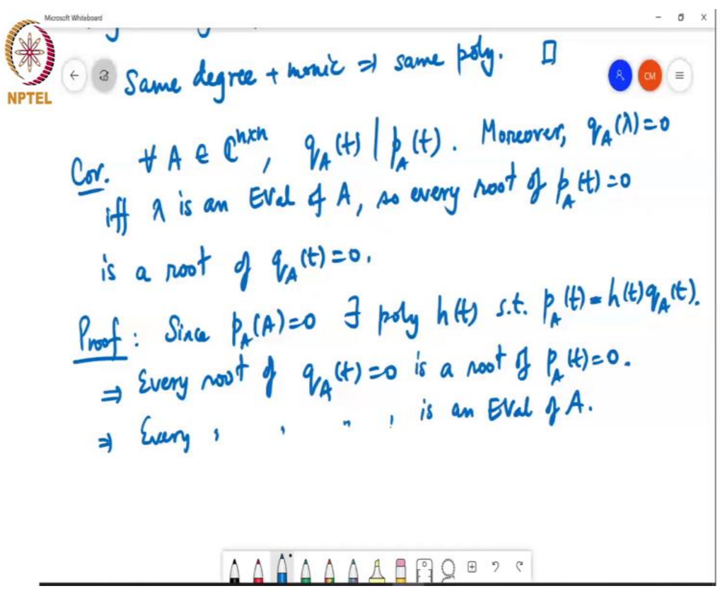

So, here is actually one more corollary. That says that for every A and C to the n cross n qA of t divides pA of t pA of t is a characteristic polynomial this of A and qA of t is a minimal polynomial. Moreover, qA of lambda if and only if lambda is a Eigen value of A. So, that means that every root of so, every root of pA of t equals 0 is also a root of qA of t equals 0. So, there are not any additional 0's in pA of t which are not roots of qA of t equals to 0.

So, since pA of A equals 0 there exists the polynomial h of t such that pA of t equals h of t times qA of t. This is just because of the previous theorem that we saw. If pA of t is some other polynomial that annihilates A then qA of t divides that other polynomial which is pA of t in this case.

So, that means that the so because it is like this every root of qA of t equals 0 is root of pA of t equals 0. Obviously, because if I say qA of lambda as equal to 0 then I have that pA of lambda as h of lambda times qA of lambda and qA of lambda equals 0 so, pA of lambda is also equals to 0. So, that means that every root of qA of t equals 0 is am Eigen value of A is an Eigen value of A. Because, pA of that root is equal to 0.

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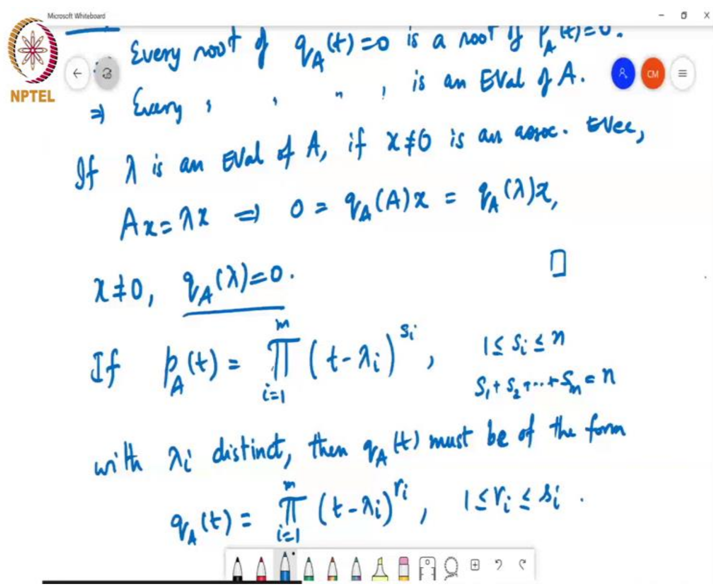

Now, then in and in if x is not equal to 0 is an associated Eigen vector then from Ax equals lambda x we have that 0 which is equal to qA of A times x is also equal to qA of lambda times x. Because, Ax equals to lambda. So, particular (())(18:33). And if x is not equal to 0 its associated Eigen vector x equals lambda x implies that qA of A times x which is equal to qA of lambda times x is equal to qA of lambda times x and since x is not equal to 0 this must mean that for qA times lambda of x to be equal to 0 we must have that qA of lambda equals to 0.

And therefore, it is through that if lambda is Eigen value of A then qA of lambda equals to 0 which proves the result. And so basically, the what this implies is that if pA of t if I write it out in its product form, i equal to 1 to t minus lambda i power say Si we have 1 less than or equal to Si less than or equal to n and S1 plus S2 plus etc plus Sm equals n.

Then with lambda i being distinct then qA of t must have the so that is to say that for every factor that appears in the characteristic polynomial the product form of the characteristic polynomial there should be a corresponding factor in the minimal polynomial and this ri here is atleast equal to 1 but it may not be equal to Si all the way.

So, the summation of ri could potentially be less than n but you will have some factors like this. So, this is one way to even to actually search for minimal polynomials that is you take ri equal to 1 to all the way up to Si for each i and then see what is the minimal degree of polynomial that you can generate out of this for which qA of A equals 0.

Of course, for very large dimensional systems this is difficult to conclude and also not that if si equals to 1 for that is any of that particular Eigen value that is appearing with multiplicity 1 that same factor was certainly appearing in the minimal polynomial. Only if in the case where there is a factor here that appears with power 2 or more then it is possible for ri to be less than si.

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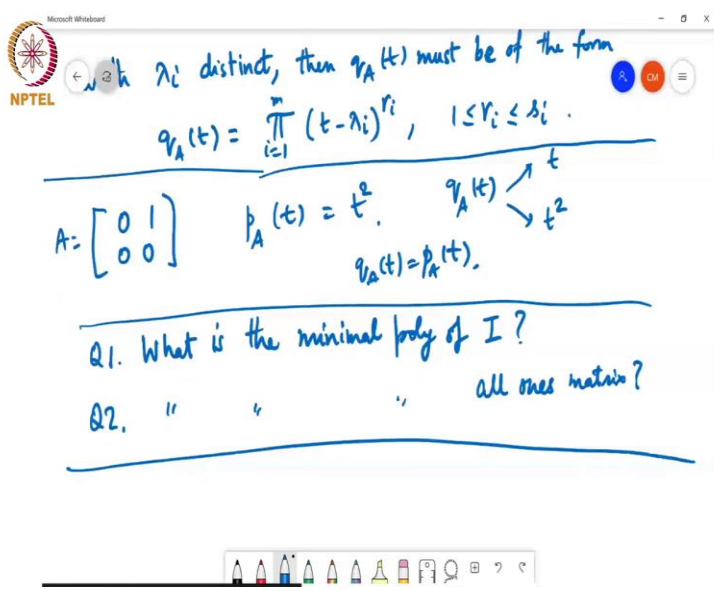

Professor: So, here is something here for you to think about they will not ask you because you would not answer, so, first question is

Student: Yes? Only if ri equals to 1 we will get the minimal polynomial?

Professor: It may not be so let me no it is not true so it is possible that you do have to take higher powers of ri in order to annihilate A. So, for example that means if I can give you a very quick example. So, if we take our favorite defective matrix as 0 1 0 0 then if I take the polynomial so both its Eigen values are equal to 0 and its characteristic polynomial so this is A pA of t is equal to t square.

Now, basically you can see that in this case if I take so my choice is for q qA of t t and t square because I have to get that here and get this factor for each Eigen value I must get a factor here and the powers available for me is only 1 or 2. Obviously, if I take qA of t equals t and I compute qA of A I will not get the 0 of matrix I will get the matrix A. But if I take qA of t equals to t square then qA of A equals 0. So, in this case qA of t equals pA of t.

Student: Yes sir, okay. Thank you.

Professor: So, it is plausible that you do have to take higher powers here it is not always true you have to take ri equal to 1. So, here are some questions so, question 1 is what is the polynomial, what is the minimal polynomial of the identity matrix. What is minimal polynomial of the all-ones matrix. So, you can think about this it is very easy questions but I will let you think about that.