Matric Theory
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Fundamental subspaces of a matrix

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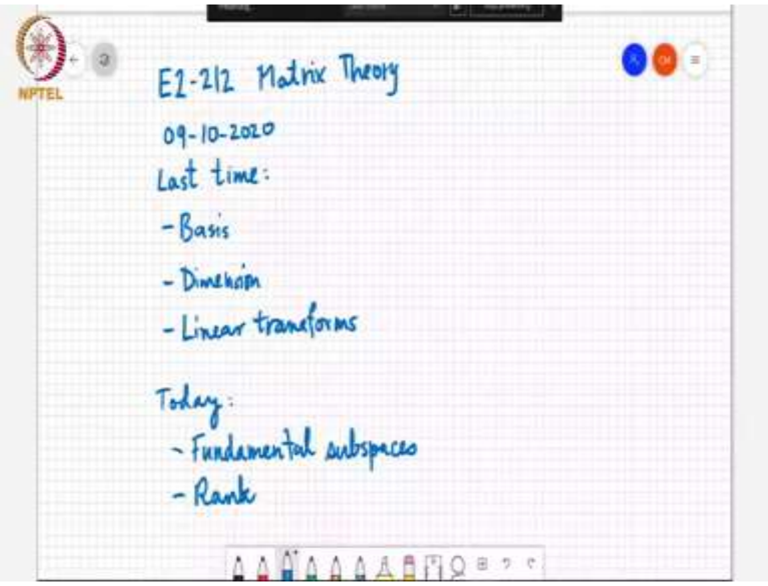

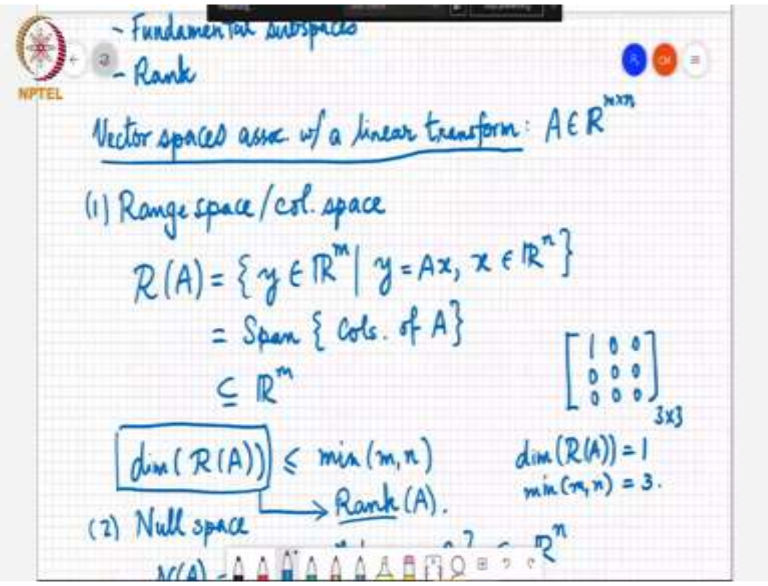

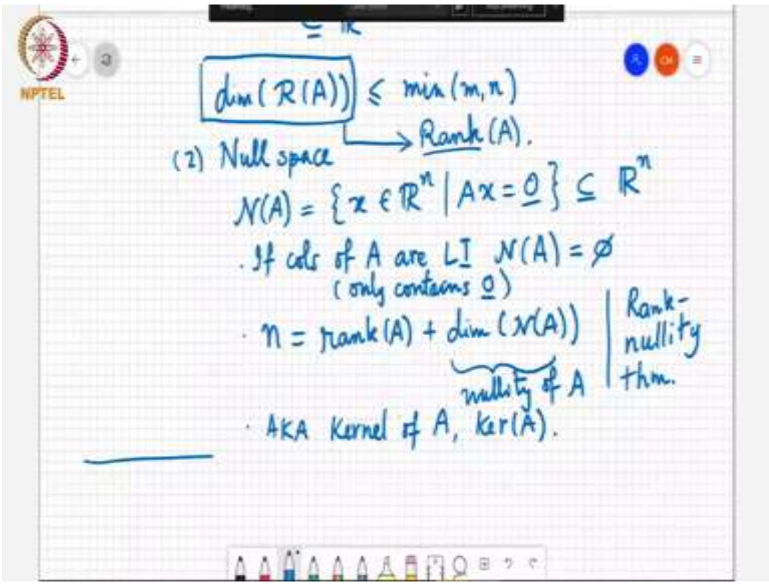

So, we will begin. The last time we discussed three concepts basis, dimension and linear transforms and we started discussing fundamental subspaces associated with the linear transform. We will continue that discussion today. So, just to recap, we started with the range space. So, the vector spaces associated with the linear, vector space is also known as the column space.

So, it is defined as R of A which is the set of vectors y in R to the M, so for all of this A is a matrix in m by n and we are defining the vector spaces associated with the linear transform represented as a matrix A. So, y can be written as a times x for some x in R to the n. It is the, it is also the span of substrates or it is a subset of R to the m and we said that the dimension of this range space of A is less than or equal to the minimum of m and n.

It is at most m because it is a subspace of R to the m or a subset of R to the m and at most n because A has only n columns and so you cannot span a dimension greater than n when you are using, when you are considering the span of n vectors. So, the second space is the null space of A. So, this is n of A, is a set of all vectors in R to the n which map to 0 under A.

So, this 0 with an underline underneath it is the 0 vector, but going forward I will not always draw the underline beneath it, you just know that if the left hand side is a vector, then the right hand side when I say it is 0, I just mean the 0 vector. The null space is just the set of vectors that map to 0. So, it is the set of x such that Ax equals 0 and if the columns of A are linearly independent, we know that the only linear combination of the columns of A that will get you the 0 vector is the all 0 combination.

So, the null space of A contains only one vector which is the all 0 vector. A very fundamental result in linear algebra is that this number n which is the number of columns in A equals the rank of A plus the dimension of the null space of A, so these two dimensions together always add up to n. So, if the rank of A equals n which is true, and the columns of A are linearly independent then the dimension of the null space of A will be 0.

But if the rank of A is less than n then the dimension of the null space of A can be greater than 0 or it will be greater than 0.

Student: Sir, if the null space contains the 0 vector, will it matrix dimension be 1 or 0?

Professor: 0.

Student: But sir how can a vector exist without a dimension.

Professor: So, that is what I was trying to say the thing is that 0 vector is a point and a point has no dimension. So, the way to think about it is take the two dimensional plane, if I consider a line, then these points on this line form a one dimensional subspace of this two dimensional plane, but if I take just the origin, a single point that has no dimension, it has 0 dimension, but it has one…

If I take the entire two dimensional plane then it has two dimensions. Does that answer your question?

Student: Yes, sir. Thank you, sir.

Professor: This is also known as the kernel of A and written as, so this dimension of the null space of A is also called the nullity of A and this result is called the rank nullity theorem. So, before I define the other two null spaces associated with the linear transform, I need to define the notion of orthogonal complement subspaces.

Student: Sir, what is the kernel of A? Can you please repeat?

Professor: It is the same as the null space, so AKA means also known as…

Student: Okay, sir.

TA: The dimension of the range space of A should be equal to minimum of m and n, not less than or equal to.

Professor: So, let me take a matrix 3 cross 3. So, what is the dimension of the range space of A here?

Student: 2.

Professor: Come on.

Student: Oh, sorry, 1.

Student: 1.

Student: 1 sir.

Professor: Equals 1 and min of m, n equals 3. So, that is easy to find matrices for which the dimension of the range space of A is less than min of m, n that is called a rank deficient matrix, if you know it already. So, that is when the rank of the matrix is smaller than the dimension, the smaller of the two dimensions of the matrix.

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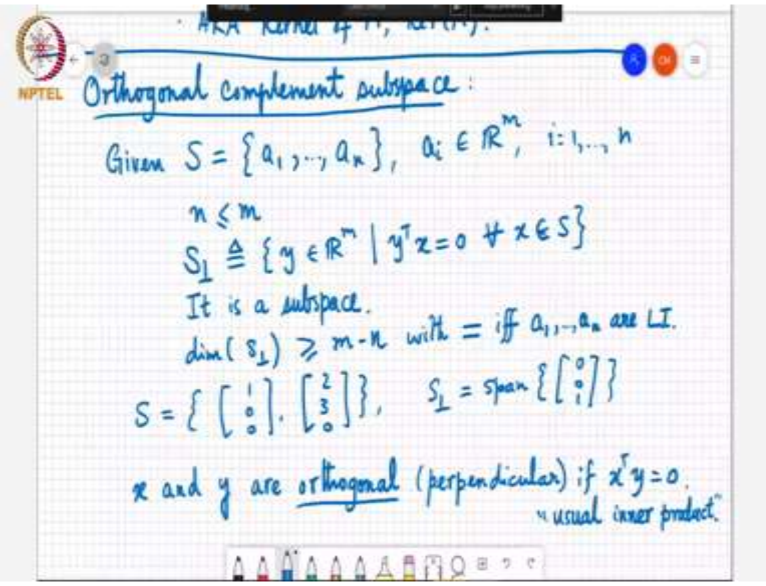

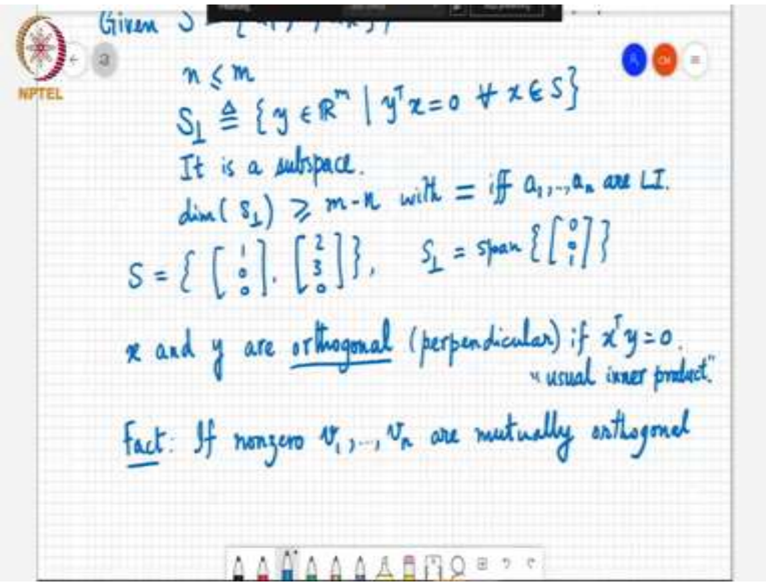

So, orthogonal complement subspace, so given a set of vectors.

Student: How do you say that null space is…?

Professor: How do you say that null space is what?

Student: R to the power of n.

Professor: Sorry, the null space of a matrix A which is of size m by n is a collection of vectors that are sitting in R to the n, because these x's they have to multiply A, so they are all of dimension R to the n. I did not follow your question.

Student: Okay, I got it.

Professor: So, take a collection of vectors, all of them are in R to the m and suppose that n is less than or equal to m, that is you have fewer vectors than the dimension of each of those vectors, then we define S perp to be the set of all vectors y which is also in R to the m, such that y transpose x equal to 0 for every x belonging to the set S.

So, it is a set of vectors that are orthogonal to every vector in this set S. Clearly this is a subspace; this is another small exercise that you can show. If you do take two different y's which, for which satisfy this, then the sum of those two y's will also satisfy this and of course alpha times y also satisfies this and so it is a subspace and we can show that the dimension of S perp is at least equal to m minus n.

And with equality if and only if these vectors A1 to An are linearly independent. So, the dimension of this, now the orthogonal complement subspace is at least m minus n and it is equal to m minus n if and only if A1 to An are linearly independent. So, for example, if I take S to be equal to the set of vectors say 1, 0, 0 and then 2, 3, 0, then S perp would be span of 0, 0, 1. Any vector proportional to this would be orthogonal to both these vectors.

So, in this context I would say and say when things like this get satisfied we call these vectors orthogonal. So, x and y are orthogonal. It is also perpendicular if x transpose y equals 0, this is for real vectors, but for complex vectors we normally use x Hermitian y equals 0. This is what is known as the usual inner product.

Student: Sir?

Professor: Yes, please.

Student: Whether S and S perp will span three dimensional space in this example.

Professor: That is a good point, S and S perp together span the three dimensional space. So, hold that thought, we will come back to it in a few minutes and this is actually one of the, what you have just said is the basis for, in fact, this rank nullity theorem here. So, the rank of A is the span of the columns of A and then the null space of A is actually the set of vectors that map to 0 and so they together, their dimensions is equal to n.

So, this is in fact related to this, so I will come back to that point in a few minutes. So, another fact is that any set of non-zero orthogonal vectors are linearly independent, so mutually orthogonal meaning that I will take pairs of these vectors and then find their inner product and I always get 0, for every pair.

Student: Why do we require n to be less than or equal to m in the first line that is visible?

Professor: No, it is not required. I just said that for convenience here, so this is not required to define S perp.

Student: Okay.

Professor: No, the reason I said that is just because it is easier to imagine an orthogonal complement subspace when the each of these vectors are higher dimensional vectors than the number of vectors you are beginning with. Clearly if you have n vectors and n is less than or equal to m, these vectors together cannot span R to the m.

If n is greater than or equal to m, then it is possible that these vectors A1 to An already span R to the m, then if you look for what are all those vectors which are going to be orthogonal to all these vectors which are spanning R to the m, then you will be left with only one point which is the 0 vector and then S perp will be a null set. So, it is easier to imagine S perp if you start out by assuming n is less than or equal to m.

Obviously, I can say for example, here in this example I can add any number of vectors here and still S perp would be span of 0, 0, 1. So, now I am in a position to define the other two subspaces.

Student: Sir, is a collection of vector subspace, like is S a subspace?

Professor: S perp is not a subspace. See, I told you earlier that a collection of vectors is a subspace only if it is hotel California, you can never leave, you would add any two vectors in that collection you will get another vector which is also in that collection and you scale any vector in that collection, you will get another vector that is also in the collection. So, in fact you can see that in the field of real numbers, you cannot have a finite number of vectors that form a subspace.

Student: Okay sir.

Professor: Whereas when you talk about things like Galois fields, then you can have a finite collection of vectors that span a, that form a subspace.

Student: Sir, basically a subspace is whatever linear combination you can get with some vectors, right?

Professor: That is right, so that was the starting point for us to define the dimension and the basis for a subspace, every subspace has a basis and the basis is the smallest number of vectors that are needed, that are linearly independent and span the subspace.

Student: Okay sir, thank you.

Professor: So, the first, the third fundamental subspace is the ortho…

Student: Sir?

Professor: Yeah please.

Student: I have a question, so can I say that if two subspaces are complements of each other they have to be orthogonal or the vice-versa?

Professor: So, by the definition the orthogonal complement subspace is, or any vector in the orthogonal complement subspace is orthogonal to any vector in S. So, at this point I have not defined orthogonal subspaces but that is not that, the only reason for that is because that is not needed to define the orthogonal complement of a set of vectors, but you can also define an orthogonal complement of a subspace.

And then that does have the property that any vector in the first subspace is going to be orthogonal to any vector in the second subspace.

Student: Okay.

Professor: One thing you can probably see is that if I take the basis of a subspace and I find the orthogonal complement subspace of it then the any vector in the orthogonal complement subspace is orthogonal to the subspace spanned by that basis. The reason being that any vector in the subspace can be represented as a linear combination of the bases.

And any vector in the orthogonal complement subspace is orthogonal to every one of those vectors that are in the basis and therefore it will be orthogonal to any other vector which is spanned by that basis. So, in fact, the third subspace that I am defining here is the orthogonal complement of the column space.

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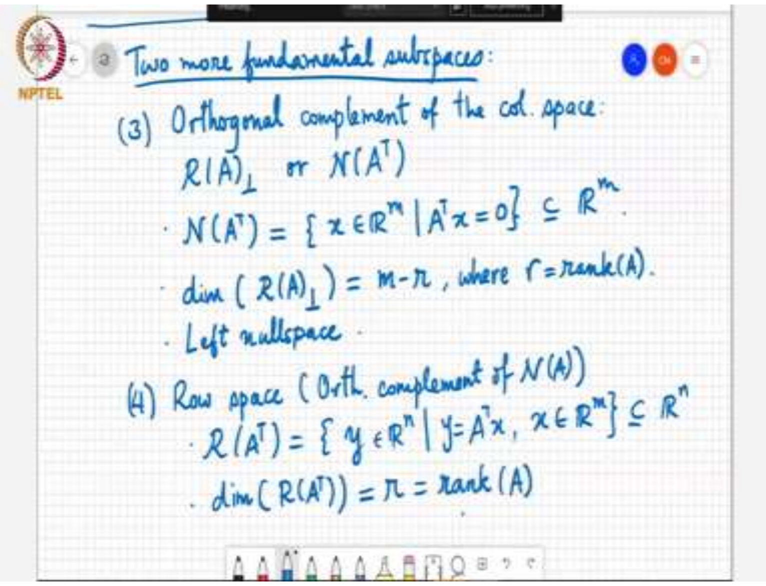

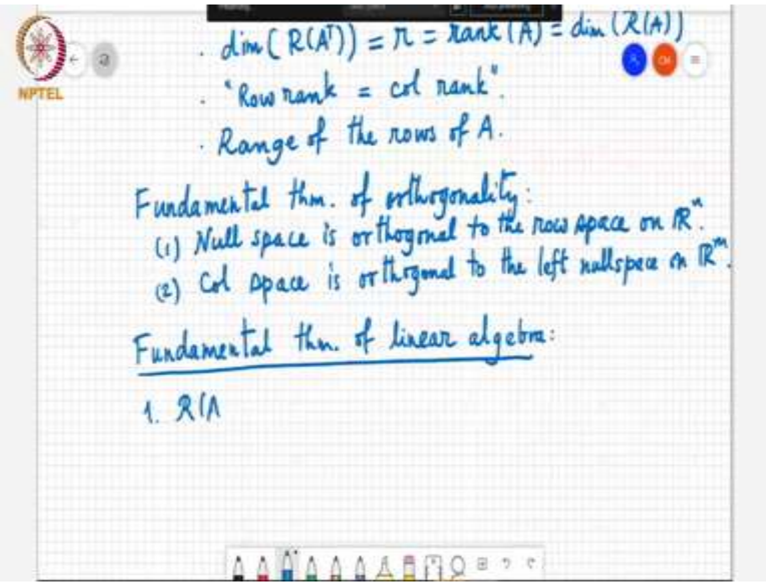

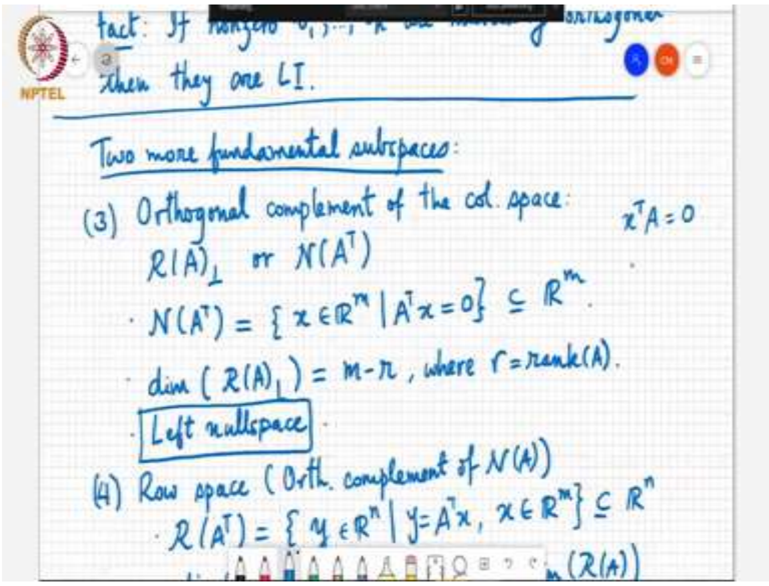

So, that is denoted by R of A perp or it is also called the null space of A transpose and formally n of A transpose is the set of all x such that, let us write it more clearly x belonging to R to the m such that A transpose x equals 0. This is of course sitting in R to the m. So, this is the set of vectors that are orthogonal to all the columns in A.

And if an x is orthogonal to all the columns in A, it is also going to be orthogonal to any linear combination of the columns of A and therefore it will be orthogonal to any vector that lies in the column space of A. And the dimension of R of A perp is equal to m minus R where R is the rank of A. This is exactly the point that somebody just made that if you take a subspace like the column space and you take its orthogonal complement subspace the two together have a dimensionality equal to the space that they are lying in which is m.

And so if the column space has a dimension R, then the orthogonal complement of the column space must have m minus R and this is also called the left null space and the fourth subspace is called the row space. And this is the orthogonal complement of the null space of it, so the row space it is denoted by R of A transpose, is the set of all Y in R to the n, such that y equals A transpose x for some x belonging to R to the m.

And yes this is a sitting in R to the n and the dimension of the row space is equal to what, so what can we say about the dimension of the row space or the orthogonal complement of the null space?

Student: n minus R.

Student: Rank.

Professor: No.

Student: It is rank of this A.

Professor: Exactly, is the same as the dimension of R of A. So, basically the dimension of the column space and the row space of a matrix are equal regardless of the dimension or the size of this matrix or even the rank of the matrix and or what kind of matrix it is, so this is often called, this particular statement that I will also write it like this, dimension of R of A, this is also written as row rank equals column rank.

So, this is true regardless of the size of A and this to me, this is maybe the first result in linear algebra that I am discussing which to me is not intuitively obvious and I cannot give you an intuitive reason as to why this should be true. It is no simple argument I can make which will convince you that the row space of a matrix and the column space of the matrix must always have the same dimension.

So, no matter how hard you try you will never be able to construct a matrix where the row space has a different dimensionality than the column space. Of course, when you, in the next class we will discuss about the row reduced echelon form, which essentially does a series of elementary row operations and the form of the matrix that comes out of these elementary row operations is such that you can see that the row rank will be equal to the column rank.

So, there are ways to see it but just by intuition of looking at the matrix it is very hard to understand why this statement must be true, at least to me. So, this is basically the range of the rows of A.

Student: Sir?

Professor: Yes, please.

Student: Sir, is orthogonality the only way of defining the complement of that subspace or is there any other way that complement of null space of A or row space of A can be defined?

Professor: See there are many other definitions of complement but for defining these fundamental subspaces we use, I mean, we need to use this notion of the usual inner product and define orthogonal subspaces.

Student: Okay.

Professor: So, basically this leads me to the fundamental theorem of orthogonality, which says that the null space of a matrix is orthogonal through the row space, so this follows directly from the way these subspace, and while constructing a transpose x. So, if you think about it by definition these are two orthogonal subspaces and the second is that the column space is orthogonal to the left null space on R to the m.

So, basically the dimension of this plus the dimension of this should be n, the dimension of this plus the dimension of this should be m, so these 4 subspaces and these 2 theorems, these two points are actually collected together in what is known as the fundamental theorem of linear algebra.

Student: Sir?

Professor: So, it basically says that… Yeah?

Student: So, what is the left null space?

Professor: I just defined it, it is right here. Here so the null space of A transpose or the orthogonal complement of the column space is known as the left null space, it is the set of vectors. So, the way to think about it is, if i look for all vectors x such that x transpose A equals 0. Now, this is a row vector containing 0s in it.

Then if I take the transpose of this, this is the same as saying A transpose x equals 0. But this part here is corresponds to multiple A from the left by a vector x and that is why it is called the left null space. It is the set of all vectors when you multiply by A from the left, it maps to the 0 vector. Is that fine?

Student: Yes sir, yes.