**Matrix Theory**
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**Lecture 49**
**Properties of the Jordan Canonical Form (part 2)**

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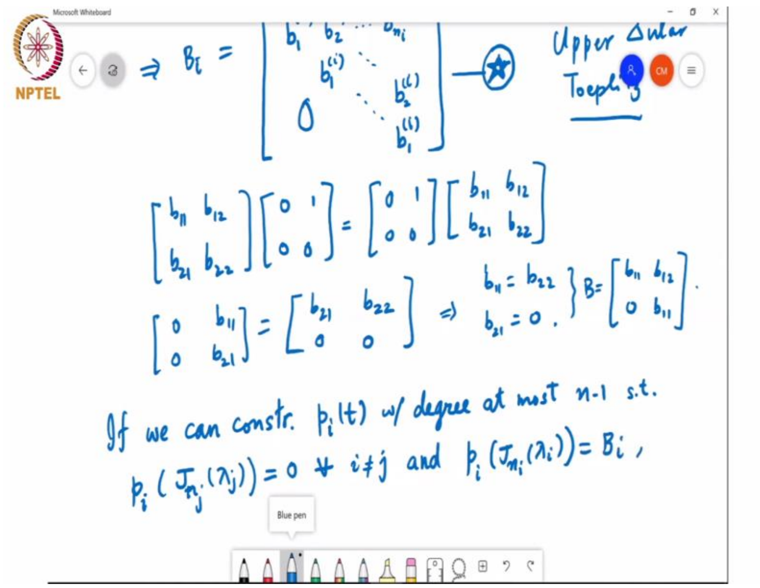


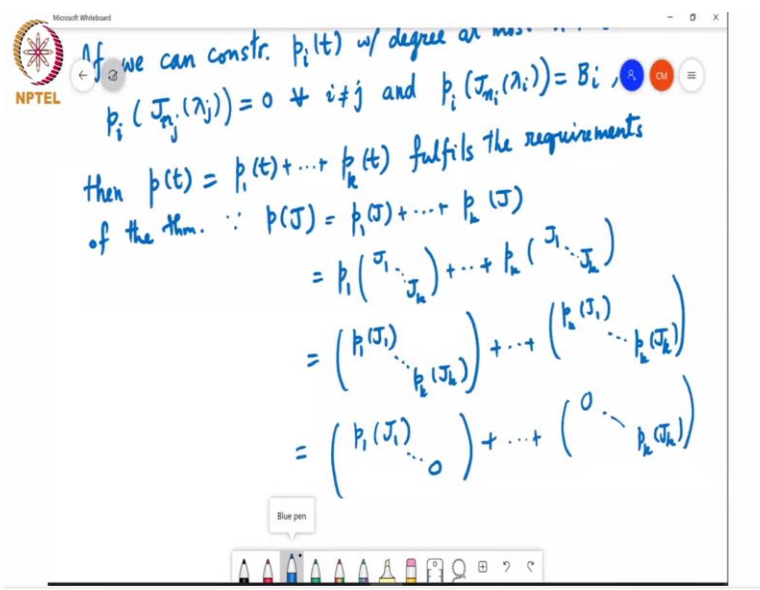

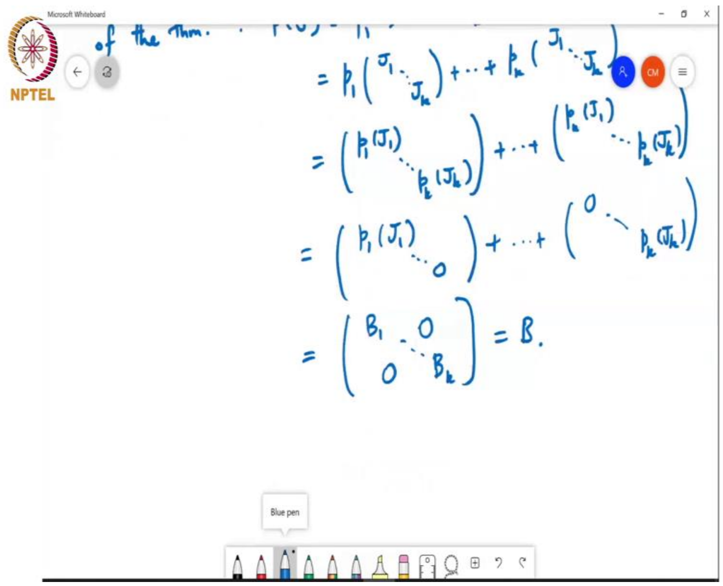


So, now we still have to construct this polynomial. So, if we can construct a polynomial pi of t with so, so with degree at most n minus 1 such that pi have jni of, so let me write this out, then it will be clear. So, if we can construct pi of t with degree at most n minus 1 such that pi of Jnj, Jnj of lambda j. So, bear with me I am just hypothesizing some things and then I will show that this is something desirable for us and then I will show how to construct these polynomials. So, this is equal to 0 for every i not equal to j and then i equals j pi have Jni of lambda i equals Bi.

Then p of t which I will define as p1 of t plus etc plus the pk of t, k is the number of distinct eigenvalues of this matrix A. So, if I consider p1, p1 of t through pk of t will fulfill the requirements of the theorem, is the polynomial or we just say the requirements of the theorem. What do I mean by that I mean that if I consider p of J, this will turn out to be equal to B. Why because if I consider p of J, this is equal to p1 of J plus etc plus pk of J which is equal to p1 of now j is a matrix which is of this block, block diagonal form J1 through Jk plus etc plus pk of J1 through Jk

And a polynomial of a block diagonal form you can apply the polynomial inside this block for each of these block diagonals and that is exactly equal to the polynomial applied to the entire block diagonal form what I mean is that this is equal to p1 of J1 pk of Jk plus etc plus pk of J1 pk of Jk. This is in general not true you cannot apply you cannot push the polynomial inside each element of a matrix but for a block diagonal matrix you can push the polynomial into each of the blocks.

Now, we already said that p is a polynomial such that this is equal to p1 and all these other terms in this block table form are equal to 0. And here all these things will be equal to 0 pk of Jk will be pk and so this is equal to p1 of J1 actually, I should have said Jn1, but anyway, J1 up to all the other things are 0 plus, etc. plus here everything is 0 except pk of Jk and this is equal to p1

And so all these are non overlapping blocks and so I will get B1, BK on the diagonal and 0s everywhere else so, which is equal to my matrix B. So, basically, if I can find, and these polynomials pi of t, the degree at most n minus 1 such that these two properties, one pi of Jnj or lambda j equals 0 for all i not equal to j and pi of Jni of lambda i equals Bi, that is when i equals j, then I am all set. So, now we just need to figure out how to construct these polynomials.

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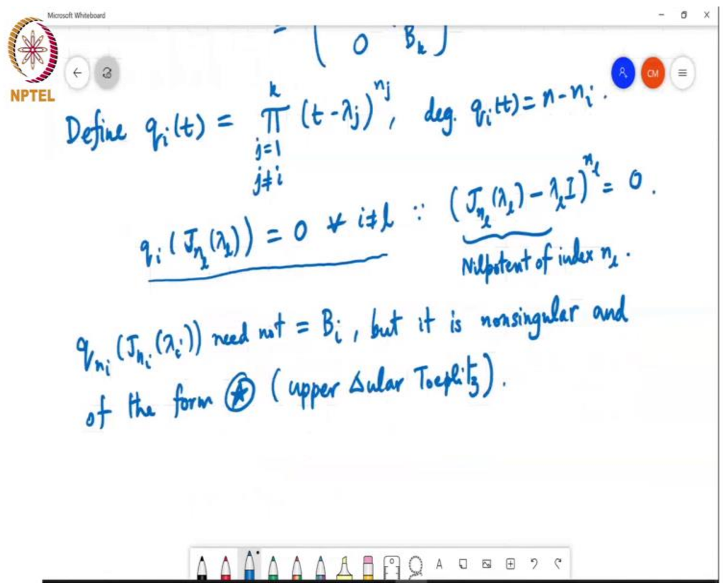


So, here it is. qi of t is equal to the product from j equals 1 to k as a product except the item of t minus lambda j power nj then the degree of this polynomial, qi of t equals what is the sum of all these nj's except the ith term that would be the degree of this polynomial, but the sum of all these nj's equals n because that is the sum of the sizes of all the Jordan blocks. And that should be of size n. So, this is equal to n minus ni.

So now, one thing we can note immediately is that, qi of Jnj of lambda j, if I compute this, this is actually going to be equal to 0 for every i not equal to j. Why is that? It is because this has this kind of form. So, if I substitute Jnj of lambda j, I will get Jnj of lambda j so, the jth term. So, the this is kind of bad notation because j is also the index of the summation here, but whatever this j is, for example, for a moment think of it as l.

So, I will just write it as l so, that it is not confusing. Jnl and lambda l equals 0 for all i not equal to l. And now, if I consider this thing, one of the terms here will be the I am belaboring the point, but there is a lambda l term Jnl of lambda l minus lambda l times the identity matrix. And then I am raising that to the power nl, but this difference is just going to be that nilpotent matrix of size ni cross ni raised to the power nl sorry nl cross nl raised to the power nl and nilpotent matrix when you raise it to the power nl you will get the all 0 matrix.

And so, this is always equal to 0. Jnl of lambda l minus lambda l times the identity matrix power nl is equal to 0. So, one of the terms in this product will be equal to 0 which will make the whole product equal to 0. This is just the nilpotent matrix of index nl. So, now this satisfies one part of what I want qi of lambda j, qi of Jnl of lambda l equals 0 for i not equal to j, but I also need that qi of Jni of lambda i should be equal to Bi I need that property also, but qni of j, qi of Jni of lambda i need not equal Bi, but, one thing we can say about it is that because the i equal to j j equal to ith term is not included here it is the all its eigenvalues will necessarily be non-zero because these lambda J's are all distinct so it is non singular.

And furthermore if you examine this matrix it will actually be so, each of these matrix matrices when I take Jni of lambda i minus lambda j so, when I substitute j ni of lambda i in here one of these terms that is the Jth term will be Jni small jth term will be Jni of lambda i minus lambda j times the identity matrix and this matrix will have non zero entries on the diagonal and nonzero entries on the first super diagonal and it is been raised to the power nj and when you raise it to the power nj, the super diagonal terms may get fully occupied, but it will remain upper triangular, and it will also retain this toeplitz structure that it has that is the diagonal entries are the same, the first Super diagonal entries are the same, the second super diagonal entries are the same and so on.

So, it is of the form star that is the and of the form that is it is upper triangular toeplitz. So, basically the point is that if I take an upper triangular toeplitz matrix, its inverse is upper triangular toeplitz and the product of upper triangular toeplitz matrices is also upper triangular toeplitz. So, here is you are seeing a product of such upper triangular toeplitz matrices, which will also remain upper triangular and toeplitz. And in fact, its inverse is also upper triangular and toeplitz.

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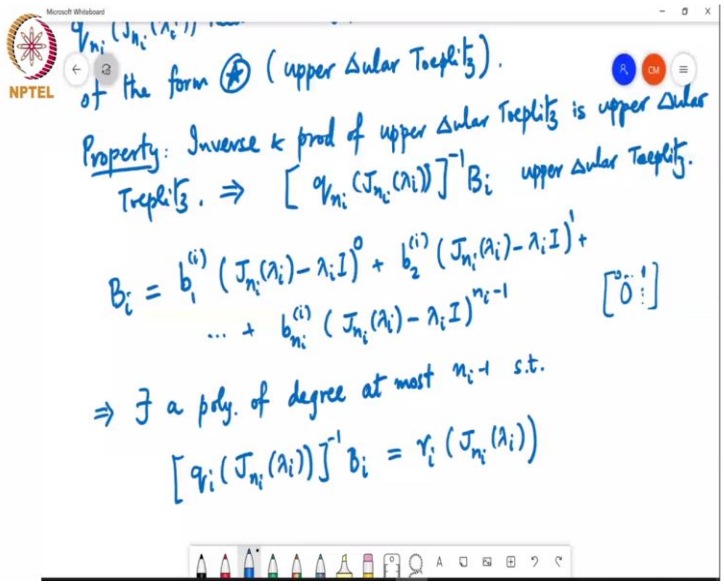


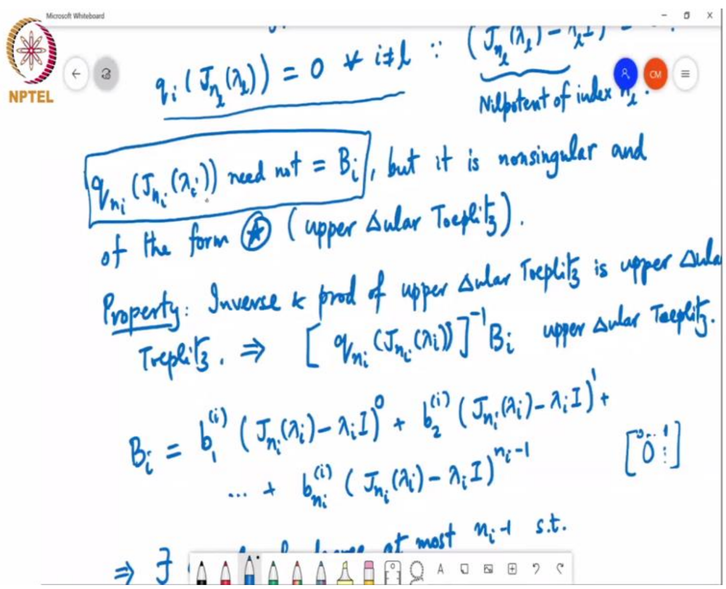

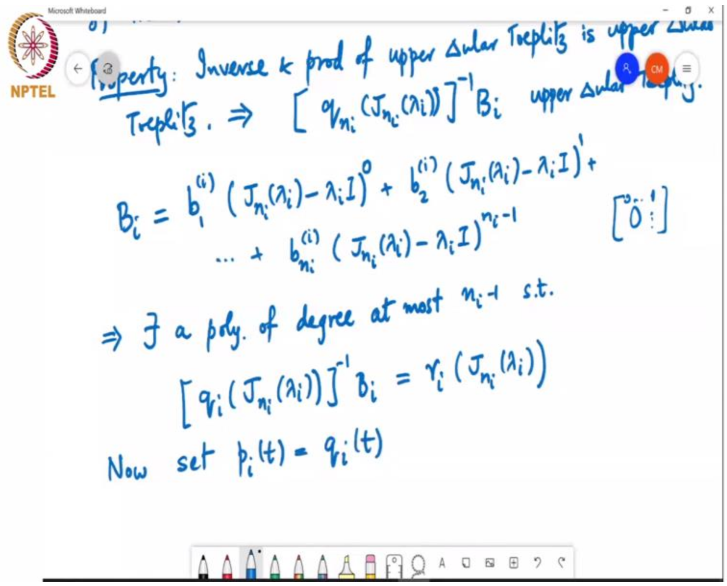

So, we will use that property next. So, this implies that if I consider the matrix J sorry qni Jni of lambda i inverse, this is also these times if I multiply this by bi, bi is also upper triangular and toeplitz. So, this is upper triangular and toeplitz so, now I am pretty close. So, basically a matrix so, basically what I am trying to say is the following.

So, we have Bi, I can write that as b1 of i, this is the first diagonal entries in this matrix Bi times Jni of lambda i minus lambda i times the identity matrix power 0 plus b2 of i it is the first super diagonal entry in the matrix Bi times Jni of lambda i minus lambda i times the identity matrix power 1 plus, and so on plus bni of i times Jni of lambda i minus lambda i identity matrix power and i minus 1.

So, when I raise this to the power 1, this, this will have 1s only on the first super diagonal entry, and those 1s are getting multiplied by b2 of i and so, they are placing b2 of i in the first super diagonal entry of this Bi and similarly, when I raise it to the power of ni minus 1, I will get a matrix which has zeros everywhere else, but a 1 at the top left entry and that is getting multiplied by bni of i and then that is getting placed at the top right entry of this Bi.

So, I can I can always write it like this. So, this implies that there exists now, this itself is a polynomial of degree at most ni minus 1. So, there exists a polynomial degree at most ni minus 1 such that qi of so, Jni of lambda i inverse times Bi, which is an upper triangular toeplitz matrix this can be written as this particular polynomial that I have written here a similar polynomial, but using the entries of this matrix instead of b1i to bni some polynomial of degree at most ni minus 1 ri I will call of Jni of lambda i.

So, I have to do all this circus, because whatever I used earlier this qni of this thing this need not equal Bi and I have to do some circus to make this to find another polynomial such that this property continues to hold, but qni of Jni of lambda i will be equal to Bi that is why I am doing all this all these steps here.

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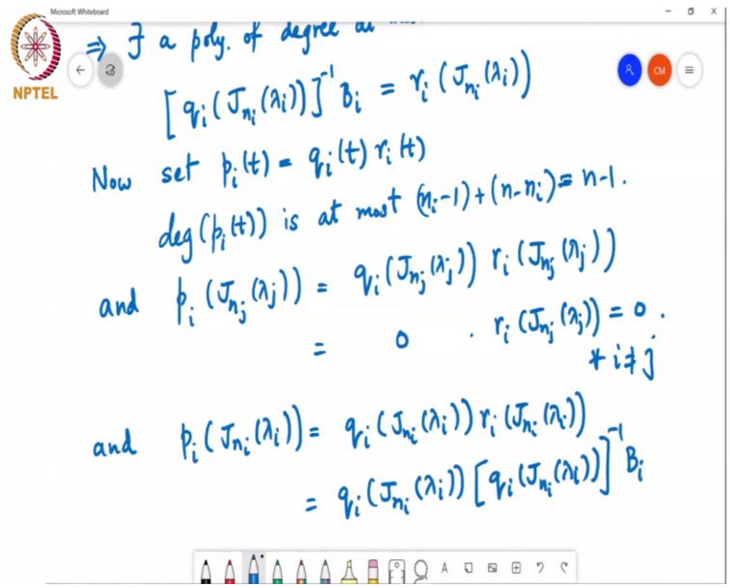


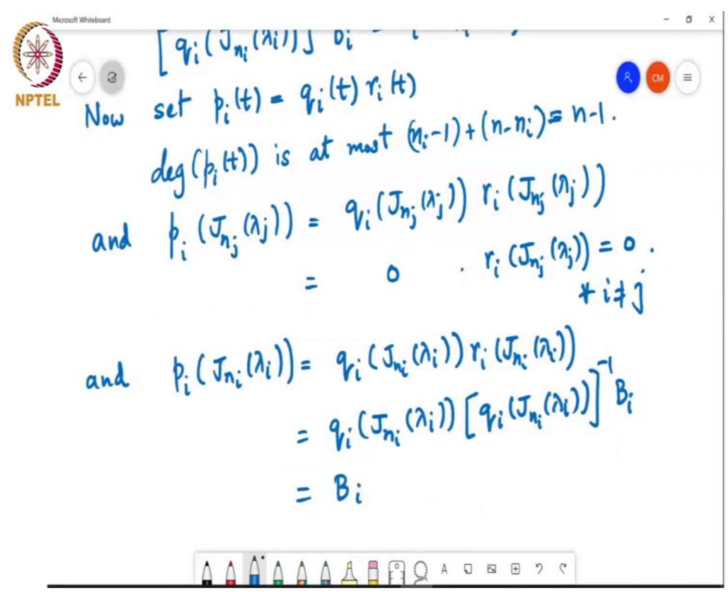

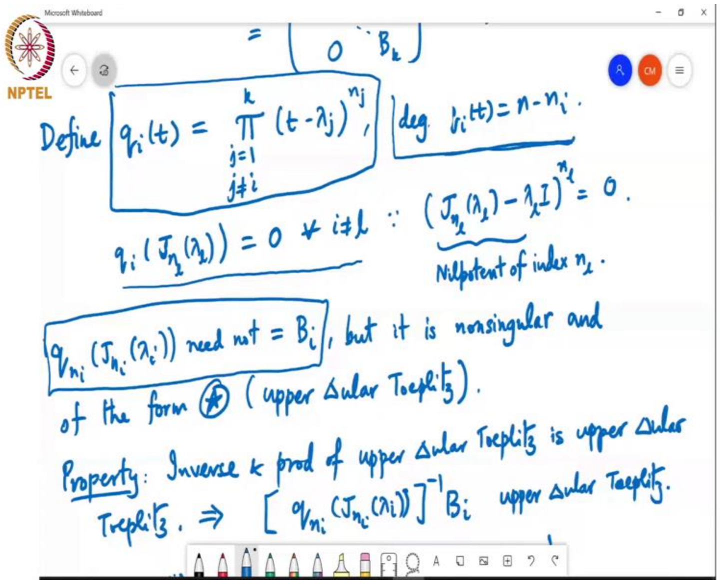


So, now, I am almost done with the proof what I will do is now set pi of t equal to qi of t which is what I defined earlier this qi of t which is of degree at most n minus ni. So, the degree is at most ni minus 1 plus n minus ni n minus 1. So, then this, the claim is basically that this if I did pi of J this is that is all I need. So, this matrix satisfies nj of lambda j is equal to qi of Jnj of lambda j times ri of Jnj of lambda j and this is equal to 0 by our construction.

So, 0 times ri of Jnj of lambda j which is equal to the all 0 matrix for all i not equal to j and pi Jni of lambda i is equal to qi of Jni of lambda i times ri of Jni of lambda i and ri of Jni of lambda i is this matrix here and qi of Jni of so, I get qi of Jni of lambda i times qi of Jni of lambda i inverse times Bi. So, these 2 just cancel with each other and this is equal to Bi, which is all that we were looking for.

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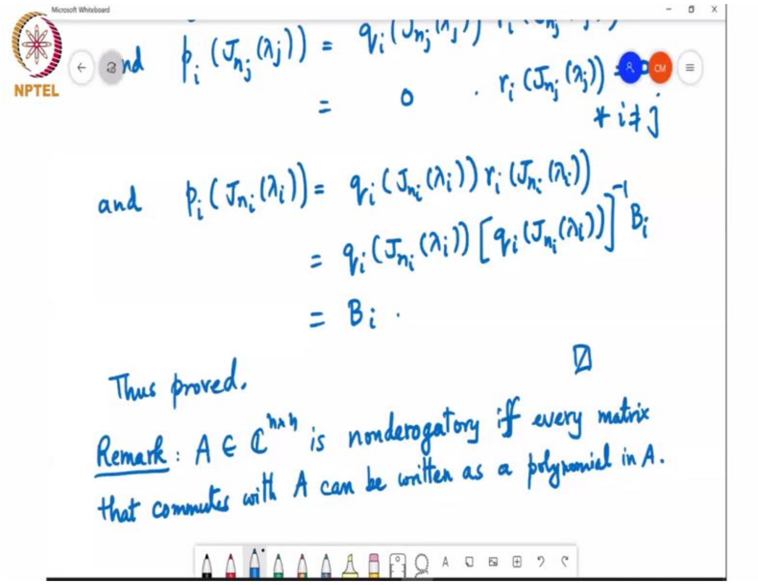


So, so, what we just showed is that, if the matrix A is non derogatory, then any other matrix B commutes with a if and only if there exists a polynomial p of degree at most n minus 1 such that B equals B of A. Of course, if B equals p of A then A commutes with B that is trivial, but showing the other way around was a little bit more involved, we had to do quite a few steps to show that if A and B commute and A is non derogatory, then there must exist a polynomial of degree at most n minus 1 such that B can be written as a polynomial of A.

In fact, the converse is also true namely that A in C to the n cross n is non derogatory if and only if every matrix that commutes with A can be written as a polynomial in A. So, that is all we have time for today, I wanted to talk, I wanted to also talk about convergent matrices and their properties. So, we have already seen that a matrix A is convergent if all the elements of A power m go to 0 as n goes to infinity.

And if the matrix A is diagonal, it means that it is convergent if and only if all the diagonal entries in magnitude are less than 1, which means that all the eigenvalues of this matrix A are less than 1 in magnitude. And this directly extends to diagonalizable matrices because A power m can be written as V lambda power m times v hermitian, where V is the mate or v hermitian lambda power m times v, where V is a matrix containing the eigenvalues of this matrix A.

And so basically that we have also seen this before that diagonalizable matrices are convergent if the magnitude of all the eigenvalues of the matrix are less than 1. Now, what we will discuss the next time is the extension of this idea to non diagonal matrices also, which of course we will do through the Jordan canonical form. So that is it for today and we will continue next on in, continue in the next class.